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Exercise Sheet 6: Solutions

1. The quantized Klein-Gordon field

In this exercise we consider the quantized charged Klein-Gordon field.

(a) The charge operator is given by

$$\mathbf{Q} = \mathrm{i}q \int_{\mathbb{R}^3} \mathrm{d}^3\vec{y} : \mathbf{\Phi}^\dagger(\underline{y}) \overleftrightarrow{\partial_{t_y}} \mathbf{\Phi}(\underline{y}). \tag{1}$$

Show that it generates the phase transformation, which is the symmetry corresponding to this conserved charge via No-ether's theorem,

$$\exp(\mathrm{i}\mathbf{Q}\alpha)\mathbf{\Phi}(\underline{x})\exp(-\mathrm{i}\mathbf{Q}) = \exp(-\mathrm{i}\alpha q)\mathbf{\Phi}(\underline{x}). \tag{2}$$

To this end calculate

$$\mathbf{\Phi}_\alpha(\underline{x}) = \exp(\mathrm{i}\mathbf{Q}\alpha)\mathbf{\Phi}(\underline{x})\exp(-\mathrm{i}\mathbf{Q}). \tag{3}$$

Hint: Take the derivative of this wrt. α and derive a differential equation for $\mathbf{\Phi}_\alpha(\underline{x})$ and then solve it with the initial condition $\mathbf{\Phi}_{\alpha=0}(\underline{x}) = \mathbf{\Phi}(\underline{x})$.

Solution:

$$\partial_\alpha \mathbf{\Phi}(\underline{x}) = \mathrm{i}\exp(\mathrm{i}\mathbf{Q}\alpha)\left[\mathbf{Q}, \mathbf{\Phi}(\underline{x})\right]\exp(-\mathrm{i}\mathbf{Q}\alpha).$$

Using the equal-time commutation relations of the field operators by using (1) with $t_y = t_x$:

$$\begin{aligned}
\left[\mathbf{Q}, \mathbf{\Phi}(\underline{x})\right] &= \mathrm{i}q \int_{\mathbb{R}^3} \mathrm{d}^3\vec{y} \left[\mathbf{\Phi}^\dagger(\underline{y}) \overleftrightarrow{\partial_{t_y}} \mathbf{\Phi}(\underline{y}), \mathbf{\Phi}(\underline{x})\right] \\
&= -\mathrm{i}q \int_{\mathbb{R}^3} \mathrm{d}^3\vec{y} \left[\partial_{t_y} \mathbf{\Phi}^\dagger(\underline{y}), \mathbf{\Phi}(\underline{x})\right] \mathbf{\Phi}(\underline{y}) \\
&= -\mathrm{i}q \int_{\mathbb{R}^3} \mathrm{d}^3\vec{y} (-\mathrm{i}) \delta^{(3)}(\vec{x} - \vec{y}) \mathbf{\Phi}(\vec{y}) = -q\mathbf{\Phi}(\underline{x}).
\end{aligned} \tag{4}$$

Using this ins (1) we find the differential equation,

$$\partial_\alpha \mathbf{\Phi}_\alpha(\underline{x}) = -\mathrm{i}q\mathbf{\Phi}_\alpha(\underline{x}). \tag{5}$$

The solution with the given initial condition indeed is

$$\mathbf{\Phi}_\alpha(\underline{x}) = \exp(-\mathrm{i}q\alpha)\mathbf{\Phi}(\underline{x}). \tag{6}$$

(b) Calculate the commutator function

$$\mathrm{i}\Delta(\underline{x} - \underline{y}) = \left[\mathbf{\Phi}(\underline{x}), \mathbf{\Phi}^\dagger(\underline{y})\right] \tag{7}$$

by using the mode decomposition of the field,

$$\mathbf{\Phi}(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \left[\mathbf{a}(\vec{p}) u_{\vec{p}}(\underline{x}) + \mathbf{b}^\dagger(\vec{p}) u^*_{\vec{p}}(\vec{p})\right]. \tag{8}$$

Solution:

$$\begin{aligned}
\mathrm{i}\Delta(\underline{x}-\underline{y}) &= \left[\boldsymbol{\Phi}(\underline{x}),\boldsymbol{\Phi}^\dagger(\underline{y})\right] = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \int_{\mathbb{R}^3} \mathrm{d}\vec{q} \left[[\mathbf{a}(\vec{p})u_{\vec{p}}(\underline{x}) + \mathbf{b}^\dagger u^*_{\vec{p}}(\underline{x})], [\mathbf{a}^\dagger(\vec{p})u^*_{\vec{p}}(\underline{y}) + \mathbf{b}(\vec{p})u_{\vec{p}}(\underline{y})]\right] \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \int_{\mathbb{R}^3} \mathrm{d}\vec{q}\,\delta^{(3)}(\vec{p}-\vec{q}) \left[u_{\vec{p}}(\underline{x})u^*_{\vec{q}}(\underline{y}) - u^*_{\vec{p}}(\underline{x})u_{\vec{q}}(\underline{y})\right] \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \frac{1}{(2\pi)^3 2E_p} \left\{\exp[-\mathrm{i}\underline{p}\cdot(\underline{x}-\underline{y}) - \exp[+\mathrm{i}\underline{p}\cdot(\underline{x}-\underline{y})\right\}_{p^0=E_p}.
\end{aligned} \tag{9}$$

(c) Show that $\Delta(\underline{x}-\underline{y})$ is a scalar field under proper orthochronous Lorentz transformations.

Solution: We can write (9) in the manifestly covariant form, setting $\underline{z}=\underline{x}-\underline{y}$,

$$\mathrm{i}\Delta(\underline{z}) = \int_{\mathbb{R}^4} \mathrm{d}^4\underline{p} \frac{1}{(2\pi)^3} \Theta(p^0)\delta(\underline{p}^2-m^2)\left[\exp(-\mathrm{i}\underline{p}\cdot\underline{z}) - \exp[+\mathrm{i}\underline{p}\cdot\underline{z}\right\}. \tag{10}$$

For $\hat{\Lambda}\in\mathrm{SO}(1,3)^\uparrow$ we get

$$\mathrm{i}\Delta(\hat{\Lambda}\underline{x}) = \int_{\mathbb{R}^4} \mathrm{d}^4\underline{p} \frac{1}{(2\pi)^3} \Theta(p^0)\delta(\underline{p}^2-m^2)\left[\exp(-\mathrm{i}\underline{p}\cdot\hat{\Lambda}\underline{z}) - \exp[+\mathrm{i}\underline{p}\cdot\hat{\Lambda}\underline{z}\right\}. \tag{11}$$

Substitution of $\underline{p}=\hat{\Lambda}\underline{p}'$, $\mathrm{d}^4\underline{p}'=\mathrm{d}^3\underline{p}$, $\Theta(p'^0)=\Theta(p^0)$, leads to

$$\begin{aligned}
\mathrm{i}\Delta(\hat{\Lambda}\underline{x}) &= \int_{\mathbb{R}^4} \mathrm{d}^4\underline{p}' \frac{1}{(2\pi)^3} \Theta(p'^0)\delta(\underline{p}'^2-m^2)\left[\exp(-\mathrm{i}\hat{\Lambda}uvec\,p'\cdot\hat{\Lambda}\underline{z}) - \exp[+\mathrm{i}\hat{\Lambda}\underline{p}'\cdot\hat{\Lambda}\underline{z}\right\} \\
&= \int_{\mathbb{R}^4} \mathrm{d}^4\underline{p}' \frac{1}{(2\pi)^3} \Theta(p'^0)\delta(\underline{p}'^2-m^2)\left[\exp(-\mathrm{i}\underline{p}'\cdot\underline{z}) - \exp[+\mathrm{i}\underline{p}'\cdot\underline{z}\right\} \\
&= \mathrm{i}\Delta(\underline{x}).
\end{aligned} \tag{12}$$

(d) From the equal-time commutation relations of the field operators, it follows that $\Delta(x-y)|_{t_x=t_y}=0$. Use this to prove the **micro-causality property**

$$\Delta(\underline{z})=0 \quad \text{if} \quad \underline{z}\cdot\underline{z}<0. \tag{13}$$

To this end show that you can always find an η in the Lorentz boost $\hat{\Lambda}(\vec{n},\eta)$ with $\vec{n}=\vec{z}/|\vec{z}|$

$$\underline{z}'=\hat{\Lambda}(\vec{n},\eta)\underline{z}=\begin{pmatrix}0\\ \vec{z}'\end{pmatrix} \Rightarrow \Delta(\underline{z})=\Delta(\underline{z}')=0. \tag{14}$$

Solution:

$$\underline{z}'=\begin{pmatrix}\cosh\eta & -\vec{n}^{\mathrm{T}}\sinh\eta \\ \vec{n}\sinh\eta & (\cosh\eta-1)\vec{n}\vec{n}^{\mathrm{T}}+\mathbb{1}\end{pmatrix}\begin{pmatrix}z^0\\ \vec{z}\end{pmatrix}=\begin{pmatrix}z^0\cosh\eta-\vec{n}\cdot\vec{z}\sinh\eta\\ \cdots\end{pmatrix}. \tag{15}$$

Now $\vec{n}\cdot\vec{z}=|\vec{z}|>|z^0|$ and thus we indeed can make the time component z'^0 vanish:

$$z^0\cosh\eta-|\vec{z}|\sinh\eta=0 \Rightarrow \eta=\operatorname{artanh}(z^0/|\vec{z}|). \tag{16}$$

2. Ideal relativistic Bose gas

In this exercise we investigate an ideal relativistic Bose gas in the grand-canonical ensemble of quantum-statistical physics. To this end consider the finite-volume box regularization for the quantized charged Klein-Gordon field discussed in the lecture.

The grand-canonical statistical operator for the ideal gas is given by

$$\hat{\rho} = \frac{1}{Z} \exp(-\beta \mathbf{H} - \alpha \mathbf{Q}), \quad Z = \operatorname{Tr} \exp(-\beta \mathbf{H} - \alpha \mathbf{Q}). \tag{17}$$

The Hamilton operator and conserved charge (with $q = 1$, i.e., the "net-particle number" $N_a - N_b$) is given in terms of the number operators

$$\begin{aligned} \mathbf{H} &= \sum_{\vec{p}} E_{\vec{p}} \left[\mathbf{N}_a(\vec{p}) + \mathbf{N}_b(\vec{p}) \right], \\ \mathbf{Q} &= \sum_{\vec{p}} \left[\mathbf{N}_a(\vec{p}) - \mathbf{N}_b(\vec{p}) \right]. \end{aligned} \tag{18}$$

The momenta run over $\vec{p} \in 2\pi\mathbb{Z}^3/L$. Note that the momenta depend on $L = V^{1/3}$, where V is the volume of the cubic box.

(a) Calculate the partition sum, Z, as defined in (17). Use the occupation-number basis to take the trace, i.e., for an operator $\mathbf{A}$

$$\operatorname{Tr} \mathbf{A} = \prod_{\vec{p}} \sum_{N_a(\vec{p})=0}^{\infty} \sum_{N_b(\vec{p})=0}^{\infty} \left\langle \{N_a(\vec{p}), N_b(\vec{p})\}_{\vec{p}} \middle| \mathbf{A} \middle| \{N_a(\vec{p}), N_b(\vec{p})\}_{\vec{p}} \right\rangle. \tag{19}$$

What are the physically meaningful ranges for the parameters β and α?

Solution:

$$\begin{aligned} Z &= \operatorname{Tr} \exp(-\beta \mathbf{H} - \alpha \mathbf{Q}) \\ &= \prod_{\vec{p}} \sum_{N_a(\vec{p})=0}^{\infty} \exp[-N_a(\vec{p})(\beta E_p + \alpha)] \sum_{N_b(\vec{p})=0}^{\infty} \exp[-N_b(\vec{p})(\beta E_p - \alpha)] \\ &= \prod_{\vec{p}} \frac{1}{1 - \exp(-\beta E_p - \alpha)} \frac{1}{1 - \exp(-\beta E_p + \alpha)}. \end{aligned} \tag{20}$$

The sum over the geometric series only exists if both $-\beta E_p - \alpha < 0$ and $-\beta E_p + \alpha < 0$ for all $\vec{p}$. Because $E_p \geq m > 0$ this implies that $\beta > 0$ and $|\alpha| < m$.

(b) Calculate the internal energy U and mean net-particle number Q by showing that with $\Omega(\beta, V, \alpha) = \ln Z(\beta, V, \alpha)$

$$U = -\partial_\beta \Omega(\beta, V, \alpha), \quad Q = -\partial_\alpha \Omega(\beta, V, \alpha). \tag{21}$$

Solution:

$$\begin{aligned} \partial_\beta \Omega &= \partial_\beta \ln Z = \frac{1}{Z} \partial_\beta Z = -\frac{1}{Z} \operatorname{Tr}[\mathbf{H} \exp(-\beta \mathbf{H} - \alpha \mathbf{Q})] = -U, \\ \partial_\alpha \Omega &= \partial_\alpha \ln Z = \frac{1}{Z} \partial_\alpha Z = -\frac{1}{Z} \operatorname{Tr}[\mathbf{Q} \exp(-\beta \mathbf{H} - \alpha \mathbf{Q})] = -Q. \end{aligned} \tag{22}$$

From (20) we get

$$\Omega = -\sum_{\vec{p}} \left\{ \ln\left[1 - \exp(-\beta E_p - \alpha)\right] + \ln\left[1 - \exp(-\beta E_p + \alpha)\right] \right\} \tag{23}$$

and from (22)

$$\begin{aligned} U &= \sum_{\vec{p}} E_p \left[\frac{1}{\exp(\beta E_p + \alpha) - 1} + \frac{1}{\exp(\beta E_p - \alpha) - 1} \right], \\ Q &= \sum_{\vec{p}} \left[\frac{1}{\exp(\beta E_p + \alpha) - 1} - \frac{1}{\exp(\beta E_p - \alpha) - 1} \right]. \end{aligned} \tag{24}$$

(c) Prove that the entropy (with $k_B = 1$) is given by

$$S = -\mathrm{Tr}(\rho \ln \rho) = \Omega + \beta U + \alpha Q \tag{25}$$

Solution:

$$S = -\mathrm{Tr}[\rho(-\ln Z - \beta \mathbf{H} - \alpha \mathbf{Q})] = \Omega + \beta U + \alpha Q. \tag{26}$$

(d) Prove that the total differential of S is given by

$$\mathrm{d}S = \beta \mathrm{d}U + \mathrm{d}V \partial_V \Omega(\beta, V, \alpha) + \alpha \mathrm{d}Q. \tag{27}$$

Solution

$$\begin{aligned} \mathrm{d}S &= \mathrm{d}\Omega + \mathrm{d}\beta U + \beta \mathrm{d}U + \mathrm{d}\alpha Q + \alpha \mathrm{d}Q \\ &= \mathrm{d}\beta \partial_\beta \Omega + \mathrm{d}V \partial_V \Omega + \mathrm{d}\alpha \partial_\alpha \Omega + \mathrm{d}\beta U + \beta \mathrm{d}U + \mathrm{d}\alpha Q + \alpha \mathrm{d}Q \\ &= -\mathrm{d}\beta U + \mathrm{d}V \partial_V \Omega - \mathrm{d}\alpha Q + \mathrm{d}\beta U + \beta \mathrm{d}U + \mathrm{d}\alpha Q + \alpha \mathrm{d}Q \\ &= \beta \mathrm{d}U + \mathrm{d}V \partial_V \Omega + \alpha \mathrm{d}Q. \end{aligned} \tag{28}$$

(e) Identify the quantities β, $\partial_V \Omega$, and α with the usual thermodynamic variables by comparing

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \mu \mathrm{d}Q. \tag{29}$$

Solution: Solving (27) for $\mathrm{d}U$ gives

$$\mathrm{d}U = \frac{1}{\beta}(\mathrm{d}S - \mathrm{d}V \partial_v \Omega - \alpha \mathrm{d}Q) \Rightarrow T = \frac{1}{\beta}, \quad p = \frac{1}{\beta}\partial_V \Omega, \quad \mu = -\frac{\alpha}{\beta}. \tag{30}$$

(f) Calculate the pressure. NB: Of course, you can't calculate the sums over $\vec{p}$ in closed form!

Solution: The dependence of Ω on V is due to the dependence of E_p on V:

$$\partial_V E_p = \partial_V \vec{p} \cdot \vec{\nabla}_p E_p. \tag{31}$$

Since $\vec{p} = 2\pi \vec{n} V^{-1/3}$ with $\vec{n} \in \mathbb{Z}^3$

$$\partial_V \vec{p} = -\frac{1}{3} V^{-4/3} 2\pi \vec{n} = -\frac{V}{3}\vec{p} \Rightarrow \partial_V E_p = -\frac{1}{3V}\frac{\vec{p}^2}{E_p}. \tag{32}$$

From this we get, using again (23)

$$P = \frac{1}{\beta}\partial_V \Omega = \frac{1}{3V}\sum_{\vec{p}} \frac{\vec{p}^2}{E_p}\left[\frac{1}{\exp(\beta E_p + \alpha) - 1} + \frac{1}{\exp(\beta E_p - \alpha) - 1}\right]. \tag{33}$$

We note that from (26) we get

$$\Omega = S - \beta U - \alpha Q, \tag{34}$$

which tells us that the somewhat unusual thermodynamic potential Ω is a Legendre transform of the entropy and thus known as a **Massieu function**. As shown above, it is pretty convenient to directly get the internal energy $U = \langle \mathbf{H} \rangle$ and conserved charge $Q = \langle \mathbf{Q} \rangle$ (net-particle number). The only drawback is that it is not so convenient to derive the zero-temperature limit, but this can be remedied by just setting $\alpha = -\beta\mu$ and then, after taking the thermodynamic limit, i.e., the infinite-volume limit $V \to \infty$ at some fixed finite inverse temperature β and then the limit $T \to 0^+$ at fixed charge density $q = Q/V$. This we shall do below, when discussing the additional puzzle problems.

A more common thermodynamical potential, closely related to the Massieu function $\Omega = \Omega(\beta, V, \alpha)$ is the **grand (canonical) potential** or **Landau potential**

$$\Phi(T, V, \mu) = -T\Omega(1/T, V, -\mu\beta) = U - TS - \mu N \tag{35}$$

Then

$$\mathrm{d}\Phi = \underbrace{T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N}_{\mathrm{d}U} - T\mathrm{d}S - S\mathrm{d}T - \mu\mathrm{d}N - N\mathrm{d}\mu = -S\mathrm{d}T - p\mathrm{d}V - N\mathrm{d}\mu. \tag{36}$$

So the "natural independent thermodynamical variables" for the grand potential Φ are indeed T, V, and μ, and we get from the total differential (36)

$$S = -\partial_T \Phi(T, V, \mu), \quad p = -\partial_V \Phi(T, V, \mu), \quad N = -\partial_\mu \Phi(T, V, \mu). \tag{37}$$

Extra (pretty hard!) puzzles:

- Take the "thermodynamic limit", i.e., $V \to \infty$. Consider without restriction of generality the case $\mu > 0$ (and thus $Q > 0$). Consider the limit at fixed $\beta = 1/T$ and be aware that you need to treat the contribution from the single-particle ground state $\vec{p} = 0$ separately. Show that taking the naive prescription

  $$\sum_{\vec{p}} \xrightarrow[V \to \infty]{} V \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tag{38}$$

  provides only the contributions to the thermodynamic quantities for the "excited states" $\vec{p} \neq 0$ in the finite-box description.

  Solution: We start again with the Massieu function Ω, which at finite β is most convenient to derive U and Q. The naive infinite-Volume limit with the substitution (38) of (23) reads

  $$\Omega_{\text{exc}} = -V \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \left\{ \ln\left[1 - \exp(-\beta E_p - \alpha)\right] + \ln\left[1 - \exp(-\beta E_p + \alpha)\right] \right\}. \tag{39}$$

  Let us now consider the limiting case, $\mu = -\alpha T \to +m$ at fixed finite β. In the box-regularized version (23) this would make Ω divergent because of the contribution from the single-particle ground state $\vec{p} = 0$. This means for fixed temperature (no matter how small it might be!) you can always find a chemical potential μ to accomodate any net-particle number N. Now the corresponding contribution in (39) comes from the first term in the integrand, but now for $\vec{p} \to 0$ this part of the integrand is proportional to[1]

  $$p^2 \ln\left[1 - \exp[-\beta(E_p - \mu)]\right] \simeq p^2 \ln\left[\beta(m - \mu + p^2/(2m))\right], \tag{40}$$

  where we expanded the exponential function for $\beta(E_p - \mu) \ll 1$. Setting now $\mu = m$ we get

  $$p^2 \ln\left[1 - \exp[-\beta(E_p - \mu)]\right] \simeq p^2 \ln\left[\beta p^2/(2m)\right] \xrightarrow[p \to 0]{} 0 \tag{41}$$

  i.e., in the limit $\mu \to m$ the contribution from the ground-state, i.e., in the infinite-volume limit from some region of small momenta is not contained in the naive substitution (39), i.e., it gives only the contribution from the single-particle excited states in this limit. That's why we labeled this contribution as Ω_{exc}.

  The same analysis holds, of course for the antiparticle contribution, which become relevant for $T \to 0$ for negative $\mu \to -m$.

[1] Because the integrand depends only on $p = |\vec{p}|$, we can introduce spherical coordinates and take the trivial integral over the angles, leading to $\mathrm{d}^3\vec{p} \to 4\pi p^2 \mathrm{d}p$.

The full Massieu function thus is

$$\begin{aligned}\Omega=\Omega_0+\Omega_{\text{exc}} &= -\ln[1-\exp[-\beta(m-\mu)]]-\ln[1-\exp[-\beta(m+\mu)]]\\ &-V\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left\{\ln\Big[1-\exp[-\beta(E_p-\mu)]\Big]+\ln\Big[1-\exp[-\beta(E_p+\mu)]\Big]\right\}\end{aligned} \tag{42}$$

- The infinite-volume limit has to be taken such that the charge density $q=Q/V$ is kept fixed.
- In the so established "thermodynamic limit". Discuss that for $T\to 0^+$ keeping the total charge density fixed, you always get Bose-Einstein condensation, i.e., all particles occupy the single-particle ground state. Which limit is implied for μ for $T\to 0$?
- How is the critical temperature determined, i.e., what is the temperature T_c such that a finite density of particles occupying the single-particle ground state $q_0\neq 0$ is necessary for $T<T_c$. **Solution:** From (22) we get from (43

$$\begin{aligned}q=-\frac{1}{V}\partial_\alpha\Omega=&\frac{1}{V}\frac{1}{\exp(\beta m+\alpha)-1}-\frac{1}{V}\frac{1}{\exp(\beta m-\alpha)-1}\\ &+\underbrace{\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left[\frac{1}{\exp(\beta E_p+\alpha)-1}-\frac{1}{\exp(\beta E_p-\alpha)-1}\right]}_{q_{\text{exc}}}\end{aligned} \tag{43}$$

Now for a given fixed temperature $T=1/\beta$ and charge density $q>0$[2] there are two possibilities

(1) We can find an $\alpha=-\beta\mu$ with $\mu\in[0,m]$ such that $q=q_{\text{exc}}$. Then taking $V\to\infty$ we simply have

$$q=q_{\text{exc}}=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left[\frac{1}{\exp(\beta E_p+\alpha)-1}-\frac{1}{\exp(\beta E_p-\alpha)-1}\right]. \tag{44}$$

(2) For a given temperature we get the maximal possible value for $q_{\text{exc}}>0$ by setting $\mu=m$, i.e.,

$$q_{\text{exc}}^{(\text{max})}=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left[\frac{1}{\exp[\beta(E_p-m)]-1}-\frac{1}{\exp[\beta(E_p+m)-1}\right]. \tag{45}$$

Since this vanishes in the limit $T\to 0^+$, i.e., $\beta\to\infty$, inevitably for a given q there is some T_{crit} at which $q=q_{\text{exc}}^{(\text{max})}$. For all $0<T<T_{\text{crit}}$ one then has $q>q_{\text{exc}}^{(\text{max})}$, and thus there must be a non-vanishing contribution $q_0=q-q_{\text{exc}}^{(\text{max})}$ from a macroscopic number of particles all occupying the single-particle ground state. According to (43) this contribution is (still at a finite volume, V and $0\leq\mu<m$!)

$$q_0=\frac{1}{V}\frac{1}{\exp[\beta(m-\mu)]-1} \tag{46}$$

In this case this means we have to take infinite-volume limit such that the chemical potential becomes volume dependent to enforce

$$\lim_{V\to\infty}\mu(V)=m \tag{47}$$

such that

$$q_0=q-q_{\text{exc}}^{(\text{max})}=\lim_{V\to 0}\frac{1}{V}\frac{1}{\exp[\beta(m-\mu)]-1}. \tag{48}$$

[2]for $q<0$ the analysis is completely analogous. One simply has to consider μ_0.

This phenomenon is known as **Bose-Einstein condensation** and was predicted by Einstein in 1924 [Ein24, Ein25]. This is an example for a **phase transition**. The order parameter is the density in the single-particle ground state, q_0, i.e., in the normal phase (case (1)) we have $q_0 = 0$. In the Bose-Einstein condensate (BEC) phase (case (2)) $q_0 \neq 0$. One also says the gas is in a **degenerate state**, i.e., its behavior drastically changes compared to the usual experience with dilute gases at not too low temperatures, where the gas is described by classical statistics, i.e., by a **Maxwell-Boltzmann distribution**. If the gas is dilute, then $\exp[\beta(E_p - \mu)] \gg 1$, and one can neglect the (−1) in the denominator of the Bose-Einstein distribution function,

$$f_{\mathrm{B}}(E) = \frac{1}{\exp[\beta(E-\mu)]-1} \simeq \exp[-\beta(E-\mu)] \tag{49}$$

In the relativistic context, where the relativistic energy-momentum relation $E = \sqrt{\vec{p}^2 + m^2}$ is used instead of the Newtonian one, $E = \vec{p}^2/(2m)$, one also calls it the **Maxwell-J"uttner distribution** [Jüt11].

Now we can consider the limit $T \to 0^+$. In this limit all gas particles occupy the single-particle ground state, i.e., to take the limit $T \to 0$ we first have to take the thermodynamic limit $V \to \infty$ as discussed above and then the zero-temperature limit $T \to 0^+$, i.e., $\beta \to \infty$.

Now we look at the zero-temperature limit of the various thermodynamic quantities. Particularly simple is the density, which simply is $q_0 = q$, $q_{\mathrm{exc}} = 0$, by the very construction of these limits described above.

The **energy density** in the thermodynamic limit is given by (24)

$$\begin{aligned} u = \frac{U}{V} = u_0 + u_{\mathrm{exc}} = &\frac{m}{V}\left[\frac{1}{\exp[\beta(m-\mu)]-1} + \frac{1}{\exp[\beta(E_p+\mu)]-1}\right] \\ &+ \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} E_p \left[\frac{1}{\exp(\beta E_p + \alpha)-1} + \frac{1}{\exp(\beta E_p - \alpha)-1}\right] \end{aligned} \tag{50}$$

For the same reason discussed above for the Massieu function the above integral u_{exc} is convergent for all $\mu \in [-m, m]$ and vanishes in the limit $T \to 0^+$, and the same arguments as for the Massieu function applies here too, i.e., for $T > T_c$ the condensate contribution vanishes and the energy density $u = u_{\mathrm{exc}}$.

With our detailed analysis above also the case $T < T_c$, when Bose-Einstein condensation occurs, it is easy to see that

$$u_0 = m q_0. \tag{51}$$

This is also intuitively clear: each boson "condensed" in the single-particle ground state, $\vec{p} = 0$, contributes its rest energy $m(c^2)$ to the internal energy (density). That's different from the non-relativistic case, where one counts only the kinetic energy $\vec{p}^2/(2m)$, i.e., the condensate does not contribute to the energy in that case.

Particularly simple is the pressure. Here a glance at (33) shows that at any temperature and particle density the ground-state contribution from the $\vec{p} = 0$ modes vanishes, and the pressure thus is only due to the contribution from the excited single-particle states, i.e.,

$$P = P_{\mathrm{exc}} = \frac{1}{3}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\vec{p}^2}{E_p}\left[\frac{1}{\exp[\beta(E_p-\mu)]-1} + \frac{1}{\exp[\beta(E_p+\mu)-1}\right]. \tag{52}$$

For $T \to 0^+$ the pressure vanishes, as to be expected since then all particles are in the BEC.

Finally we address the **entropy**. Particularly for the entropy it's crucial to *first* take the infinite-volume limit. Here only the condensate contribution is non-trivial. The contribution from the excited states is given by the naive sum-to-integral prescription, i.e., for the corresponding contribution to the entropy density,

$$s_{\text{exc}} = \Omega_{\text{exc}} + \beta u_{\text{exc}} + \mu q_{\text{exc}} \tag{53}$$

The limit $T \to 0^+$, i.e., $\beta \to \infty$ is unproblematic, even for the 2nd term, because the the Bose distributions goes exponentially to 0 compensating the explicit factor β.

Now we have to consider the infinite-volume limit for the condensate contribution to the entropy density (again at fixed temperature $T > 0$)

$$s_0 = \frac{1}{V}\Omega_0 + \beta u_0 - \beta \mu q_0 \tag{54}$$

For the last two contributions the infinite-volume limit is simple. We have to make $\mu \to \mu(V)$ as discussed above, and $\mu(V) \to m$ for $V \to \infty$ and according to (51) $u_0 = q_0 m$, so that this contribution vanishes in the infinite-volume limit. So only the first term is left, and we need to consider only the contribution from the particles,

$$\begin{aligned} \frac{1}{V}\Omega_0 &= -\frac{1}{V}\ln\left[1 - \exp[-\beta(m-\mu)]\right] \\ &= -\frac{1}{V}\ln\left\{\exp[-\beta(m-\mu)]\left[\exp[\beta(m-\mu)] - 1\right]\right\} \\ &= \frac{\beta(m-\mu)}{V} + \frac{1}{V}\ln(q_0 V) \xrightarrow[V\to\infty]{} 0. \end{aligned} \tag{55}$$

So the contribution of the condensate to the entropy density vanishes in the infinite-volume limit, and we have simply $s = s_{\text{exc}}$ for $T > 0$, no matter whether in the normal or the BEC phase. For $T \to 0^+$ also this contribution vanishes. Thus in accordance with Nernst's Third Law of Thermodynamics, in the zero-temperature limit the entropy vanishes. This is also clear by counting the microstates for $T \to 0^+$. Since then all gas particles are in the single-particle ground state, i.e., the ground state is simply the pure Fock state $\left|N_a(\vec{p} = 0) = Q\right\rangle$, but a pure state means "complete information" about the quantum state of the gas, and thus $S = 0$ for this state.

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