Exercise Sheet 7

From the Langevin to the Fokker-Planck equation

The relativistic stochastic Langevin equation for the heavy-quark motion in a thermally equilibrated medium is defined in the (local) rest frame of the heat bath, by the time step,

$$\mathrm{d}x_j = \frac{p_j}{E}\mathrm{d}t, \quad \mathrm{d}p_j = -\Gamma p_j \mathrm{d}t + C_{jk}^{(\xi)} \rho_k \sqrt{\mathrm{d}t}. \tag{1}$$

The ρ_k are uncorrelated Gaussian norml-distributed random numbers with the statistical properties

$$\left\langle \rho_j \right\rangle = 0, \quad \left\langle \rho_j \rho_k \right\rangle = \delta_{jk}. \tag{2}$$

$\Gamma = \Gamma(\vec{p})$ is the drag (or friction) coefficient and the covariance matrix $\hat{C} = \hat{C}^{\mathrm{T}} = \sqrt{\hat{B}(\vec{p})}$ with the diffusion matrix $\hat{B}$, as derived in the lecture.
In (1) the realization of the stochastic process depends on the choice, at which momentum value between $\vec{p}$ and $\vec{p} + \mathrm{d}\vec{p}$ the covariance matrix of the fluctuatin force $\hat{C}$ in (1) is evaluated, parametrized by the parameter $\xi \in [0,1]$, i.e., one has to use

$$\Gamma_{jk}^{(\xi)} = \Gamma_{jk}(\vec{p} + \xi \mathrm{d}\vec{p}) = \xi \mathrm{d}\vec{p}_l \frac{\partial}{\partial p_l} \Gamma(\vec{p}). \tag{3}$$

We want to derive the Fokker"=Planck equation for the phase-space distribution function of the heavy quarks that describes the statistics of the heavy-quark evolution due to the stochastic dynamics given by the Lange equation (1). By definition with the phase-space distribution function $f(t,\vec{x},\vec{p})$ the expectation value for an arbitrary phase-space function $g(\vec{x},\vec{p})$ at time t is given by

$$\left\langle g(\vec{x},\vec{p}) \right\rangle_t = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\, g(\vec{x},\vec{p}) f(t,\vec{x},\vec{p}). \tag{4}$$

The Fokker-Planck equation to be derived is

$$\partial_t f + \frac{p_j}{E}\frac{\partial}{\partial x_j} f = \left(\Gamma p_j - \xi C_{lk} \frac{\partial C_{jk}}{\partial p_l}\right) f + \frac{1}{2}\frac{\partial^2}{\partial p_j \partial p_k}\left(C_{jl} C_{kl} f\right). \tag{5}$$

To prove it, follow the following steps to calculate the expression for the time derivative $\mathrm{d}_t \left\langle g(\vec{x},\vec{p}) \right\rangle_t$.

(a) Expand
$$g[\vec{x}(t+\mathrm{d}t), \vec{p}(t+\mathrm{d}t)] - g(\vec{x},\vec{p}) = g(\vec{x}+\mathrm{d}\vec{x}, \vec{p}+\mathrm{d}\vec{p}) - g(\vec{x},\vec{p}), \tag{6}$$
with $\mathrm{d}\vec{x}$ and $\mathrm{d}\vec{p}$ given by (1) in powers of $\mathrm{d}t$ up to order $\mathrm{d}t$ (note that the next order to be omitted is $\mathscr{O}(\mathrm{d}t^{3/2})$ due to the $\sqrt{\mathrm{d}t}$ dependence of fluctuating-force contribution).

(b) Take the expectation value of this expression,
$$\left\langle g(\vec{x}+\mathrm{d}\vec{x}, \vec{p}+\mathrm{d}\vec{p}) - g(\vec{x},\vec{p}) \right\rangle \tag{7}$$
using the statistical properties of the fluctuating force, defined by (2).

(c) Calculate the expectation value, on the other hand, by using the phase-space distribution function, f:

$$\left\langle g(\vec{x}+\mathrm{d}\vec{x},\vec{p}+\mathrm{d}\vec{p})-g(\vec{x},\vec{p})\right\rangle = \int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\,f(t,\vec{x},\vec{p})\left[\left\langle g(\vec{x}+\mathrm{d}\vec{x},\vec{p}+\mathrm{d}\vec{p})-g(\vec{x},\vec{p})\right\rangle\right]. \tag{8}$$

Use integration by parts to shuffle all partial derivative wrt. to $\vec{x}$ and $\vec{p}$ to the distribution function.

(d) Divide the found expression by $\mathrm{d}t$ and take the limit $\mathrm{d}t \to 0$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle g(\vec{x},\vec{p})\right\rangle_t = \mathrm{d}_t\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\,f(t,\vec{x},\vec{p})g(\vec{x},\vec{p}) = \int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{p}\,g(\vec{x},\vec{p})\partial_t f(t,\vec{x},\vec{p}). \tag{9}$$

By comparing this with the result from the previous step, conclude the validity of the Fokker"=Planck equation (5).