Einführung in die Theoretische
Kern- und Elementarteilchenphysik 2

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Exercise Sheet 5

(1) Ideal fluid

Consider the phase-space distribution function of local thermal equilibrium

$$f(\underline{x},\underline{p}) = \exp\{-\beta(\underline{x})[\underline{u}(\underline{x})\cdot\underline{p}-\mu(\underline{x})]\}\Big|_{p^0=E_p=\sqrt{\vec{p}^2+m^2}}. \tag{1}$$

The particle-number current density and the energy-momentum tensor are defined by

$$J^{\mu}(\underline{x}) = \frac{g}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{E_p}p^{\mu}f(\underline{x},\underline{p}), \tag{2}$$

$$T^{\mu\nu}(\underline{x}) = \frac{g}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{E_p}p^{\mu}p^{\nu}f(\underline{x},\underline{p}). \tag{3}$$

(a) Calculate these fields in the local rest frame, where $\underline{u}^*(\underline{x}^*) = (1,0,0,0)^{(\mathrm{T})}$ and define the scalar fields $n(\underline{x})$ (the particle-number density in the local rest frame), $\epsilon(\underline{x})$ (the internal-energy density in the local rest frame), and $P(\underline{x})$ (the pressure in the local rest frame).

(b) Use the fact that $\underline{u}$ is a four-vector field to derive the expressions for $J^{\mu}(\underline{x})$ and $T^{\mu\nu}(\underline{x})$ in the lab frame, where the fluid element at $\underline{x}$ moves with four-velocity $\underline{u}(\underline{x}) = \gamma_v(1,\vec{v})^{\mathrm{T}}$ to show that

$$J^{\mu} = n u^{\mu}, \quad T^{\mu\nu} = (\epsilon+P)u^{\mu}u^{\nu} - P\eta^{\mu\nu}. \tag{4}$$

(c) Use the (rotation-free) Lorentz boost with velocity $\vec{v} = \vec{u}/u^0$ (note that $u^0 = \gamma_v$, and $\vec{u} = \gamma_v\vec{v}$),

$$\hat{\Lambda} = ({\Lambda^{\mu}}_{\nu}) = \begin{pmatrix} u^0 & \vec{u}^{\mathrm{T}} \\ \vec{u} & \mathbb{1}_3 + (u^0-1)\vec{n}\otimes\vec{n} \end{pmatrix} \quad \text{with} \quad \vec{n} = \vec{u}/|\vec{u}| = \vec{v}/|\vec{v}|, \tag{5}$$

for which

$$\underline{u} = \hat{\Lambda}\underline{u}^*, \quad \underline{x} = \hat{\Lambda}\underline{x}^*, \quad \ldots \tag{6}$$

to verify your finding in (b).

(d) Use the continuity equation (particle-number conservation) and the ideal-fluid equation of motion,

$$\partial_{\mu}J^{\mu} = \partial_{\mu}(n u^{\mu}) = 0, \quad \partial_{\mu}T^{\mu\nu} = 0 \Rightarrow u_{\nu}\partial_{\mu}T^{\mu\nu} = 0, \tag{7}$$

to verify that the entropy is (locally) conserved,

$$\partial_{\mu}(s u^{\mu}) = 0. \tag{8}$$

Hint: Use the Gibbs-Duhem relation

$$H = U + pV = TS + \mu N \tag{9}$$

and

$$\mathrm{d}H = \mathrm{d}U + p\mathrm{d}V + V\mathrm{d}p = T\mathrm{d}S + p\mathrm{d}V + \mu\mathrm{d}N \tag{10}$$

and

$$h = \frac{H}{V} = \frac{U + pV}{V} = \epsilon + p, \quad s = \frac{S}{V}, \quad n = \frac{N}{V} \tag{11}$$

to prove that

$$s\mathrm{d}T + n\mathrm{d}\mu + \mathrm{d}P \tag{12}$$

and with that

$$\mathrm{d}\left(\frac{h}{n}\right) = T\mathrm{d}\left(\frac{s}{n}\right) + \frac{1}{n}\mathrm{d}P. \tag{13}$$

Also note that

$$u_\nu u^\nu = 1 \Rightarrow \partial_\mu (u_\nu u^\nu) = 2u_\nu \partial_\mu u^\nu = 0. \tag{14}$$

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(2) Bjorken flow

The Bjorken flow is an exact solution of relativistic hydrodynamics, which describes roughly the expansion of a hot and dense fireball produced in ultra-relativistic heavy-ion collisions [Bjo83]. It uses the fact that in ultra-relativistic collions of nucleons (and thus also nuclei) in the center-momentum frame the produced hadron-number density n (and thus also ϵ and P) in the mid-rapidity region are independent of spatial rapidity, η [Fey69], where

$$\underline{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tau \cosh \eta \\ x \\ y \\ \tau \sinh \eta \end{pmatrix}, \quad \tau = \sqrt{t^2 - z^2}, \tag{15}$$

with the z-direction the beam direction of the colliding nuclei (in the center-momentum rame).

Then one has $n = n(\tau)$, $\epsilon = \epsilon(\tau)$, and $P = P(\tau)$, and we assume that this holds for the entire (early-time) evolution of the fireball.

The incoming nuclei are highly Lorentz contracted in z direction, and when hitting each other at $z = 0$ the "leading partons" run nearly undisturbed through each other ("color transparency"), while through the secondary collisions many new partons are produced. After a very short "formation time" (after about 0.2-1 fm/c) this hot and dense "QGP" comes to local thermal equilibrium and streams as a (nearly) perfect fluid.

The Bjorken-flow solution of the ideal-fluid dynamics makes the ansatz that the fluid four-velocity is given by $v^1 = v^2 = 0$ (i.e., one neglects the "radial flow" completely) and $v^3(\underline{x}) = z/t$. In the following we only consider the longitudinal expansion of this cylindrical fireball. The four-velocity field thus is

$$\underline{u} = \begin{pmatrix} u^0 \\ u^3 \end{pmatrix} = \gamma_v \begin{pmatrix} 1 \\ z/t \end{pmatrix}, \quad \gamma_v = \frac{1}{\sqrt{1 - z^2/t^2}}. \tag{16}$$

(a) For the evaluation of the idea-fluid equations,

$$\partial_\mu (n u^\mu) = 0, \quad \partial_\mu T^{\mu\nu} = 0 \quad \text{with} \quad T^{\mu\nu} = h u^\mu u^\nu - P\eta^{\mu\nu}, \quad h = \epsilon + P, \tag{17}$$

it is convenient to introduce Milne coordinates $(q^\alpha) = (\tau, \eta)$

$$\underline{x} = \begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} \tau \cosh \eta \\ \tau \sinh \eta \end{pmatrix}. \tag{18}$$

Show that because of

$$\partial_\mu = \frac{\partial q^\alpha}{\partial x^\mu} \frac{\partial}{\partial q^\alpha} \tag{19}$$

one has

$$(\partial_\mu) = \begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} \tau \cosh\eta & -\tau \sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \partial_\eta \end{pmatrix} = \begin{pmatrix} \cosh\eta \partial_\tau - (1/\tau)\sinh\eta \partial_\eta \\ -\sinh\eta \partial_\tau + (1/\tau)\cosh\eta \partial_\eta \end{pmatrix}. \tag{20}$$

Hint: Defining the matrices

$$U^\mu{}_\alpha = \frac{\partial x^\mu}{\partial q^\alpha} \quad \text{and} \quad T^\alpha{}_\mu = \frac{\partial q^\alpha}{\partial x^\mu} \tag{21}$$

one has $\hat{U}\hat{T} = \hat{T}\hat{U} = \mathbb{1}_2$.

(b) Show that the above defined fluid four-velocity is given by

$$\underline{u} = \begin{pmatrix} u^0 \\ u^3 \end{pmatrix} = \begin{pmatrix} \cosh\eta \\ \sinh\eta \end{pmatrix}. \tag{22}$$

(c) Show that

$$\partial_\mu u^\mu = \frac{1}{\tau}. \tag{23}$$

(d) What follows for the (proper) density of conserved quantities like $n = n(\tau)$ (particle-number density) and $s(\tau)$ (entropy density) by using the continuity equation

$$\partial_\mu(n u^\mu) = 0, \quad \partial_\mu(s u^\mu) = 0. \tag{24}$$

Note: The ideal-fluid dynamics are only valid for some initial proper time, $\tau_0 > 0$, which we take to be the "formation time" of the locally equilibrated fluid.

(e) Now also consider the equation

$$u_\nu \partial_\mu T^{\mu\nu} = 0 \tag{25}$$

under the assumption of a constant speed of sound c_s

$$P = c_s^2 \epsilon. \tag{26}$$

What follows for ϵ, P, and the temperature T (as a function of τ)?

References

[Bjo83] J. D. Bjorken, Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region, Phys. Rev. D **27**, 140 (1983), `https://doi.org/10.1103/PhysRevD.27.140`.

[Fey69] R. P. Feynman, Very high-energy collisions of hadrons, Phys. Rev. Lett. **23**, 1415 (1969), `https://doi.org/10.1103/PhysRevLett.23.1415`.