Exercise 1.1: Fundamental Forces

What are the known fundamental interactions? Consider the following objects and discuss which interaction is essential for both its interaction and structure:

- **Galaxy:**
  - **Essential Interaction:** Gravitational
  - **Explanation:** Gravity binds stars and other matter in a galaxy, determining its structure and dynamics over vast distances.
- **Solar System:**
  - **Essential Interaction:** Gravitational
  - **Explanation:** The gravitational force from the Sun governs the orbits of the planets and maintains the structure of the solar system.
- **Planet:**
  - **Essential Interaction:** Gravitational
  - **Explanation:** Gravity holds a planet's material together, giving it shape and enabling it to maintain an atmosphere.
- **Basketball:**
  - **Essential Interaction:** Gravitational
  - **Explanation:** Gravity influences the basketball's motion, especially when thrown or dropped, affecting its trajectory and interactions with the ground.
- **Bacterium:**
  - **Essential Interaction:** Electromagnetic
  - **Explanation:** Electromagnetic forces govern biochemical processes and molecular interactions within the bacterium.
- **Molecule:**
  - **Essential Interaction:** Electromagnetic
  - **Explanation:** Electromagnetic interactions (chemical bonds) define the structure and stability of molecules.
- **Atom:**
  - **Essential Interaction:** Electromagnetic (for electrons), Strong (for the nucleus)
  - **Explanation:** Electromagnetic forces hold electrons in orbit around the nucleus, while the strong force binds protons and neutrons together in the nucleus.

- **Atomic Nucleus:**
  - **Essential Interaction:** Strong
  - **Explanation:** The strong force is essential for binding protons and neutrons together in the nucleus, overcoming their electromagnetic repulsion.
- **Proton:**
  - **Essential Interaction:** Strong
  - **Explanation:** The strong force binds quarks together to form protons, defining their structure and interactions.
- **Quark:**
  - **Essential Interaction:** Strong
  - **Explanation:** Quarks interact primarily through the strong force, which governs their behavior and holds them together to form protons and neutrons.

Exercise 1.2: Natural Units

In the course, so-called natural units are used. That means we set $c = \hbar = k_B = 1$. As the name suggests, this does not change the physics, but only the units of our quantities. This exercise is about determining conversion factors between natural units and SI units.

(a) What is a second in GeV^{-1}?

Solution: $1\mathrm{s}c/(\hbar c) = 1.519 \cdot 10^{24}/\mathrm{GeV}$ oder $t = 1/\mathrm{GeV}$ in natural units means $t = 6.582 \cdot 10^{-25}\mathrm{s}$ in SI units.

(b) What is a meter in GeV^{-1}?

Solution: $1\mathrm{m}/(\hbar c) = 10^{15}\ \mathrm{fm}/\hbar c = 5.068 \cdot 10^{15}/\mathrm{GeV}$ or $L = 1/\mathrm{GeV}$ in natural units means $0.1973\ \mathrm{fm} = 1.973 \cdot 10^{-16}\mathrm{m}$

(c) What is the unit of momentum in the SI system and in natural units? Determine the conversion factor.

Solution: $p = 1\mathrm{GeV}/c = 5.344 \cdot 10^{-19}\mathrm{kgm/s}$ which is written as $p = 1\,\mathrm{GeV}$ in natural units (since $c = 1$)

(d) What is the unit of temperature in natural units? What is a Kelvin in this unit?

Solution: $1\mathrm{K}k_\mathrm{B} = 8.617 \cdot 10^{-14}\ \mathrm{GeV} = 8.617 \cdot 10^{-5}\ \mathrm{eV}$ or $T = 1\mathrm{GeV}$ in natural units means $T = 1.6 \cdot 10^{13}\mathrm{K}$ in SI units.

(f) What is a second in fm? Use the results of parts (a) and (e).

Solution: $1\ \mathrm{s}c = 2.998{\cdot}10^{23}\mathrm{fm}$ or $1\ \mathrm{s}c/(\hbar c) = 1.519{\cdot}10^{24}/\mathrm{GeV}$; $t = 1/\mathrm{GeV}$ in natural units thus means $t = 6.582 \cdot 10^{-25}\mathrm{s}$ in SI units.

Exercise 1.3: Form factor and charge radius

The form factor for the scattering of an electron with a nucleus is given by

$$F(\vec{q}) = \frac{1}{Ze} \int_{\mathbb{R}^3} \mathrm{d}^3 r \exp(\mathrm{i}\vec{r} \cdot \vec{q}) \rho(\vec{r}),$$

where ρ is the charge density of the nucleus, and $\vec{q}$ is the momentum transfer in the scattering. Assume a spherically symmetric charge distribution, i.e., $\rho(\vec{r}) = \rho(r)$ (with $r = |\vec{r}|$) and that the typical scale of the nucleus's size R_{n} is such that one can assume $qR_{\mathrm{nucl}} \ll 1$ within the integral. Show that for these small momentum transfers, the form factor is given by

$$F(\vec{q}) = F(|\vec{q}|) = 1 - \frac{1}{6}\vec{q}^2 \langle r^2 \rangle$$

with

$$\langle r^2 \rangle = \int_{R^3} \mathrm{d}^3 r r^2 \rho(r),$$

i.e., from measuring the form factor you can deduce the "root-mean-square charge radius" $R_{\mathrm{rms}} = \sqrt{\langle r^2 \rangle} \simeq R_{\mathrm{nucl}}$.

Solution: We can expand the exponential under the integral up to 2nd order. In spherical coordinates we have $\vec{r} \cdot \vec{q} = rq \cos\vartheta$ (taking the polar axis in direction of $\vec{q}$). Then

$$\begin{aligned} F(\vec{q}) &\simeq \frac{1}{Ze} \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi r^2 \sin\vartheta \left(1 + \mathrm{i}rq\cos\vartheta - \frac{1}{2} r^2 q^2 \cos^2\vartheta\right) \rho(r) \\ &= 1 - \frac{1}{6Ze} \int_0^\infty \mathrm{d}r (4\pi r^2) r^2 \rho(r) = 1 - \frac{1}{6} q^2 \langle r^2 \rangle. \end{aligned}$$

The integral over ϑ is done by substitution of $u = \cos\vartheta$, $\mathrm{d}u = -\mathrm{d}\vartheta \sin\vartheta$. Also we used that the total charge is

$$\int_{\mathbb{R}^3} \mathrm{d}^3 r \rho(r) = \int_0^\infty \mathrm{d}r 4\pi r^2 \rho(r) = Ze.$$