Einführung in die Theoretische Kern- und Elementarteilchenphysik 2

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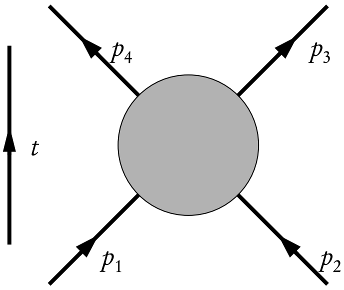
Exercise Sheet 2

(1) Scattering kinematics for $2 \rightarrow 2$ scattering



We consider the scattering process $A+B \rightarrow C+D$ as depicted in the above Feynman diagram, where the incoming particles have four-momenta $\underline{p}_1$ and $\underline{p}_2$ and the outgoing particles $\underline{p}_3$ and $\underline{p}_4$. We work in natural units with $\hbar = c = 1$. Then the particle momenta are "on shell", i.e., $\underline{p}_j^2 = E_j^2 - \vec{p}_j^{\,2} = m_j^2$ with the invariant masses, m_j. Also energy-momentum conservation holds in the scattering process, $\underline{p}_1 + \underline{p}_2 = \underline{p}_3 + \underline{p}_4$.

Except the four invariant masses there are three Mandelstam variables, which characterize the scattering kinematics in a Lorentz-invariant way, defined by

$$s = (\underline{p}_1 + \underline{p}_2)^2 = (\underline{p}_3 + \underline{p}_4)^2, \tag{1}$$

$$t = (\underline{p}_1 - \underline{p}_3)^2 = (\underline{p}_2 - \underline{p}_4)^2, \tag{2}$$

$$u = (\underline{p}_1 - \underline{p}_4)^2 = (\underline{p}_2 - \underline{p}_3)^2. \tag{3}$$

(a) Prove that the Mandelstam variables are not independent, but that the equation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{4}$$

holds.

(b) Now consider the kinematics in the "**lab frame**"[1], where particle A is at rest, $\underline{p}_1 = (m_1, 0, 0, 0)^{\mathrm{T}}$ and particle B moves with a four-momentum $\underline{p}_2 = (E_2, 0, 0, P_2)^{\mathrm{T}}$. The **relative velocity** is defined as the (magnitude of the) three-velocity of particle B in this lab frame, where particle A is at rest, i.e., $v_{\text{rel}} = P_2/E_2$. Show that

$$v_{\text{rel}} = \frac{\sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - m_1^2 m_2^2}}{E_1 E_2}. \tag{5}$$

Further show that one can express the energies in three-momenta of the particles in the incoming and

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[1] This terminology originates from the early days of accelerator expriments, where one could only manage fixed-target experiments, i.e., one had a beam of particles of kind B, which were scattered with particles of kind A at rest in some fixed target like some metal foil, as in Rutherford's seminal "gold-foil experiment", where α particles from some radioactive sample have been scattered by gold atoms within a gold foil.

outgoing channel as

$$E_2^{(\text{lab})}=\frac{s-m_1^2-m_2^2}{2m_1}, \tag{6}$$

$$P_2^{(\text{lab})}=\sqrt{(E_2^{(\text{lab})})^2-m_2^2}=\frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}}{2m_1}, \tag{7}$$

$$E_3^{(\text{lab})}=\frac{m_1^2+m_3^2-t}{2m_1}, \tag{8}$$

$$E_4^{(\text{lab})}=\frac{m_1^2+m_4^2-u}{2m_1}, \tag{9}$$

$$P_3^{(\text{lab})}=\frac{\sqrt{[(m_1+m_3)^2-t][(m_1-m_3)^2-t]}}{2m_1}, \tag{10}$$

$$P_4^{(\text{lab})}=\frac{\sqrt{[(m_1+m_4)^2-u][(m_1-m_4)^2-u]}}{2m_1}. \tag{11}$$

(c) Now consider the “**center-momentum system**”, where the total three-momentum in the incoming (and thus also in the outgoing) channel vanishes,

$$\underline{p}_1^{(\text{cm})}=\begin{pmatrix} E_1^{(\text{cm})} \\ \vec{p}^{\,(\text{cm})} \end{pmatrix}, \quad \underline{p}_2^{(\text{cm})}=\left(E_2^{(\text{cm})}-\vec{p}^{\,(\text{cm})}\right), \quad \underline{p}_3^{(\text{cm})}=\begin{pmatrix} E_3^{(\text{cm})} \\ \vec{p}^{\,\prime(\text{cm})} \end{pmatrix}, \quad \underline{p}_4^{(\text{cm})}=\begin{pmatrix} E_4^{(\text{cm})} \\ -\vec{p}^{\,\prime(\text{cm})} \end{pmatrix}. \tag{12}$$

Calculate the Mandelstam variable s and interpret the physical meaning of $\sqrt{s}$. Further show that for the magnitudes of the three-momentum in the cm frame

$$P^{(\text{cm})}=\frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}}{2\sqrt{s}}, \tag{13}$$

$$P^{\prime(\text{cm})}=\frac{\sqrt{[s-(m_3+m_4)^2][s-(m_3-m_4)^2]}}{2\sqrt{s}}. \tag{14}$$

What follows from the physical constraints that $P^{(\text{cm})} \geq 0$ and $P^{\prime(\text{cm})} \geq 0$.
