Exercise Sheet 3 (Solutions)

Task 3.1: Deformed nucleus

For a deformed nucleus with a surface given by the multipole expansion with coefficients $\alpha_{\ell m}$, calculate:

1. The nuclear volume in second order of $\alpha_{\lambda\mu}$. How can we ensure that it is unafected by deformations from a sphere?

   Solution: The boundary of the nucleus's surface is given by

$$R(t,\vartheta,\varphi) = R_0\left[1+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\alpha_{\ell m}(t)\mathrm{Y}_{\ell m}(\vartheta,\varphi)\right]. \tag{1}$$

   The volume is (neglecting terms of order $\alpha_{\ell m}^3$ and $\alpha_{\ell m}^4$)

$$\begin{aligned}
V &= \int_\Omega \mathrm{d}^2 f \int_0^{R(t,\vartheta,\varphi)} \mathrm{d}r\, r^2 = \frac{R_0^3}{3}\int_\Omega \mathrm{d}^2 f \left[1+\sum_{\ell,m}\alpha_{\ell m}(t)\mathrm{Y}_{\ell m}(\vartheta,\varphi)\right]^3 \\
&= \frac{R_0^3}{3}\int_\Omega \mathrm{d}^2 f\left[1+3\left(\sum_{\ell,m}\alpha_{\ell m}(t)\mathrm{Y}_{\ell m}(\vartheta,\varphi)\right)+3\left(\sum_{\ell,m}\alpha_{\ell m}\mathrm{Y}_{\ell m}(\vartheta,\varphi)\right)\left(\sum_{\ell' m'}\alpha_{\ell' m'}\mathrm{Y}_{\ell' m'}(\vartheta,\varphi)\right)\right] \\
&= \frac{4\pi R_0^3}{3} + R_0^3\left[\sum_{\ell m}\int_\Omega \mathrm{d}^2 f\,\alpha_{\ell m}(t)\mathrm{Y}_{\ell m}(\vartheta,\varphi)+\sum_{\ell,m}\sum_{\ell' m'}\alpha_{\ell m}\alpha_{\ell' m'}^*\int_\Omega \mathrm{d}^2 f\,\mathrm{Y}_{\ell m}(\vartheta,\varphi)\mathrm{Y}_{\ell' m'}^*(\vartheta,\varphi)\right].
\end{aligned} \tag{2}$$

   In the final step we have interchanged the integrals with the sums and used the fact that $R(t,\vartheta,\varphi)=R^*(t,\vartheta,\varphi)$, from which also follows that

$$\alpha_{\ell m}^*(t) = \int_\Omega \mathrm{d}^2 f\,\mathrm{Y}_{\ell m}(\vartheta,\varphi)R(t,\vartheta,\varphi) = (-1)^m\int_\Omega \mathrm{d}^2 f\,\mathrm{Y}_{\ell,-m}^*(\vartheta,\varphi)R(t,\vartheta,\varphi) = (-1)^m\alpha_{\ell,-m}(t), \tag{3}$$

   where we have used that $\mathrm{Y}_{\ell m} = (-1)^m \mathrm{Y}_{\ell,-m}^*$.

   For integrating over the angles we use the orthonormality of the spherical harmonics. For the first sum we use $\mathrm{Y}_{00}^* = 1/\sqrt{4\pi} = \text{const}$:

$$\int_\Omega \mathrm{d}^2 f\,\alpha_{\ell m}(t)\mathrm{Y}_{\ell m}(\vartheta,\varphi) = \sqrt{4\pi}\int_\Omega \mathrm{d}^2 f\,\alpha_{\ell m}(t)\mathrm{Y}_{\ell m}(\vartheta,\varphi)\mathrm{Y}_{00}^* = \sqrt{4\pi}\delta_{\ell 0}\delta_{m0}. \tag{4}$$

   In the second sum we can simply apply the orthonormality of the spherical harmonics and do the sum over (ℓ', m'), finally leading to

$$V = \frac{4\pi R_0^3}{3} = \frac{4\pi R_0^3}{3} + R_0^3\left(\alpha_{00}\sqrt{4\pi}+\sum_{\ell m}|\alpha_{\ell m}(t)|^2\right).$$

   So up to contributions of order $\mathscr{O}(\alpha_{\ell m}^3)$ the volume stays unchanged if one chooses

$$\alpha_{00}(t) = -\frac{1}{\sqrt{4\pi}}\sum_{\ell m}|\alpha_{\ell m}(t)|^2. \tag{5}$$

2. The center of mass vector in first order of $\alpha_{\lambda\mu}$. What is the physical interpretation?

   Solution: The center of mass is defined by

$$\vec{R}_{\mathrm{cm}} = \frac{\int_V \mathrm{d}^3 r\,\vec{r}\rho(\vec{r})}{\int_V \mathrm{d}^3 r\,\rho(\vec{r})} = \frac{\int_V \mathrm{d}^3 r\,\vec{r}}{V}.$$

In the last step we have used that $\rho(\vec{r}) = M/V = 3m/(4\pi R_0^3) = \text{const}$. To evaluate the numerator, we note that

$$\begin{aligned}
Y_{11}(\vartheta,\varphi) &= \sqrt{\frac{-3}{8\pi}}\sin\vartheta\exp(\mathrm{i}\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\vartheta(\cos\varphi + \mathrm{i}\sin\varphi) = -Y_{1,-1}^*,\\
Y_{10} &= \sqrt{\frac{3}{4\pi}}\cos\vartheta = Y_{10}^*.
\end{aligned} \tag{6}$$

Now

$$\vec{x} = r\begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}. \tag{7}$$

So we have

$$x_1 + \mathrm{i}x_2 = \sqrt{\frac{8\pi}{3}}\, r Y_{1,-1}^*, \quad x_3 = \sqrt{\frac{4\pi}{3}}\, r Y_{10}^*, \tag{8}$$

and thus

$$\begin{aligned}
\int_V \mathrm{d}^3 r\begin{pmatrix}x_1 + \mathrm{i}x_2\\ x_3\end{pmatrix} &= \sqrt{\frac{4\pi}{3}}\int_\Omega \mathrm{d}^2 f\int_0^{R(t,\vartheta,\varphi)}\mathrm{d}r\, r^3\begin{pmatrix}\sqrt{2}Y_{1,-1}^*(\vartheta,\varphi)\\ Y_{10}^*(\vartheta,\varphi)\end{pmatrix}\\
&= \sqrt{\frac{\pi}{12}}\int_\Omega \mathrm{d}^2 f R^4(t,\vartheta,\varphi)\begin{pmatrix}\sqrt{2}Y_{1,-1}^*(\vartheta,\varphi)\\ Y_{10}^*(\vartheta,\varphi)\end{pmatrix}\\
&= \sqrt{\frac{\pi}{12}}R_0^4\int_\Omega \mathrm{d}^2 f\left[1 + \sum_{\ell m}\alpha_{\ell m}Y_{\ell m}(\vartheta,\varphi)\right]^4\begin{pmatrix}\sqrt{2}Y_{1,-1}^*(\vartheta,\varphi)\\ Y_{10}^*(\vartheta,\varphi)\end{pmatrix}\\
&= \sqrt{\frac{\pi}{12}}R_0^4\int_\Omega \mathrm{d}^2 f\left[1 + 4\sum_{\ell m}\alpha_{\ell m}Y_{\ell m}(\vartheta,\varphi) + \mathcal{O}(\alpha_{\ell m}^2)\right]\begin{pmatrix}\sqrt{2}Y_{1,-1}^*(\vartheta,\varphi)\\ Y_{10}^*(\vartheta,\varphi)\end{pmatrix}\\
&= \sqrt{\frac{4\pi}{3}}R_0^4\int_\Omega \mathrm{d}^2 f\sum_{\ell m}\alpha_{\ell m}Y_{\ell m}(\vartheta,\varphi)\begin{pmatrix}\sqrt{2}Y_{1,-1}^*(\vartheta,\varphi)\\ Y_{10}^*(\vartheta,\varphi)\end{pmatrix} + \mathcal{O}(\alpha_{\ell m}^2)\\
&= \sqrt{\frac{4\pi}{3}}R_0^4\begin{pmatrix}\sqrt{2}\alpha_{11}\\ \alpha_{10}\end{pmatrix}.
\end{aligned} \tag{9}$$

With this

$$\left\langle\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right\rangle = \sqrt{\frac{3}{4\pi}}R_0\begin{pmatrix}(\alpha_{11} + \alpha_{11}^*)/\sqrt{2}\\ (\alpha_{11} - \alpha_{11}^*)/(\mathrm{i}\sqrt{2})\\ \alpha_{10}\end{pmatrix}. \tag{10}$$

This shows that in leading order $\mathcal{O}(\alpha_{\ell m})$ the shift of the center of mass is determined by the dipole contribution of the deformation.

Task 3.2: Uranium nucleus

In Cartesian coordinates, the radius of a uranium-238 nucleus with a quadrupole deformation is given by

$$R(x,y,z) = R_0\Big(1 + \sum_{i,j\in\{x,y,z\}}\alpha_{ij}x_i x_j\Big) \tag{11}$$

where

$$\alpha_{ij} = \begin{pmatrix}0.0974076 & -0.03602963 & 0.08174144\\ -0.03602963 & -0.06457203 & -0.01736175\\ 0.08174144 & -0.01736175 & -0.03283557\end{pmatrix}_{ij}. \tag{12}$$

1. Calculate the eigenvalues of this matrix. What do they tell you about the symmetries of the nucleus?

   Solution: Using `Mathematica` one finds the eigenvalues,

$$\lambda_1 = \lambda_2 = -0.0722247, \quad \lambda_3 = 0.144449 \tag{13}$$

and eigenvectors

$$\vec{e}_1 = \begin{pmatrix} -0.0365068 \\ 0.88428 \\ 0.465529 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} -0.464517 \\ -0.427471 \\ 0.77556 \end{pmatrix}, \quad \begin{pmatrix} 0.884812 \\ -0.187933 \\ 0.426368 \end{pmatrix}, \tag{14}$$

choosen such that they form a right-handed Cartesian basis. The shape of the nucleus is a axially symmetric ellipsoid with the symmetry axis in direction $\vec{e}_3$ the long axis, because $\lambda_1 = \lambda_2 < 0 < \lambda_3$. Thus ^{238}U is a *prolate* nucleus.

2. Calculate the deformation parameters a_0 and a_2. Make sure to choose your major axes such that you exploit any symmetries.

   Solution: The multipole deformation coefficients in terms of the spherical harmonics, $\tilde{\alpha}_{\ell m}$ with $\ell = 2$ (quadrupole), are related to the Cartesian coefficients in the principal-axes basis (i.e., the eigenbasis (14)), $(\alpha'_{jk}) = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ by

$$a_0 = \tilde{\alpha}_{20} = \sqrt{\frac{4\pi}{45}}(2\lambda_3 - \lambda_1 - \lambda_2) = 0.229, \quad a_2 = \alpha_{22} = \sqrt{\frac{2\pi}{15}}(\lambda_1 - \lambda_2) = 0.$$

   a_0 describes the stretch along the $\vec{e}_3$ principal axis with respect to the $\vec{e}_1$ and $\vec{e}_2$ principal axes, and a_2 the stretch along $\vec{e}_1$ with respect to the $\vec{e}_2$ direction, which is 0 in our case, because the deformation is symmetric in the plane perpendicular to $\vec{e}_3$.

3. Measuring these deformation parameters is not trivial, since the body-fixed frame is not usually accessible in experiment. Read the first 3 pages of this Nature paper [A+24] (also accessible in the OLAT) and summarize: what are the differences between high and low-energy collisions, when it comes to measuring the excitations of ^{238}U? How can we understand the effect that the nuclear shape has on the observables v_2 and δp_T?

   Solutions: At low collition energies the duration of the interaction in the collision, τ_{int} is much longer than the rotation-time scale $\tau_{\text{rot}} = I/\hbar$, i.e., in such experiments the rotation-energy levels can be measured, which gives access to moments of inertia and thus the mass distribution within the nuclei.

   At high collision energies $\tau_{\text{int}} \ll \tau_{\text{rot}}$ and in central collisions rotational states are not excited. The prolate nuclei can hit each other either in a "tip-tip" or "body body configurtation" (see Fig. 1d in the paper). Now, in a heavy-ion collisin at ultrarelativistic energies a collectively flowing medium, behaving like a nearly perfect fluid, is formed, consisting of the quarks and gluons produced in the collision. Now the initial shape of this "fireball" produced in a tip-tip collision is pretty circular with than the one of a fireball produced in a body-body collision, where the initial shape is elliptical with a larger area than the former one. Due to the smaller size of the tip-tip-produced fireballs the pressure gradients are large than in those of the body-body-produced ones. Due to the collective fluid-like behavior that implies that the radial flow (i.e., the momentum components of the observed particles perpendicular to the beam direction, p_{T}) are larger in the tip-tip than in the body-body collisions.

   On the other hand the eccentricity of the initial fireball state in configuration space translates in a larger asymmetry of the p_{T} distribution, which is measured in the so-called elliptic-flow parameter v_2, which is the order-2 coefficient in a Fourier expansion of the p_{T} angular distribution in the transverse plane of the reaction.

   Thus in a tip-tip collision thus one expects to find a large radial flow with small v_2, while in the body-body collision a smaller radial flow with a large v_2. Thus correlating the radial flow with the v_2 of the observed transverse-momentum spectra, together with model calculations simulating the time-evolution of the entire collision allows for parametrizing the nuclear shape.

References

[A+24] M. I. Abdulhamid, et al., Imaging shapes of atomic nuclei in high-energy nuclear collisions, Nature **635**, 67 (2024).
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