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Exercise Sheet 5: Solutions

The complex Klein-Gordon field

Consider a complex-valued Lorentz-scalar field. The Lagrangian, defining the free field equations is given by

$$\mathscr{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi. \tag{1}$$

Obviously the Lagrangian is a Lorentz scalar and thus the action too. The Lagrangian is not explicitly dependent on the space-time coordinates $\underline{x} = (x^\mu)$, with the fields transforming under both translations and Lorentz transformations as a scalar field.

1. Derive the equations of motion from the Lagrange equations. The complex scalar field has to be interpreted as two real field-degrees of freedom, i.e., you can vary Φ and Φ^* as independent fields, and thus you have the Euler-Lagrange equations for these two fields,

$$\Pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}, \quad \Pi^{*\mu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^*)}, \quad \partial_\mu \Pi^\mu = \frac{\partial \mathscr{L}}{\partial \Phi}, \quad \partial_\mu \Pi^{*\mu} = \frac{\partial \mathscr{L}}{\partial \Phi^*}. \tag{2}$$

   Show that the result is the **Klein-Gordon equation** for both Φ and Φ^*.

   Solution: Taking the derivatives (beware the "vertical index placement"!) gives

$$\begin{gathered} \Pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \Phi^*, \quad \Pi^{*\mu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \Phi, \\ \partial_\mu \Pi^\mu = \Box \Phi^* = \frac{\partial \mathscr{L}}{\partial \phi} = -m^2 \phi^*, \quad \partial_\mu \Pi^{*\mu} = \Box \Phi = \frac{\partial \mathscr{L}}{\partial \phi} = -m^2 \phi. \end{gathered} \tag{3}$$

   As expected we obtain the Klein-Gordon equation

$$(\Box + m^2)\Phi = 0, \quad (\Box + m^2)\Phi^* = 0. \tag{4}$$

2. Use the Fourier ansatz for the field Φ

$$\Phi(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{p} A(t, \vec{p}) \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \tag{5}$$

   to show that the general solution of the Klein-Gordon equation can be written in the form

$$\Phi(\vec{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{p} [a(\vec{p}) u_{\vec{p}}(\underline{x}) + b^*(\vec{p}) u^*_{\vec{p}}(\underline{x})] \tag{6}$$

   with the "relativistic plane-wave mode functions"

$$u_{\vec{p}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} \exp(-\mathrm{i}\underline{x} \cdot \underline{p})|_{p^0 = E_p}, \quad E_p = \sqrt{m^2 + \vec{p}^2} \tag{7}$$

   and arbitrary (square-integrable) $\mathbb{C}$-valued functions $a(\vec{p})$ and $b(\vec{p})$.

Solution: With the ansatz (5) we get

$$\begin{aligned}(\Box+m^2)\Phi(\underline{x}) &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[\partial_t^2 A(t,\vec{p})+(\vec{p}^2+m^2)A(t,\vec{p})\exp(\mathrm{i}\vec{p}\cdot\vec{x}) \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[\partial_t^2 A(t,\vec{p})+E_p^2 A(t,\vec{p})]\exp(\mathrm{i}\vec{p}\cdot\vec{x}) \stackrel{!}{=}.\end{aligned} \tag{8}$$

This implies

$$\partial_t^2 A(t,\vec{p}) = -E_p^2 A(t,\vec{p}) \Rightarrow A(t,\vec{p}) = A_1(\vec{p})\exp(-\mathrm{i}E_p t)+A_2(\vec{p})\exp(+\mathrm{i}E_p t). \tag{9}$$

So the general solution for the Klein-Gordon equation reads

$$\Phi(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[A_1(\vec{p})\exp(-\mathrm{i}E_p t)+A_2(\vec{p})\exp(+\mathrm{i}E_p t)]\exp(\mathrm{i}\vec{p}\cdot\vec{x}). \tag{10}$$

In the 2nd term we substitute $\vec{p}\to-\vec{p}$. Then we can write

$$\Phi(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[A_1(\vec{p})\exp(-\mathrm{i}\underline{x}\cdot\vec{p})+A_2(-\vec{p})\exp(+\mathrm{i}\underline{x}\cdot\underline{p})]_{p^0=E_p}. \tag{11}$$

Setting

$$A_1(\vec{p}) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} a(\vec{p}), \quad A_2(-\vec{p}) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} b^*(\vec{p}),$$

(11) takes the form (6).

3. For two scalar fields Φ_1 and Φ_2 we define

$$\Phi_1 \overleftrightarrow{\partial_\mu} \Phi_2 = \Phi_1\partial_\mu\Phi_2 - (\partial_\mu\Phi_1)\Phi_2 \tag{12}$$

and the *non-definite* bilinear form

$$(\Phi_1,\Phi_2) = \mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3x \Phi_1 \overleftrightarrow{\partial_t} \Phi_2. \tag{13}$$

Show that for the mode functions (7) and $\vec{p},\vec{q}\in\mathbb{R}^3$

$$(u_{\vec{p}},u_{\vec{q}}) = (u_{\vec{p}}^*,u_{\vec{q}}^*) = 0, \quad (u_{\vec{p}}^*,u_{\vec{q}}) = -(u_{\vec{q}},u_{\vec{p}}^*) = \delta^{(3)}(\vec{p}-\vec{q}). \tag{14}$$

Solution: Using the definition of the $u_{\vec{p}}$ we get (understanding that $p^0=E_p$ and $q^0=E_1$)

$$\begin{aligned}(u_{\vec{p}},u_{\vec{q}}) &= \mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \frac{1}{(2\pi)^3 2\sqrt{E_p E_q}} \exp(-\mathrm{i}\underline{x}\cdot\underline{p}) \overleftrightarrow{\partial_t} \exp(-\mathrm{i}\underline{x}\cdot\underline{q}) \\ &= \mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \frac{1}{(2\pi)^3 2\sqrt{E_p E_q}} \exp[-\mathrm{i}\underline{x}\cdot(\underline{p}+\underline{q})](-\mathrm{i})(E_q-E_p) \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \frac{1}{(2\pi)^3 2\sqrt{E_p E_q}} \exp[-\mathrm{i}(E_p+E_q)t]\exp[\mathrm{i}\vec{x}\cdot(\vec{p}+\vec{q})](E_q-E_p) \\ &= \frac{1}{2\sqrt{E_p E_q}} \exp[-\mathrm{i}(E_p+E_q)](E_q-E_p)\delta^{(3)}(\vec{p}+\vec{q}) = 0.\end{aligned} \tag{15}$$

Further $(u^*_{\vec{p}}, u^*_{\vec{q}}) = -(u_{\vec{p}}, u_{\vec{q}})^* = 0$. For the other equation the analogous calculation leads to

$$\begin{aligned}(u^*_{\vec{p}}, u_{\vec{q}}) &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \frac{1}{(2\pi)^3 2\sqrt{E_p E_q}} \exp[\mathrm{i}(E_p - E_q)t] \exp[\mathrm{i}\vec{x}\cdot(\vec{q}-\vec{p})](E_q + E_p) \\ &= \frac{1}{2\sqrt{E_p E_q}} \exp[\mathrm{i}(E_p - E_q)](E_q + E_p)\delta^{(3)}(\vec{p}-\vec{q}) = \delta^{(3)}(\vec{p}-\vec{q}).\end{aligned} \tag{16}$$

4. Calculate the canonical energy-momentum tensor,

$$\Theta^{\mu\nu} = \Pi^{\mu}\partial^{\nu}\Phi + \Pi^{*\mu}\partial^{\nu}\Phi^* - \mathscr{L}\eta^{\mu\nu}$$

and express the total energy and momentum

$$P^{\nu} = \int_{\mathbb{R}^3} \mathrm{d}^3 x \Theta^{0\nu} \tag{17}$$

in terms of the Fourier components $a(\vec{p})$ and $b(\vec{p})$ defined in (6).

Solution: Using (2) we get

$$\Theta^{\mu\nu} = (\partial^{\mu}\Phi^*)(\partial^{\nu}\Phi) + (\partial^{\mu}\Phi)(\partial^{\nu}\Phi^*) - \mathscr{L}\eta^{\mu\nu}. \tag{18}$$

From this we find, using the Lagrangian (1)

$$\begin{aligned}\varepsilon = \Theta^{00} &= (\partial_t\Phi)^*(\partial_t\Phi) + (\vec{\nabla}\Phi^*)\cdot(\vec{\nabla}\Phi) + m^2\Phi^*\Phi, \\ \vec{g} &= -(\partial_t\Phi^*)(\vec{\nabla}\Phi) - (\partial_t\Phi)(\vec{\nabla}\Phi^*).\end{aligned} \tag{19}$$

NB: Here we have used that $(\partial^{\nu}) = (\partial_0, -\partial_1, -\partial_2, -\partial_3) = (\partial_t, -\vec{\nabla})$.

For the total energy we first consider the contribution from the 2nd term in (19), we find by integration by parts and using the Klein"=Gordon equation

$$\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} (\vec{\nabla}\Phi^*)\cdot(\vec{\nabla}\Phi) = -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\Phi^*\Delta\Phi = -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\Phi^*(\partial_t^2 + m^2)\Phi. \tag{20}$$

This gives

$$P^0 = E = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\epsilon = -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\Phi^* \overleftrightarrow{\partial_t} \partial_t\Phi. \tag{21}$$

Using the mode decomposition (6) and (13)

$$\begin{aligned}E &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{q} [a^*(\vec{p})(\mathrm{i}E_p)u^*_{\vec{p}} + b(\vec{p})(-\mathrm{i}E_p)u_{\vec{p}}] \overleftrightarrow{\partial_t} [a(\vec{q})u_{\vec{q}} + b^*(\vec{q})u^*_{\vec{q}}] \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{q} [a^*(\vec{p})a(\vec{q})E_p + b(\vec{p})b^*(\vec{q})]\delta^{(3)}(\vec{p}-\vec{q}) \\ &= \int_{\mathbb{R}_3} \mathrm{d}^3\vec{p} E_p [|a(\vec{p})|^2 + |b(\vec{p})|^2].\end{aligned} \tag{22}$$

The evaluation of the total momentum is quite analogous, using (19) for the momentum density, $\vec{g}$:

$$\begin{aligned}\vec{P} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,\vec{g} = -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}[(\partial_t\Phi^*)(\vec{\nabla}\Phi)+(\partial_t\Phi)(\vec{\nabla}\Phi^*)] \\ &= -\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}[(\partial_t\Phi^*)(\vec{\nabla}\Phi)-(\partial_t\vec{\nabla}\Phi)\Phi^*] \\ &= +\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}[\Phi^*\overleftrightarrow{\partial_t}\vec{\nabla}\Phi] \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{q}[a^*(\vec{p})u_{\vec{p}}^*+b(\vec{p})u_{\vec{p}}]\overleftrightarrow{\partial_t}[a(\vec{q})\mathrm{i}\vec{q}u_{\vec{q}}+b^*(\vec{p})(-\mathrm{i}\vec{q})u_{\vec{q}}^*] \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{q}\,\vec{q}[a^*(\vec{p})a(\vec{q})+b(\vec{p})b^*(\vec{q})]\delta^{(3)}(\vec{p}-\vec{q}) \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\,\vec{q}[|a(\vec{p})|^2+|b(\vec{p})|^2].\end{aligned} \tag{23}$$

As to be expected from Noether's theorem, both the energy and momentum are conserved, i.e., do not depend on time.

5. From the obvious invariance of the Lagrangian under the phase transformation

$$\Phi'(\underline{x}) = \exp(-\mathrm{i}q\alpha)\Phi(\underline{x}), \quad \Phi^{*\prime}(\underline{x}) = \exp(+\mathrm{i}q\alpha)\Phi^*(\underline{x}), \tag{24}$$

where $\alpha \in \mathbb{R}$ is the group parameter[1] the corresponding Noether current is given by

$$j_\mu = \mathrm{i}q\Phi^*\overleftrightarrow{\partial_\mu}\Phi. \tag{25}$$

Show that indeed the continuity equation,

$$\partial_\mu j^\mu = 0 \tag{26}$$

holds if Φ fulfills the Klein-Gordon equation and calculate the total charge

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3x\, j^0(\underline{x}) \tag{27}$$

in terms of the Fourier components $a(\vec{p})$ and $b(\vec{p})$.

Solution: We just evaluate the four-dimensional divergence in (26), using (25) for the four-current density and the Klein"=Gordon equation for Φ and Φ^*:

$$\begin{aligned}\partial_\mu j^\mu &= \mathrm{i}q\partial_\mu[\Phi^*\partial^\mu\Phi-(\partial^\mu\Phi^*)\Phi] \\ &= \mathrm{i}q[(\partial^\mu\Phi^*)(\partial^\mu\Phi)+\Phi^*\Box\Phi-(\partial^\mu\Phi^*)(\partial_\mu\Phi)-(\Box\Phi^*)\Phi] \\ &= \mathrm{i}q[-\Phi^*m^2\Phi+m^2\Phi^*\Phi]=0.\end{aligned} \tag{28}$$

The total charge is calculated using the charge density $\rho = j^0$ and the mode decomposition (6),

$$\begin{aligned}Q &= \mathrm{i}q\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,\Phi^*\overleftrightarrow{\partial_t}\Phi \\ &= \mathrm{i}q\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{q}[a^*(\vec{p})u_{\vec{p}}^*+b(\vec{p})u_{\vec{p}}]\overleftrightarrow{\partial_t}[a(\vec{q})u_{\vec{q}}+b^*(\vec{q})u_{\vec{q}}^*] \\ &= q\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{q}[a^*(\vec{p})a(\vec{q})-b(\vec{p})b^*(\vec{a})] \\ &= q\int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}[|a(\vec{p})|^2-|b(\vec{p})|^2].\end{aligned} \tag{29}$$

[1] The group is the U(1), i.e., the multiplication of complex numbers $z = x+\mathrm{i}y$ $(x, y \in \mathbb{R}$ with a phase factor, keeping the absolute value $|z|$ invariant, which is equivalent to the rotations of vectors $(x, y)^{\mathrm{T}} \in \mathbb{R}^2$, i.e., SO(2).

As expected from Noether's theorem the charge turns out to be constant.

Extra task: Derive the form of this Noether current (26) from the Noether formalism as shown in the presentation slides of Lect. 5.

Solution: The infinitesimal form of the transformations (24) give

$$\delta \underline{x} = 0, \quad \delta\Phi = -\mathrm{i}q\delta\alpha\Phi, \quad \delta\Phi^* = +\mathrm{i}q\delta\alpha\Phi^*. \tag{30}$$

In terms of the formalism given on Slide 44 we thus have

$$\xi = 0, \quad \Xi = -\mathrm{i}q\Phi, \quad \Xi^* = \mathrm{i}q\Phi^*. \tag{31}$$

Since not only the variation of the action is invariant under the U(1) transformation, we can set $\Omega^\mu = 0$ and then we can immediately read off the Noether current as given on Slide 48,

$$j^\mu = \Pi^\mu \Xi + \Pi^{*\mu}\Xi^* = \mathrm{i}q[\Phi^*(\partial^\mu\Phi) - (\partial^\mu\Phi^*)\Phi] = \mathrm{i}q\Phi^* \overleftrightarrow{\partial^\mu}\Phi, \tag{32}$$

as given in (25).

A Manifestly covariant Fourier transformation

Above, the mode decomposition is written in a not manifestly covariant from which is convenient in the sense that in the quantized version with this convention the particle and anti-particle states created by the operator valued coefficients, which are then creation operators of the corresponding field-oscillator modes, are normalized to δ-distributions, i.e.,

$$\left[\mathbf{a}^\dagger(\vec{p}), \mathbf{a}(\vec{q})\right] = \left[\mathbf{b}^\dagger(\vec{p}), \mathbf{b}(\vec{q})\right] = \delta^{(3)}(\vec{p} - \vec{q}). \tag{33}$$

However, the mode decomposition (10) is not written in manifestly covariant form, but of course the Klein-Gordon field $\Phi(\underline{x})$ is a scalar field. The reason is that the mode functions (7) are no scalar fields, but neither is $\mathrm{d}^3 p$ nor the $a(\vec{p})$ and $b(\vec{p})$. All together we obtain the Lorentz-covariant scalar field Φ.

Alternatively we can define manifestly covariant mode functions, as functions of the *four-vectors* $\underline{x}$ and $\underline{p}$ and make the restriction of the 3D-momentum integration along the **mass shell** $p^0 = E_p$ manifest with appropriate invariant δ-distributions. Thus we define the covariant mode functions as

$$U_{\underline{p}}(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \exp(-\mathrm{i}\underline{p}\cdot\underline{x}). \tag{34}$$

It transforms under a proper orthochronous Lorentz transformation $\underline{x}' = \hat{\Lambda}^{-1}\underline{x}$ as a scalar field,

$$\begin{aligned} U'_{\underline{p}}(\underline{x}') = U_{\underline{p}}(\underline{x}) = U_{\underline{p}}(\hat{\Lambda}\underline{x}') &= \frac{1}{(2\pi)^{3/2}} \exp(-\mathrm{i}\underline{p}\cdot\hat{\Lambda}\underline{x}') \\ = \frac{1}{(2\pi)^{3/2}} \exp(-\mathrm{i}\hat{\Lambda}\underline{p}'\cdot\hat{\Lambda}\underline{x}') &= \frac{1}{(2\pi)^{3/2}} \exp(-\mathrm{i}\underline{p}'\cdot\underline{x}') = U_{\underline{p}'}(\underline{x}'). \end{aligned} \tag{35}$$

Then defining new Fourier coefficients,

$$A(\underline{p}) = \sqrt{2E_p}\,a(\vec{p}), \quad B(\underline{p}) = \sqrt{2E_p}\,b(\vec{p}), \tag{36}$$

the mode decomposition reads

$$\begin{aligned} \Phi(\underline{x}) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{2E_p}[A(\underline{p})U_{\underline{p}}(\underline{x}) + B^*(\underline{p})U_{\underline{p}}(\underline{x})]_{p^0=E_p} \\ &= \int_{\mathbb{R}^4} \mathrm{d}^4 p \frac{1}{2E_p}\delta(p^0 - E_p)[A(\underline{p})U_{\underline{p}}(\underline{x}) + B^*(\underline{p})U_{\underline{p}}(\underline{x})] \\ &= \int_{\mathbb{R}^4} \mathrm{d}^4 p \Theta(p^0)\delta(\underline{p}^2 - m^2)[A(\underline{p})U_{\underline{p}}(\underline{x}) + B^*(\underline{p})U_{\underline{p}}(\underline{x})]. \end{aligned} \tag{37}$$

Now $\Theta(p^0)\delta(\underline{p}^2 - m^2)$ is a scalar distribution-valued field under *proper orthochronous* Lorentz transformations, because $\operatorname{sign} p^0 = \text{const}$ for time- or light-like vectors. Since $m^2 \geq 0$, the δ distribution ensures that $\underline{p}^2 = m^2 \geq 0$ (i.e., $\underline{p}$ is time- or light-like). Also the momentum four-volume element, $\mathrm{d}^4 p$, is invariant:

$$\mathrm{d}^4 p = \mathrm{d}^4 p' \det\left(\frac{\partial p^\mu}{\partial p'^\mu}\right) = \mathrm{d}^4 p' \det \hat{\Lambda} = \mathrm{d}^4 p' \quad \text{for} \quad \hat{\Lambda} \in \mathrm{SO}(1,3)^{\uparrow}. \tag{38}$$

Since $\Phi(\underline{x})$ is a scalar field, thus also $A(\underline{p})$ and $B(\underline{p})$ must be scalar fields.

Using (36) in (22), (23), and (29), one finds

$$\begin{aligned} \underline{P} = \begin{pmatrix} E \\ \vec{P} \end{pmatrix} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\, \frac{\underline{p}}{2E_p} [|A(\underline{p})|^2 + |B(\vec{p})|^2]_{p^0 = E_p} = \int_{\mathbb{R}^4} \mathrm{d}^4 p \Theta(p^0)\delta(\underline{p}^2 - m^2)\underline{p}[|A(\underline{p})|^2 + |B(\vec{p})|^2], \\ Q &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\, \frac{q}{2E_p} [|A(\underline{p})|^2 + |B(\vec{p})|^2]_{p^0 = E_p} = \int_{\mathbb{R}^4} \mathrm{d}^4 p \Theta(p^0)\delta(\underline{p}^2 - m^2) q [|A(\underline{p})|^2 + |B(\vec{p})|^2]. \end{aligned} \tag{39}$$

The latter expressions show that $\underline{P}$ indeed is a four-vector, and Q a scalar, as it should be.

Note that with this convention for the mode functions, the corresponding annihilation and creation operators in the quantized theory are normalized as

$$\left[\mathbf{A}^\dagger(\underline{p}), \mathbf{A}(\underline{q})\right] = \left[\mathbf{B}^\dagger(\underline{p}), \mathbf{B}(\underline{q})\right] = 2E_p \delta^{(3)}(\vec{p} - \vec{q}). \tag{40}$$