Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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SoSe 2026
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Exercise Sheet 4

(1) Master equations for summational invariants

In the manuscript [Hee15] we have derived the general master equation for an arbitrary phase-space function $\psi(\underline{x},\underline{p})$,

$$\begin{aligned}&\frac{\partial}{\partial x^\mu}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}p_1^\mu f_1\psi_1-\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}f_1\left(p_1^\mu\frac{\partial\psi_1}{\partial x^\mu}+mK_1^\mu\frac{\partial\psi_1}{\partial p_1^\mu}\right)\\ &\quad=\frac{1}{4}\frac{g}{(2\pi\hbar)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1}{E_1}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1{}'}{E_1'}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2{}'}{E_2'}(\psi_1'+\psi_2'-\psi_1-\psi_2)\\ &\qquad\times W(p_1',p_2'\leftarrow p_1,p_2)f_1f_2,\end{aligned}\tag{1}$$

where the notation f_1 for phase-space functions means $f(\underline{x},\underline{p}_1)$. Further we have shown that for summational invariants, which fulfill $\psi_1'+\psi_2'=\psi_1+\psi_2$ under the condition of four-momentum conservation, $\underline{p}_1+\underline{p}_2=\underline{p}_1'+\underline{p}_2'$, which is ensured in the collision integral by the corresponding factor $W(p_1',p_2'\leftarrow p_1,p_2)\propto\delta^{(4)}(\underline{p}_1+\underline{p}_2-\underline{p}_1'-\underline{p}_2')$, i.e., for summational invariants the right-hand side vanishes. Further we have shown that ψ is a summational invariant if and only if there are fields A and B_μ such that

$$\psi(\underline{x},\underline{p})=A(\underline{x})+B_\mu(\underline{x})p^\mu.\tag{2}$$

Apply (1) to the following summational invariants and interpret the results physically:

(a) $\psi=1$

(b) $\psi=p^\alpha$

(c) $\psi=x^\alpha p^\beta-x^\beta p^\alpha$.

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(2) Equilibrium distributions

The collision term, taking into account Fermi-Dirac statistics, reads

$$\begin{aligned}\mathscr{C}[f_1]=\frac{1}{2}\frac{g}{(2\pi\hbar)^3}&\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1{}'}{E_1'}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2{}'}{E_2'}\\ &\times W(p_1',p_2'\leftarrow p_1,p_2)[f_1'f_2'(1-f_1)(1-f_2)-f_1f_2(1-f_1')(1-f_2')],\end{aligned}\tag{3}$$

where the factors $(1-f)$ take "Pauli blocking" into account, i.e., that a particle cannot be scattered into a state, which is already occupied. We have also seen from the H-theorem, that the equilibrium distribution $f^{(\mathrm{eq})}$ is determined by the vanishing of the collision term, i.e., the vanishing of the square bracket under the collision integral.

Show that this condition implies that for the equilibrium distribution

$$\phi=-\ln\left(\frac{f}{1-f}\right)\tag{4}$$

is a summational invariant, for which (2) must hold. In this case set

$$A(\underline{x}) = -\beta(\underline{x})\mu(\underline{x}), \quad B^{\mu}(\underline{x}) = \beta(\underline{x}) u_{\mu}(\underline{x}). \tag{5}$$

Which equilibrium distribution follows and what is the physical meaning of $\beta(\underline{x})$ and $\mu(\underline{x})$?

References

[Hee15] H. van Hees, Introduction to relativistic transport theory (2015), `https://itp.uni-frankfurt.de/~hees/publ/kolkata.pdf`.