Einführung in die Theoretische
Kern- und Elementarteilchenphysik 2

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Exercise Sheet 3

(1) Black-Body radiation

Consider the Planck spectrum of black-body radiation. In the rest frame of the thermal source the distribution function is given by a Bose-Einstein distribution function of photons

$$f^*(\underline{k}^*,T)=2f_{\mathrm{B}}(E^*,T)=\frac{2}{\exp(E^*/T)-1}, \tag{1}$$

where T is the Lorentz-invariant temperature and $\underline{k}^*$ is the wave-four-vector of the photon as measured Four-vector in the rest frame of the source.

(a) Calculate the distribution function when the black-body radiation is measured in an inertial reference frame, where the thermal source moves with a constant velocity $\vec{v}=v\vec{e}_1$ as a function of the angle θ between $\vec{k}$ and $\vec{v}$ and show that for each θ the spectrum is described by a Bose-Einstein distribution with an “effective temperature” $T_{\text{eff}}(\theta)$.

(b) To interpret the result as a relativistic Doppler effect for photons, discuss the Lorentz-transformation properties of an electromagnetic plane-wave mode.

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(2) Unitarity of the S matrix and the Boltzmann collision term

In the lecture we have derived the Boltzmann equation as

$$\frac{p_1^\mu}{m}\frac{\partial f_1}{\partial x^\mu}=\frac{1}{2}\frac{g}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2}{E_2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_1^{\,\prime}}{E_1'}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2^{\,\prime}}{E_2'}\left[\underbrace{f_{1'}f_{2'}W(\underline{p}_1,\underline{p}_2\leftarrow\underline{p}_1',\underline{p}_2')}_{\text{“gain”}}-\underbrace{f_1f_2W(\underline{p}_1',\underline{p}_2'\leftarrow\underline{p}_1,\underline{p}_2)}_{\text{“loss”}}\right], \tag{2}$$

where $f_1=f(\underline{x},\underline{p}_1)$ etc.; g is the degeneracy factor, and the factor $1/2$ takes into account the quantum-mechanical indistinguishability of the particles.
The invariant transition-probability rate W for elastic scattering is related to the invariant matrix elements from quantum field theory by

$$W(p_1',p_2'\leftarrow p_1,p_2)=\frac{\overline{|\mathcal{M}_{fi}|^2}}{16(2\pi)^6}(2\pi)^4\delta^{(4)}(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2), \tag{3}$$

where the invariant matrix element $\mathcal{M}_{fi}$ is related to the corresponding S-matrix element by

$$S_{fi}=\mathbb{1}-\mathrm{i}(2\pi)^4\delta^{(4)}(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2)T_{fi},\quad T_{fi}=\frac{\mathcal{M}_{fi}}{\prod_{k\in\{f,i\}}[(2\pi)^3 2E_k]^{1/2}}. \tag{4}$$

To simplify the following consideration of the unitary of the S-matrix assume the “box regularization”, so that the four-momentum conserving δ distribution becomes a Kronecker-δ symbol.
For simplicity we write for the box-regularized S matrix

$$\hat{S}=\mathbb{1}-\mathrm{i}\hat{T} \tag{5}$$

(a) Use the unitarity of the S-matrix,

$$\hat{S}^\dagger \hat{S} = \hat{S}\hat{S}^\dagger = \mathbb{1}, \tag{6}$$

to derive the relation for the T-matrix elements,

$$\hat{T}^\dagger \hat{T} = \hat{T}\hat{T}^\dagger. \tag{7}$$

(b) Show that this implies, now written again in the infinite-volume limit and for the integral over the two-particle final states

$$\int \mathrm{d}^3\vec{p}_1{}' \int \mathrm{d}^3\vec{p}_2{}' \delta^{(4)}(P_f - P_i)|T_{fi}|^2 = \int \mathrm{d}^3\vec{p}_1{}' \int \mathrm{d}^3\vec{p}_2{}' \delta^{(4)}(P_f - P_i)|T_{if}|^2, \tag{8}$$

relevant for elastic scattering[1].

(c) Use this in the loss term of (2) to show that the collision term simplifies to

$$\frac{p_1^\mu}{m}\frac{\partial f_1}{\partial x^\mu} = \frac{1}{2}\frac{g}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_2 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}_1{}'}{E_1}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}_2{}'}{E_2} W(p_1', p_2' \leftarrow p_1, p_2)(f_{1'}f_{2'} - f_1 f_2). \tag{9}$$

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[1] This relation is known as the “weak principle of detailed balance”