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Exercise Sheet 6

(1) Particle-Number distribution in the grand-canonical ensemble

The grand-canonical distribution of a many-body system is defined by the statistical operator

$$\mathbf{R} = \frac{1}{Z}\exp(-\beta \mathbf{H} - \alpha \mathbf{N}), \quad Z(\beta,\alpha) = \operatorname{Tr}\exp(-\beta \mathbf{H} - \alpha \mathbf{N}). \tag{1}$$

For an ideal gas of fermions or bosons one obtains in the limit of a non-degenerate gas, i.e., where the average occupation number for the single-particle states is small compared to 1, one obtains the Maxwell"=Boltzmann approximation for the phase-space distribution function,

$$f(\vec{p}) = \exp(-\alpha)\exp(-\beta E_p), \quad E_p = \sqrt{m^2+\vec{p}^2} = \sqrt{m^2+p^2}. \tag{2}$$

The corresponding phase-space distribution for the particles is then

$$\frac{\mathrm{d}N}{\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{p}} = \frac{gV}{(2\pi)^3} f(\vec{p}), \tag{3}$$

where g is the degeneracy factor for the particles under consideration due to spin, isospin etc. In this approximation the partition sum (or rather its logarithm!) is given by

$$\ln Z = \frac{gV}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\, f(\vec{p}). \tag{4}$$

(a) Show with help of (1) that

$$\langle N \rangle = -\partial_\alpha \ln Z = \ln Z. \tag{5}$$

(b) According to (1) the probability that the gas contains exactly $\tilde{N}$ particles, is given by

$$P(\tilde{N}) = \operatorname{Tr}[\mathbf{R}\,\delta(\mathbf{N}-\tilde{N})] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\phi\, \operatorname{Tr}\{\mathbf{R}\exp[\mathrm{i}\phi(\mathbf{N}-\tilde{N})]\}. \tag{6}$$

Show that

$$P(\tilde{N}) = \frac{1}{2\pi Z}\int_{-\pi}^{\pi}\mathrm{d}\phi\, Z(\beta,\alpha-\mathrm{i}\phi)\exp(-\mathrm{i}\phi\tilde{N}). \tag{7}$$

(c) Evaluate the integral to prove that the number distribution is given by the Poisson distribution,

$$P(\tilde{N}) = \frac{\langle N\rangle^{\tilde{N}}}{\tilde{N}!}\exp(-\langle N\rangle). \tag{8}$$

(d) Evaluate its generating function,

$$g(x) = \sum_{\tilde{N}=0}^{\infty} x^{\tilde{N}} P(\tilde{N}). \tag{9}$$

(e) Determine $\langle N\rangle$ and $\langle N^2\rangle$ using

$$g'(1) = \langle N\rangle, \quad g''(1) = \langle N(N-1)\rangle = \langle N^2\rangle - \langle N\rangle. \tag{10}$$

(f) Why are the equations (10) valid?

(g) What is the k^{th} derivative, $g^{(k)}(1)$, for a generating function of an arbitrary distribution function?

(2) Rate equation for grand-canonical ($\langle N_s \rangle \gg 1$) and "ultra-canonical" treatment i($\langle N_s \rangle \ll 1$)

In the lecture we have derived the master equation for the generating function $g(\tau, x)$ in the general case, where the time evolution for a single event is described with the exact number of $s\bar{s}$ pairs,

$$\partial_\tau g = \frac{L}{V}(1-x)(x\partial_x^2 g + \partial_x g - \epsilon g), \tag{11}$$

where $\epsilon = G \langle N_A \rangle \langle N_B \rangle / L$, with $\langle N_{A,B} \rangle$ the grand-canonical expectation values for the Numbers of the "light particles" in the reaction $A + B \longleftrightarrow s + \bar{s}$, underlying the Boltzmann collision term.

The master equation for the approximation, where also the strange particles are treated in the grand-canonical approximation, which is valid in the limit $\langle N_s \rangle = \langle N_{\bar{s}} \rangle \gg 1$, i.e., when $\langle N^2 \rangle \simeq \langle N \rangle^2$,

$$\partial_\tau g_{gc} = \frac{L}{V}\sqrt{\epsilon}(1-x)(\partial_x g_{gc} - \sqrt{\epsilon} g_{gc}). \tag{12}$$

(a) Use (11) and (12) to derive the rate equations for the time evolution of the average number $\langle N \rangle (\tau)$ of $s\bar{s}$ by making use of $\partial_x g(\tau, 1) = N(\tau)$ in the exact case

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \langle N \rangle = \frac{L}{V}(\epsilon - \langle N^2 \rangle) \tag{13}$$

and in the case where the strange particles are treated by the grand-canonical approximation in the "single event",

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \langle N \rangle_{gc} = \frac{L}{V}\sqrt{\epsilon}(\sqrt{\epsilon} - \langle N \rangle_{gc}). \tag{14}$$

Note that (14) is a closed differential equation for $\langle N \rangle_{gc}$, while (14) is not closed, because on the right-hand side one has $\langle N^2 \rangle$. One can use (11) to prove that one obtains an "infinite tower" of equations for all moments $\langle N^k \rangle$, $k \in \mathbb{N}$, whose solution would be equivalent to solve (11) itself.

(b) Solve (14) and show that for $\tau \to \infty$ one finds for the equilibrium value $\langle N \rangle_{gc,eq} = \sqrt{\epsilon}$. What is the "relaxation time", i.e., the typical time scale at which $\langle N_{gc} \rangle (\tau)$ approaches $\langle N \rangle_{gc,eq}$.

(c) Now consider the case that $\langle N_s \rangle = \langle N_{\bar{s}} \rangle \ll 1$. To that end assume that $x\partial_x^2 g \ll \partial_x g$, i.e., one can simply omit this term on the right-hand side of (11).

(d) How does the rate equation for the corresponding expectation value $\langle N \rangle$ look like now? What is the equilibrium value, $\langle N \rangle_{eq}$ and the relaxation time in that case? Is the result consistent with the assumption made to derive the approximate rate equation?