Institut für Theoretische Physik Goethe
Universität Frankfurt
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PD Dr. Hendrik van Hees

Exercise Sheet 12

Problem 1: Spontaneously breaking discrete and continous symmetries

Consider the following Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \\ &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4\end{aligned} \tag{1}$$

that describes a real scalar field ϕ with a specific potential term. It is straightforward to show that $\mathscr{L}$ is invariant under $\phi \to -\phi$. Regarding the parameters (λ, μ^2), λ should be positive to ensure the existence of a ground state but, in principle, μ^2 could be positive or negative. The minimum of $V(\phi)$, ϕ_0, turns out to be:

$$\phi_0 = \begin{cases} 0 & \text{for} \quad \mu^2 > 0 \\ \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv v & \text{for} \quad \mu^2 < 0 \end{cases} \tag{2}$$

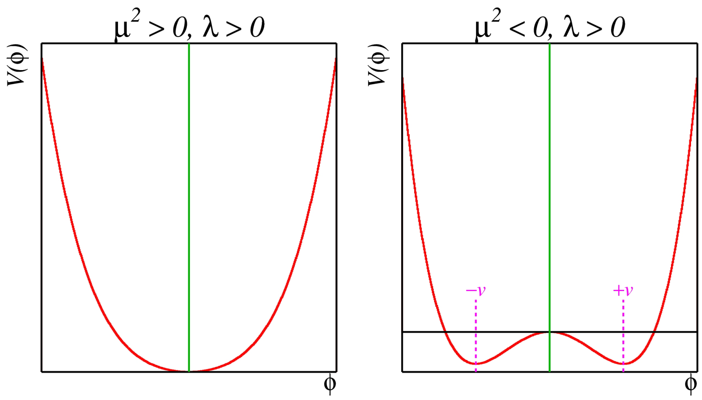

Figure 1: $V(\phi)$ as given by Eq. 1 in two different cases: $\mu^2 > 0$ (left) and $\mu^2 < 0$ (right).

The potential, $V(\phi)$, in both scenarios is shown in Fig. 1. For $\mu^2 > 0$ the quantum field must have a null vacuum expectation value (VEV) . This is not the case for ϕ when $\mu^2 < 0$

$$\langle 0|\phi|0\rangle = v \tag{3}$$

i.e. the vacuum is degenerate. Thus, a field η that is centered at the vacuum should be introduced

$$\eta = \phi - v \tag{4}$$

such that this shifted field satisfies

$$\langle 0|\eta|0\rangle = 0. \tag{5}$$

Then, at the quantum level, the same system is described by $\eta(x)$ with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 \tag{6}$$

that describes a massive scalar field ($m_\eta = \sqrt{2\lambda}v$) but is no longer invariant under $\eta \to -\eta$. $\mathscr{L}(\phi)$ had the symmetry but the parameters (λ, μ^2) can be such that the ground state of the Hamiltonian is not symmetric: the symmetry has been spontaneously broken.
The same procedure can be applied to more complex scenarios.
Consider a complex scalar field $\phi(x)$ with Lagrangian

$$\mathscr{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - V(|\phi|) = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2 |\phi|^2 - \lambda |\phi|^4 \tag{7}$$

Show that it is invariant under a global U(1) symmetry, i.e., $\phi \to \exp(-\mathrm{i}\alpha)\phi$, $\alpha = \text{const} \in \mathbb{R}$. Write down the Lagrangian in terms of (ϕ_1, ϕ_2) where

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \tag{8}$$

Repeat the previous procedure in the case of the discrete symmetry to compute ϕ_0. Write down the Lagrangian in terms of the shifted fields:

$$\begin{aligned} \eta &= \phi_1 - \nu \\ \xi &= \phi_2 \end{aligned} \tag{9}$$

How many massive and massless scalar particles does it contain? What does this fact have to do with the broken symmetry?
Hint: If we repeat the same calculation but with an SU(2) triplet instead of a scalar, the Lagrangian in terms of the shifted fields would no longer be invariant under SU(2) but under U(1) and it would contain 2 massless scalar fields.
Solution: The potential in terms of the fields η and ξ according to (9) and using $\nu = \sqrt{-\mu^2/\lambda} > 0$ reads

$$V = -\frac{\lambda}{4}\nu^4 + \frac{2\lambda v^2}{2}\eta^2 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4 + \frac{\lambda}{2}\eta^2\xi^2 + \lambda v \eta \xi^2 + \frac{\lambda}{4}\xi^4. \tag{10}$$

The field ξ is massless. Since in the Gaussian plane of ϕ this field is perpendicular to the direction of the arbitrarily chosen vacuum, i.e., choosing $v \in \mathbb{R}$. This means a little change of ϕ to $\Phi + \mathrm{i}\delta\xi$ just transforms to another vacuum state with the same potential energy as the arbitrarily choosen $\phi = v$. The mass of the η-field is $m_\eta = v\sqrt{2\lambda}$.
In the case of the spontaneous breaking of the SU(2) (dim SU(2)=3) symmetry to the U(1) (dim U(1)=1) symmetry of the vacuum, of the initial four real field-degrees of freedom (two complex fields) one has [dim SU(2)-dim U(1)]=2 generators of symmetry transformations which leave the vacuum *not* invariant, i.e., according to Goldstone's theorem there are 2 massless field excitations (for the detailed argument, see Lect. 12).

Problem 2: Spontaneously breaking gauge invariance

Consider a U(1) gauge-invariant Lagrangian for a complex scalar field $\phi(x)$

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 \tag{11}$$

with the gauge-covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i}qA_\mu(\underline{x}) \tag{12}$$

and the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{13}$$

that is invariant under

$$\begin{aligned}\phi'(\underline{x})\phi(x) &= \exp[-\mathrm{i}q\Theta(\underline{x})]\phi(\underline{x}) \\ A'_\mu(\underline{x}) &= A_\mu(x) + \partial_\mu \Theta(x)\end{aligned} \tag{14}$$

Compute ϕ_0 for $\mu^2 < 0$. In this case instead of writing ϕ in terms of the shifted fields (η, ξ) as

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[\nu + \eta(x) + \mathrm{i}\xi(x)], \quad \langle 0|\eta|0\rangle = \langle 0|\xi|0\rangle = 0 \tag{15}$$

it is more transparent to write

$$\phi(x) \equiv \exp(-\mathrm{i}q\xi(x)/\nu)\frac{1}{\sqrt{2}}[\nu + \eta(x)], \quad \langle 0|\eta|0\rangle = \langle 0|\xi|0\rangle = 0 \tag{16}$$

and exploit local gauge invariance so that

$$\phi'(x)\exp(-\mathrm{i}q\xi(x)/\nu)\phi(x) = \frac{1}{\sqrt{2}}[\nu + \eta(x)] \tag{17}$$

that is equivalent to choose $\Theta(x) = \xi(x)/\nu$. This choice is called unitary gauge. Rewrite $\mathscr{L}$ in terms of $\eta(x)$ and analyse the different terms that it contains. Is the gauge boson massless?

Discuss, how the various "field-degrees of freedom" are redistributed when comparing the cases with and without broken *local* gauge invariance, and discuss, why there are no massless Goldstone modes in this case in contradistinction to the case of a spontaneously broken *global* symmetry, as considered in Problem 1.

Solution: Using the gauge invariance of $\mathscr{L}$ the Lagrangian written in the new fields

$$\phi' = \frac{1}{\sqrt{2}}(\nu + \eta) \tag{18}$$

with the vacuum-expectation value $\nu = \sqrt{(-\mu^2/\lambda)} > 0$ (since $-\mu^2 > 0$) and A'_μ we find

$$\mathrm{D}'_\mu \Phi' = \frac{1}{\sqrt{2}}\left[\partial_\mu \eta + \mathrm{i}qA'_\mu(v + \eta)\right] \tag{19}$$

and thus

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_A^2}{2}A_\mu A^\mu + \frac{1}{2}(\partial_\mu\eta)(\partial_\mu\eta) - \frac{m_\eta^2}{2}\eta^2 \\ &+ q^2 v\eta A'_\mu A'^\mu + \frac{q^2}{2}A'_\mu A'^\mu \eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{\lambda}{4}\nu^4\end{aligned} \tag{20}$$

with the masses $m_A = |eqv|$ and $m_\eta = v\sqrt{2\lambda}$.

The non-interacting part (first line) shows that we have now a massive vector field and only 1 real scalar field, η. Thus the "would-be-Goldstone boson", ξ, is absorbed into the massive vector field, which has 3 physical polarization-degrees of freedom in contradistinction to the "unbroken gauge invariance", where A_μ is massless and thus has only 2 physical polarization-degrees of freedom.

There is no massless Goldstone boson, because a *local* gauge transformation does not describe a change of the physical state but describes the redundance of an underdetermined system of equations of motion, i.e., the gauge fields (e.g., the electromagnetic four-potential in electrodynamics) are only determined up to a gauge transformation, but all physical quantities (like $\vec{E}$ and $\vec{B}$, or $F_{\mu\nu}$) are gauge independent and thus completely determined by the equations of motion (the Maxwell fields).

This is the celebrated Higgs mechanism, which also works for non-Abelian gauge theories and is one of the pillars of the electroweak sector of the Standard Model (Nobel prize for Peter Higgs and Francois Englert in 2013 after the discovery of the Higgs boson by the ATLAS and CMS collaborations at CERN).

Originally the Higgs mechanism has been discovered in the context of the theory of superconductivity, where the formation of Cooper pairs of electrons due to an effective attractive interaction mediated by the lattice vibrations of the solid (phonons). This leads to an effective mass of the electromagnetic field in the medium and explains the phenomenology of superconductivity like the vanishing resistance and the perfect diamagnetism of a superconducting Material (Meissner-Ochsenfeld effect).