Institut für Theoretische Physik Goethe
Universität Frankfurt
WiSe 2024
PD Dr. Hendrik van Hees

Exercise Sheet 2

Exercise 2.1: Fermi Gas Model

Assuming $Z/A \sim 1/2$ and $R_0 = 1.2$ fm, determine the average momentum and average energy of a nucleon in a nucleus using the Fermi Gas Model. How can this information be measured experimentally? Read and summarize the following paper **Phys. Rev. Lett. 26, 445** [MSW+71], which can be found in the Olat directory.

Solution: For details of the Fermi-gas model for nuclei, where one can use the non-relativistic theory for a Fermi gas at 0 temperature, see the presentation for Lecture 2 available at the course webpage. Using the Fermi momentum, $k_{\mathrm{F}} = \sqrt{2mE_{\mathrm{F}}}$, one finds the average momentum and energy per nucleon

$$\langle p \rangle = \frac{3}{4} k_{\mathrm{F}}, \quad \langle E \rangle = \frac{3k_{\mathrm{F}}^2}{10 m_N} = \frac{3E_{\mathrm{F}}}{5}, \tag{1}$$

where $m_N c^2 = m_{\mathrm{p}} c^2 \simeq m_{\mathrm{n}} c^2 \simeq 938$ MeV. Using $R = A r_0^{1/3}$ with $r_0 \simeq 1.25$ fm, in the lecture we derived $E_{\mathrm{F}} \simeq 30$ MeV, leading to $k_{\mathrm{F}} = \sqrt{2m_{\mathrm{p}}c^2 E_{\mathrm{F}}}/c = 237\ \mathrm{MeV}/c$. With (1) we find $\langle p \rangle = 179$ MeV and $\langle E \rangle = 18$ MeV.

In the experiment described in the paper electrons with a fixed energy of 500 MeV were scattered on various nuclei. The scattered electrons were observed at a constant scattering angle of $\theta = 60°$. Considering only events, where only a nucleon is knocked out with no other particles being created (e.g., pions as discussed in the paper). This explains the name "quasi-elastic scattering" of this process. If the nucleons were free, it would be simply elastic electron-nucleon scattering, $\mathrm{e}^- + N \rightarrow \mathrm{e}^- + N$.

However the nucleons are bound within the nucleus, and that's why the scattering is only quasi-elastic. One needs a overcome a certain average nucleon interaction energy ("binding energy"), $\bar{\epsilon}$, which turns out to be approximately independent of the energy level the nucleon occupies within the nucleus.

Due to the Fermi motion of the nucleons in the nucleus the energy of the nucleons is distributed uniformly within the Fermi sphere, $E < k_{\mathrm{F}}^2/(2m_N)$. The energy conservation of the scattering process reads

$$\omega + \left(\frac{k^2}{2m_N} - \bar{\epsilon} \right) = \frac{1}{2m_N} (\vec{k} + \vec{q})^2, \tag{2}$$

where ω is the energy loss of the incoming electron, k the momentum of the nucleon within the nucleus and $\bar{\epsilon}$ the above described nucleon interaction energy; $\vec{q}$ is the momentum transfer of the scattered electron to the nucleon.

The mean interaction energy $\bar{\epsilon}$ has been determined from the peak position of the measured cross section, and from its width the Fermi momentum of the nucleons within the nucleus can be determined, according to the quoted Fermi-gas model by Moniz [Mon69] (Ref. 6 of the discussed paper), which was used to fit the quasi-elastic cross section by determining the parameters k_{F} and $\bar{\epsilon}$.

Exercise 2.2: White Dwarfs

a) Why are there no nuclei composed only of neutrons? Then, how can neutron stars exist?

Solution: A hint is the empirical knowledge about the deuteron, i.e., the hydrogen nucleus with mass number 2. It is also the only known two-nucleon bound state. Theoretically this must be explained by figuring out the properties of the strong interaction between nucleons (protons and neutrons), and the experimental facts about the deuteron provide some important input:

From the nuclear Compton effect, i.e., the dissociation of the deuteron in a proton and a neutron by scattering X rays off deuterons it turns out that the binding energy is $E_{\mathrm{B}} \simeq 2.225$ MeV, which is pretty low compared with the binding energy per nucleon in larger nuclei, which is about 8MeV. Also in contrast to large nuclei the deuteron has no other bound states.

Further information can be gained from the spectral properties of deuterium, i.e., the hydrogen *atom* with a deuteron instead of a proton as its nucleus. The hyperfine splitting of the hydrogen spectroscopic lines is due to the interaction of the magnetic moments of the nucleus and the electron, which indicates that the spin of the deuteron nucleus is $S = 1$. Since there is only one bound state, which thus is the ground state of the pn-system, it has orbital angular momentum $L = 0$. So to get the total deuteron spin of $J = 1$ the spins of the proton and the neutron must add to 1.

From the rules of adding angular momenta (see, e.g. [ST93]) we know that adding two spins 1/2 leads to a total spin of either $S = 0$ with the eigenstate being given by the antisymmetric superposition of the state with $\sigma_1 = -\sigma_2$ ("singlet"),

$$|S=0,\sigma=0\rangle = \frac{1}{\sqrt{2}}(|1/2,-1/2\rangle - |-1/2,1/2\rangle). \tag{3}$$

The remaining three states ("triplet") orthogonal to this one are the symmetric superpositions, defining the $S = 1$ eigenstates,

$$|S=1,\sigma=\pm 1\rangle = |\pm 1/2,\pm 1/2\rangle, \quad |S=1,\sigma=0\rangle = \frac{1}{\sqrt{2}}(|1/2,-1/2\rangle + |-1/2,1/2\rangle). \tag{4}$$

Since the wave function of the deuteron bound state has $L = 0$, its parity is $+1$, i.e., the behavior under reflection of the proton-deuteron relative coordinates, $\vec{r} \to -\vec{r}$, i.e., the total state is symmetric under exchange of the proton and the neutron. Since proton and neutron differ in their electric charge (and thus also in their isospin $T_3 = \pm 1/2$) this is no problem although both particles are fermions.

Now the strong interaction is independent of the isospin, i.e., there is an SU(2)-symmetry in the two-flavor space of quarks and thus also for the strong interaction between protons and neutrons. Since the pn system has only the one deuteron bound state, also the pp and nn system could only have this same bound state, but this is forbidden for *indistinguishable fermions*, whose states must be totally antisymmetric under exchange of these particles. So there is no bound state for the nn system and the pp system.

All this indicates that the interaction between two nucleons is not provided by a purely central potential but that it must be spin dependent, because it obviously must be more attractive for the symmetric $S = 1$ state than for the antisymmetric $S = 0$ state. Indeed the there's a spin-dependent part in the interaction potential of the form

$$\mathbf{V}_{\text{spin}}(\vec{r}) = \mathbf{V}(\vec{r}) \frac{1}{\hbar^2} \vec{\mathbf{s}}_1 \cdot \vec{\mathbf{s}}_2. \tag{5}$$

Now

$$\vec{\mathbf{s}}_1 \cdot \vec{\mathbf{s}}_2 = \frac{1}{2}\left[\vec{\mathbf{S}}^2 - \vec{\mathbf{s}}_1^2 - \vec{\mathbf{s}}_2^2\right], \tag{6}$$

and in an S"=eigenstate this gives

$$\langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{\hbar^2}{2}[S(S+1) - s_1(s_1+1) - s_2(s_2+1)] = \frac{\hbar^2}{2}\left[S(S+1) - \frac{3}{2}\right] = \begin{cases} +\hbar^2/4 & \text{for} \quad S=1, \\ -3\hbar^2/4 & \text{for} \quad S=0. \end{cases} \tag{7}$$

Since the deuteron-bound state has $S = 1$ and there is no bound state with $S = 0$, the spin-dependent potential must be attractive for $S = 1$, so that it enables (together with the central potential) an $S = 1$-bound state. For

$S=0$ the spin-dependent potential then is necessarily repulsive, and it must be strong enough to make bound states with $S=0$ impossible.

Since now the nn system has no bound state (although it's experimentally not rigorously decided whether such a dineutron state exists or not) a stable nucleus consisting of only neutrons due to the strong interaction is very improbable, although there are recent experimental hints at a "quasi-bound state" of four neutrons ("tetraneutrons").

Neutron stars, however, exist, because for a macroscopic object gravitation becomes also significant. Any star is bound due to the gravitational interaction, which is strictly attractive, also at small distances between the particles/bodies. This implies that for a star to be stable there must be some counter action against gravitational collapse. For stars like our Sun that is provided by the thermal pressure due to the thermonuclear fusion reactions. If the corresponding fuel is exhausted, which must occur at a time since only nuclei up to iron provide energy in fusion processes, nothing withstands the gravitation, and the star collapses. There are different possible remnants, depending on the mass of the collapsing star.

For heavy stars with masses of 10—$25M_{\odot}$ ($M_{\odot}$: mass of our Sun) the nuclei are compressed such that a neutron star is formed in supernovae as its remnant (again depending on the mass of the star). Stars with even higher masses inevitably collapse to a black hole.

In a neutron star the pressure is so high that by inverse β decay most protons absorb an electron, become a neutron and emit an electron-neutrino. So the neutron star indeed consists mainly of neutrons ($\mathrm{e}^- + \mathrm{p} \rightarrow \nu_{\mathrm{e}} + \mathrm{n}$). Due to the emission of the neutrinos the "proto-neutron star" is also rather quickly cooled down, and it is stable due to the Pauli principle, i.e., it can be described as a degenerate Fermi gas of neutrons at nearly 0 temperature with the Fermi pressure counteracting further gravitational collapse. Usually a neutron star has a mass of about $1.4M_{\odot}$ but a radius of only about 10 km. It's one of the most interesting topics of nuclear astrophysics to understand the mass-radius relationship and how the observed neutron stars with masses as high as $2M_{\odot}$ can be described with the equation of state of nuclear matter and what can be learned from this about the strong interaction. Another open question is, whether there are neutron stars which have a core with "quark matter", i.e., that the nucleons are so much compressed that they dissolve into a "quark soup" with quarks as the relevant "thermodynamical degrees of freedom".

Lighter stars like our Sun first blow up to a "red giant" and finally collapse to a white dwarf, which we discuss in this exercise. White dwarfs consist of the nuclei bread in the fusion processes within the star, i.e., He, C, O, N, Mg and various mixtures thereof, and electrons. It is stabilized against gravitational collapse due to the Fermi pressure of the electrons. As we shall discuss in this problem, the mass of white dwarfs must be lower than the so-called Chandrasekhar limit (currently accepted to be about $1.4M_{\odot}$).

b) A white dwarf consists of helium nuclei with a temperature of $T \sim 10^7\mathrm{K} \sim \mathcal{O}(100)\mathrm{eV}/\mathrm{k_B}$. Since the ionization energy of the electrons is significantly lower ($\mathcal{O}(1)$ eV), a white dwarf can be greatly simplified as a gas of α-particles and a relativistic gas of electrons. Let N_{e} be the number of electrons in the star and $\rho_{\mathrm{S}} = 3.8 \cdot 10^9\,\mathrm{kg/m^3}$ the total density. Determine the electron density and their Fermi momentum.

Solution: Since $k_{\mathrm{B}} T \ll m_{\mathrm{e}} \ll m_{\mathrm{He}}$ we can make the simplifying assumption that $T=0$. Since for He nuclei $Z=2$, due to charge neutrality we must have $N_{\mathrm{e}} = N_{\alpha}/2$. The total mass of the star is $M = N_{\mathrm{e}} m_{\alpha}/2 \simeq 2N_{\mathrm{e}} m_{\mathrm{p}}$. Using $m_{\mathrm{p}} = 1.672 \cdot 10^{-27}\mathrm{kg} = 938.272\mathrm{MeV}/c^2$ the electron-number density is

$$n_e = \frac{N_{\mathrm{e}}}{V} = \frac{M/(2m_{\mathrm{p}})}{N/\rho} = \frac{\rho_{\mathrm{S}}}{2m_{\mathrm{p}}} = 1.137 \cdot 10^{36}/\mathrm{m}^3 = 1.137 \cdot 10^{-9}/\mathrm{fm}^3. \tag{8}$$

With (60) (with $g=2$ for the two spin degrees of freedom of the electron) we find

$$p_{\mathrm{F}} = 0.637\,\mathrm{MeV}/c. \tag{9}$$

c) From special relativity, we know that the energy per particle is given by

$$\varepsilon = c\sqrt{m^2c^2 + p^2} \tag{10}$$

Why must relativistic calculations be used in this case at all?

Solution: Using (9) and $m_e = 0.511\,\text{MeV}/c^2$, the Fermi momentum is found to be

$$E_F = c\sqrt{m_e^2 c^2 + p_{\text{eff}}^2} = 0.817\,\text{MeV} \Rightarrow E_{\text{kin,F}} = E_F - m_e c^2 = 0.306\,\text{MeV} \sim m_e c^2. \tag{11}$$

The higher kinetic energies of the electrons in the gas are not too much less than $m_e c^2$, so that the behavior of the gas enters the region, where relativistic effects become important.

d) Use the formula for E_F and ε to calculate the pressure of the Fermi gas

$$P_0 = \frac{8\pi c}{3(2\pi\hbar)^3}\int_0^{p_F} \mathrm{d}p \frac{p^4}{\sqrt{m^2c^2 + p^2}} \tag{12}$$

[Hint: Introduce the variable $x_F = \frac{p}{mc}$.]

Solution: Substituting $x_F = p/(m_e c)$ and then $x_F = \sinh u$, in (12), we get

$$\begin{aligned} P_0 &= \frac{8\pi c}{3(2\pi\hbar)^3}(m_e c)^4 \int_0^{x_F} \mathrm{d}x \frac{x^4}{1+x^2} \\ &= \frac{8\pi c}{3(2\pi\hbar)^3}(m_e c)^4 \int_0^{u_F} \mathrm{d}u \sinh^4 u, \end{aligned} \tag{13}$$

where $u_F = \operatorname{arsinh} x_F$. The integral can be evaluated by rewriting $\sinh^4 u$ in terms of exponential functions and then rewrite the result in terms of hyperbolic functions again. Expressing finally everything in terms of p_F one finds

$$P_0 = \frac{c}{24\pi^2\hbar^3}\left[p_F\sqrt{c^2m_e^2 + p_F^2}(2p_F^2 - 3m_e^2c^2) + 3(m_e c)^4 \operatorname{arsinh}\left(\frac{p_F}{mc}\right)\right]. \tag{14}$$

This pressure is not negligible. In a white dwarf, this is simplified to be balanced by the gravity of the α-particles. Show that the gravitational binding energy of a homogeneous sphere of mass M and radius R is given by

$$U_{\text{grav}} = -\frac{3}{5}\frac{GM^2}{R}. \tag{15}$$

Solution: The gravitational energy of the α nuclei is given by the fact that the gravitational force on a mass element $\mathrm{d}m$ at the distance r from the center is given as the force between a point particle in the origin with the mass contained in the sphere of radius, r, i.e., $m(r) = Mr^3/R^3$ (assuming a homogeneous star). With $\mathrm{d}m = \mathrm{d}^3 r 3M/(4\pi R^3)$ and using spherical coordinates for the integration yields after integrating out the angles

$$U_{\text{grav}} = -\int_0^R \mathrm{d}r \frac{Gm(r)3M}{4\pi r R^3} 4\pi r^2 = -\frac{3GM^2}{R^6}\int_0^R \mathrm{d}r\, r^4 = -\frac{3GM^2}{5R}. \tag{16}$$

In the ultra-relativistic limit, where one can assume $m_e = 0$, according to (58) the energy density is $\epsilon_0 = U_0/V = 3P_0$ and thus the total internal energy of the degenerate electron gas is

$$U_{\text{gas}} = U_0 = \frac{4\pi}{3}R^3\epsilon_0 = 4\pi R^3 P_0 \tag{17}$$

The total energy should be < 0 for the star to be stable, i.e., the maximum pressure is determined by $U_{\text{gas}} = |U_{\text{grav}}|$, i.e.,

$$P_0 = \frac{3GM^2}{20\pi R^4}. \tag{18}$$

e) Determine the relationship between the radius R and the mass M in the limit $x_F = p_{\mathrm{F}}/(mc) \gg 1$ and calculate a critical mass M_0 for which a white dwarf is stable in this simplified model.

Solution: Using (58) with $g=2$ (19) implies

$$P_0 = \frac{c p_{\mathrm{F}}^4}{12\pi^2 \hbar^3} = \frac{3GM^2}{20\pi R^4}. \tag{19}$$

Using (60) with (8),

$$\frac{N_0}{V} = n_{\mathrm{e}} = \frac{\rho_{\mathrm{S}}}{2m_{\mathrm{p}}}, \tag{20}$$

we find

$$p_{\mathrm{F}} = \left(\frac{9\pi M}{8m_{\mathrm{p}}}\right)^{1/3} \frac{\hbar}{R}. \tag{21}$$

Using this in (19) and solving for M we find for the limiting mass

$$Mc^2 = \left(\frac{5\hbar c^5}{9\pi G}\right)^{3/2} \left(\frac{9\pi}{8m_{\mathrm{p}}c^2}\right)^2 = 1.1158 \cdot 10^{60}\,\mathrm{MeV} = 1.721\, M_{\odot}, \tag{22}$$

which is not too bad an estimate given the pretty much simplified model used.

The now accepted value for the limiting mass (known as the Chandrasekhar mass, named after Subrahmanyan Chandrasekhar, 1910-1995) is about $1.4\, M_{\odot}$. For more details on the theory of compact stars, which is related to nuclear physics, particularly the equation of state of nuclear matter, as well as hydrostatics, see [Wei20, SB20].

A Thermodynamics of relativistic ideal Fermi gases

The thermodynamics of the relativistic ideal Fermi gas is most easily derived by using the grand-canonical statistical operator of maximum entropy,

$$\mathbf{R} = \exp(-\Omega - \beta \mathbf{H} - \alpha \mathbf{N}). \tag{23}$$

We consider the particles to be enclosed in a cubic box with length a. To have properly defined single-particle momenta we impose periodic boundary conditions, such that the single-particle momenta are given by $\vec{p}(\vec{n}) = 2\pi/a\vec{n}$ with $\vec{n} \in \mathbb{Z}^3$. The parameters Ω, β, and α allow for adjusting the constraints

$$\mathrm{Tr}\,\mathbf{R} = 1, \quad \langle E \rangle = \mathrm{Tr}(\mathbf{H}\mathbf{R}) = U, \quad \langle N \rangle = \mathrm{Tr}(\mathbf{N}\mathbf{R}) = N. \tag{24}$$

The Hamiltonian is the quantum-field theoretical Hamiltonian, describing, e.g., Dirac fermions, but we consider only the particles, so that we have the particle number $\mathbf{N}$ rather than the conserved "net particle number" (number of particles minus number of antiparticles). We assume an arbitrary half-integer spin s and denote the corresponding degeneracy factor as $g = 2s+1$ (for protons, neutrons, and electrons we have of course $s = 1/2$ and thus $g = 2$.

The entropy is given by (using "natural units", $k_{\mathrm{B}} = 1$)

$$S = -\mathrm{Tr}(\mathbf{R} \ln \mathbf{R}) = \mathrm{Tr}[\mathbf{R}(\Omega + \beta \mathbf{H} + \alpha \mathbf{N})] = \Omega + \beta U + \alpha N, \tag{25}$$

which is known as the Gibbs-Duhem equation. Written in the form

$$\Omega = S - \beta U - \alpha N, \tag{26}$$

we see that Ω is a Legendre transform of the entropy and thus called a Massieu potential. From $\mathrm{Tr}\, R = 1$ one finds that

$$\Omega = \Omega(\beta, \alpha, V) = \ln Z = \ln \mathrm{Tr} \exp(-\beta \mathbf{H} - \alpha \mathbf{N}). \tag{27}$$

We added $V = a^3$ as an external parameter, which enters the formalism through the boundary conditions. Taking the total differential of (25) we get

$$\mathrm{d}S = \mathrm{d}V \partial_V \Omega + \mathrm{d}\beta \partial_\beta \Omega + \mathrm{d}\alpha \partial_\alpha \Omega + \mathrm{d}\beta U + \beta \mathrm{d}U + N \mathrm{d}\alpha + \alpha \mathrm{d}N. \tag{28}$$

From (25), one immediately sees that

$$\partial_\beta \Omega = -U, \quad \partial_\alpha \Omega = -N. \tag{29}$$

Thus (28) simplifies to

$$\mathrm{d}S = \mathrm{d}V \partial_V \Omega + \beta \mathrm{d}U + \alpha \mathrm{d}N. \tag{30}$$

As we see, the "natural independent thermodynamical parameters" for S are $S = S(U, N, V)$.
To identify the thermodynamical parameters we rewrite this expression as

$$\mathrm{d}U = \frac{1}{\beta} \mathrm{d}S - \frac{\alpha}{\beta} \mathrm{d}N - \frac{1}{\beta} \partial_V \Omega \mathrm{d}V \stackrel{!}{=} T \mathrm{d}S - P \mathrm{d}V + \mu \mathrm{d}N, \tag{31}$$

from which we obtain the relation of the grand-canonical parameters β and α to the more familiar ones (T: temperature, μ chemical potential).

$$\beta = \frac{1}{T}, \quad \alpha = -\frac{\mu}{T} \tag{32}$$

and the expression for the pressure

$$P = T \partial_V \Omega. \tag{33}$$

We also note from (31) that the natural independent thermodynamical variables for the internal energy U are S, V, and N.
As we shall see below, the potential Ω is inconvenient when considering the zero-temperature limit $T \to 0^+$. For this the **grand potential** (also known as **Landau potential**),

$$\Phi = U - TS - \mu N \tag{34}$$

is more convenient. From (31) it follows

$$\mathrm{d}\Phi = \mathrm{d}U - T \mathrm{d}S - S \mathrm{d}T - \mu \mathrm{d}N - N \mathrm{d}\mu = -S \mathrm{d}T - p \mathrm{d}V - N \mathrm{d}\mu. \tag{35}$$

So the "natural independent thermodynamical quantities" for Φ are $T = 1/\beta$, V, and $\mu = -\alpha T$, and from (35) we obtain the relations

$$S = -\partial_T \Phi(T, V, \mu), \quad p = -\partial_V \Phi(T, V, \mu), \quad N = -\partial_\mu \Phi(T, V, \mu). \tag{36}$$

The relation with Ω can be found by using (25):

$$\begin{aligned} \Omega = S - \beta U - \alpha N = \beta(TS - U + T\mu N) = -\beta \Phi \\ \Rightarrow \Phi(T, V, \mu) = -T \Omega(\beta, V, \alpha) = -T \Omega(1/T, V, -\mu/T). \end{aligned} \tag{37}$$

This shows that (36) is indeed consistent with (29) and (33):

$$\begin{aligned}
-\partial_T \Phi(T,V,\mu) &= +\Omega(\beta,V,\alpha) + T(\partial_T\beta)\partial_\beta\Omega(\beta,V,\alpha) + T(-\mu\partial_T\beta)\partial_\alpha\Omega(\beta,V,\alpha) \\
&= \Omega + \beta U + \alpha N \stackrel{(25)}{=} S \\
-\partial_V \Phi(T,V,\mu) &= +T\partial_V \Omega(\beta,V,\alpha) \stackrel{(33)}{=} p, \\
-\partial_\mu \Phi &= +T(-1/T)\partial_\alpha \Omega(\beta,V,\alpha) \stackrel{(29)}{=} +N
\end{aligned} \tag{38}$$

Now for a free field theory we can use the Fock basis (occupation-number representation) to evaluate the trace for the partition sum, Z. For each one-particle $(\vec{p}, m_s)$ with $\vec{p} \in \frac{2\pi}{a}\mathbb{Z}^3$ and $m_s \in \{-s, -s+1, \ldots, s-1, s\}$ the possible occupation numbers are $N(\vec{p}, m_s) \in \{0,1\}$ since due to the necessity to quantize odd-spin fields with canonical equal-time **anti-commutators**, the described particles are fermions, and each one-particle state can occupied by at most one particle. Further

$$\mathbf{N} = \sum_{\vec{p}, m_s} \mathbf{N}(\vec{p}, m_s), \quad \mathbf{H} = \sum_{\vec{p}, m_s} E_p \mathbf{N}(\vec{p}, m_S) \tag{39}$$

with the relativistic one-particle energy,

$$E_p = c\sqrt{p^2 + m^2 c^2}, \tag{40}$$

where m is the invariant mass ("rest mass") of the particle.
Thus the partition sum gets

$$Z = \prod_{\vec{p}, m_s} \{1 + \exp[-(E_p - \mu)/T]\} = \prod_{\vec{p}} [1 + \exp(-\beta E_p - \alpha)]^g. \tag{41}$$

We note that all these calculations are valid for $T = 1/\beta > 0$ and arbitrary $\alpha \in R$. This is because the energy spectrum is bounded from below, as it should be to have a well-defined ground state of minimal energy.
The Massieu potential is now obtained from (25). For large volumes $V = a^3$ we can approximate the sum over momenta by an integral, using the number of states contained in a momentum-space volume element, $\mathrm{d}^3 p V/(2\pi\hbar)^3$:

$$\begin{aligned}
\Omega = \ln Z &= g \sum_{\vec{p}} \ln[1 + \exp(-\beta E_p - \alpha)] \\
&= \frac{gV}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \mathrm{d}^3 p \ln[1 + \exp(-\beta E_p - \alpha)] \\
&= \frac{4\pi g V}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \mathrm{d}p\, p^2 \ln[1 + \exp(-\beta E_p - \alpha)].
\end{aligned} \tag{42}$$

The latter form is obtained by introducing polar coordinates and integrating over the angles. An integration by parts brings (42) in a form, which is more convenient for some applications, particularly when taking the zero-temperature limit:

$$\Omega = \frac{4\pi g V \beta}{3(2\pi\hbar)^3} \int_0^\infty \mathrm{d}p\, p^3 \frac{\mathrm{d}E_p}{\mathrm{d}p} f_{\mathrm{F}}(E_p) = \frac{4\pi g V \beta}{3(2\pi\hbar)^3} \int_0^\infty \mathrm{d}p \frac{c p^4}{\sqrt{m^2 c^2 + p^2}} f_{\mathrm{F}}(E_p), \tag{43}$$

where we have introduced the Fermi-distribution function,

$$f_{\mathrm{F}}(E) = \frac{1}{1 + \exp(\beta E + \alpha)} = \frac{1}{1 + \exp[\beta(E - \mu)]} = \frac{1}{1 + \exp[(E - \mu)/T]}. \tag{44}$$

According to (37) the grand potential is

$$\Phi(T,V,\mu) = -\frac{4\pi g V}{3(2\pi\hbar)^3}\int_0^\infty \mathrm{d}p \frac{c p^4}{\sqrt{m^2c^2+p^2}} f_{\mathrm{F}}(E_p). \tag{45}$$

In the last steps we have used (32) and the relativistic energy-momentum relation (40).

With (29) and (33) we obtain the thermodynamic quantities. For taking the derivatives of Ω to obtain U and N the form (42) is most convenient, while for the pressure (43) is preferrable:

$$U = -\partial_\beta \Omega(\beta,V,\alpha) = \frac{4\pi g V}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}p\, p^2 c\sqrt{m^2c^2+p^2} f_{\mathrm{F}}(E_p), \tag{46}$$

$$N = -\partial_\alpha \Omega(\beta,V,\alpha) = \frac{4\pi g V}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}p\, p^2 f_{\mathrm{F}}(E_p), \tag{47}$$

$$P = T\partial_V \Omega = \frac{4\pi g}{3(2\pi\hbar)^3}\int_0^\infty \mathrm{d}p \frac{c p^4}{\sqrt{m^2c^2+p^2}} f_{\mathrm{F}}(E_p). \tag{48}$$

Now we consider the special case of zero temperature. Taking $T \to 0^+$ gives

$$f_{\mathrm{F}}^{(0)}(E) = \lim_{T\to 0^+} f_{\mathrm{F}}(E) = \Theta(E_{\mathrm{F}} - E) = \begin{cases} 1 & \text{for} \quad E < E_{\mathrm{F}}, \\ 1/2 & \text{for} \quad E = E_{\mathrm{F}}, \\ 0 & \text{for} \quad E > E_{\mathrm{F}}, \end{cases} \tag{49}$$

where we have set $\mu = E_{\mathrm{F}}$, introducing the Fermi energy. It describes the ground state of the Fermi gas: All energy levels with $E < E_{\mathrm{F}}$ are filled and all other unoccupied. For the following it is convenient to also introduce the Fermi momentum p_{F} via

$$E_{\mathrm{F}} = c\sqrt{m^2c^2 + p_{\mathrm{F}}^2}. \tag{50}$$

Of course then $f_{\mathrm{F}}^{(0)}(E) = \Theta(p_{\mathrm{F}} - p)$.

This simplifies (46-48) to

$$U_0 = -\partial_\beta \Omega = \frac{4\pi g V}{(2\pi\hbar)^3}\int_0^{p_{\mathrm{F}}} \mathrm{d}p\, p^2 c\sqrt{m^2c^2+p^2}, \tag{51}$$

$$N_0 = -\partial_\alpha \Omega = \frac{4\pi g V}{(2\pi\hbar)^3}\int_0^{p_{\mathrm{F}}} \mathrm{d}p\, p^2 = \frac{4\pi g V}{3(2\pi\hbar)^3} p_{\mathrm{F}}^3 = \frac{gV}{6\pi^2\hbar^3} p_{\mathrm{F}}^3, \tag{52}$$

$$P_0 = T\partial_V \Omega = \frac{4\pi g}{3(2\pi\hbar)^3}\int_0^{p_{\mathrm{F}}} \mathrm{d}p \frac{c p^4}{\sqrt{m^2c^2+p^2}} = \frac{U_0}{3V}. \tag{53}$$

To calculate also the entropy in the zero-temperature limit, we have to use Ω in the form (42) and must set $\alpha = -\beta\mu$. We also set $K_p = E_p - \mu$. The entropy is given by the Gibbs-Duhem relation (25), i.e.,

$$S = \Omega(\beta, V, -\beta\mu) + \beta(U - \mu N) = \frac{4\pi g V}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}p\, p^2 \left\{ [\ln[1 + \exp(-\beta K_p] + \beta K_p f_{\mathrm{F}}(E_p) \right\}. \tag{54}$$

Now we can take the limit $T \to 0^+$, i.e., $\beta = 1/T \to +\infty$. For $K_p > 0$ it is immediately clear that the limit is 0. For $K_p < 0$ we set $x = -\beta K_p$. Then the limit is

$$\begin{aligned} \lim_{x\to 0}\left[\ln(1+\exp x) - \frac{x}{1+\exp(-x)}\right] &= \lim_{x\to 0}[\ln(1+\exp x) - x] \\ &= \lim_{x\to\infty}\{\ln[(\exp(-x)+1)\exp x] - x\} \\ &= \lim_{x\to\infty}\ln[\exp(-x)+1] = 0. \end{aligned} \tag{55}$$

Thus we finally get

$$S_0 = 0 \tag{56}$$

as it should be, according to Nernst's Theorem. This is also seen without complicated calculations from the Planck"=Boltzmann relation of entropy with the number of "micro states" being compatible with the "macro state". As we have just seen, for $T \to 0^+$ the macro state, given by the fixed number of particles (or equivalently by fixed p_{F} or $\mu = E_{\mathrm{F}}$ implies that all states with $E_p < E_{\mathrm{F}}$ must be occupied with $g = 2s + 1$ particles, i.e., at zero temperature there is only one quantum state compatible with the macro state, and thus $S = \ln 1 = 0$.

For $p_{\mathrm{F}} \gg mc$ we can consider the ultra-relativistic limit, i.e., set $m = 0$. In this case we get simple closed forms for the thermodynamical quantities:

$$U_0 = \frac{4\pi g V c}{4(2\pi\hbar)^3} p_{\mathrm{F}}^4 = \frac{g V c}{8\pi^2\hbar^2} p_{\mathrm{F}}^4, \tag{57}$$

$$P_0 = \frac{4\pi g c}{12(2\pi\hbar)^3} p_{\mathrm{F}}^4 = \frac{g c}{24\pi^2\hbar^3} p_{\mathrm{F}}^4 = \frac{U_0}{3V}. \tag{58}$$

We also note that the thermodynamical relations, following from (31) for $T = 0$

$$P_0 = -\partial_V U(N_0, V), \quad \mu = E_{\mathrm{F}} = \partial_{N_0} U(N_0, V) \tag{59}$$

are satisfied. To see this explicitly, we first have to express U_0 in terms of N_0. With (52) we find

$$p_{\mathrm{F}} = \hbar \left(\frac{6\pi^2 N_0}{g V} \right)^{1/3} \hbar. \tag{60}$$

Using this in (57) we get U_0 in terms of its "natural" thermodynamic variables, N_0 and V:

$$U_0(N_0, V) = \frac{3\hbar c}{4} \left(\frac{6\pi^2}{g} \right)^{1/3} \frac{N_0^{4/3}}{V^{1/3}}. \tag{61}$$

Taking the derivatives in (59) gives

$$P_0 = -\partial_V U_0(N_0, V) = \frac{\hbar c}{4} \left(\frac{6\pi^2}{g} \right)^{1/3} \left(\frac{N_0}{V} \right)^{4/3}, \tag{62}$$

$$\mu = \partial_{N_0} U_0(N_0, V) = c\hbar \left(\frac{6\pi^2 N_0}{g V} \right)^{1/3} = c\, p_{\mathrm{F}} = E_{\mathrm{F}}. \tag{63}$$

Using (52) in (62) indeed leads to (53), which shows the consistency of the thermodynamical formalism.

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