Institut für Theoretische Physik Goethe
Universität Frankfurt
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PD Dr. Hendrik van Hees

Exercise Sheet 7

Poincaré transformations of the quantized Dirac field

In this exercise we consider the quantized Dirac field and the unitary representation of the Poincré group induced by the field-operator algebra. For the following one needs the generally valid equations for commutators of operator products involving anticommutators:

$$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}\{\mathbf{B}, \mathbf{C}\} - \{\mathbf{A}, \mathbf{C}\}\mathbf{B}. \tag{1}$$

which one proves by just writing out the corresponding operator products explicitly.
We use the simplified Lagrangian

$$\mathscr{L} = \overline{\Psi}(\mathrm{i}\not{\partial} - m)\Psi. \tag{2}$$

From Noether's theorem, applied to space-time translations and boosts (see Lecture 5) one finds the expressions for the corresponding conserved quantities, which are total energy, momentum (forming together the four-vector operator $\underline{\mathbf{P}}$), angular momentum, $\vec{\mathbf{J}}$, and the "boost operators", $\vec{\mathbf{K}}$, (whose conservation says that the "center of energy" moves with constant velocity), written down as normal-ordered operator expressions of the quantized theory:

$$\begin{aligned}
\mathbf{P}_\nu &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Theta}^\mu{}_\nu := \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger(\underline{x})\mathrm{i}\partial_\nu \mathbf{\Psi} : \\
\vec{\mathbf{J}} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger \left[\vec{x} \times (-\mathrm{i}\vec{\nabla}) + \hat{\vec{s}}_{\mathrm{D}}\right] \mathbf{\Psi} : \\
\vec{\mathbf{K}} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger \left[-\vec{x}\mathrm{i}\partial_t - \mathrm{i}t\vec{\nabla} - \gamma^0 \hat{\vec{k}}_{\mathrm{D}}\right] \mathbf{\Psi} :
\end{aligned} \tag{3}$$

The colons indicate normal ordering[1], and the field operators obey the Dirac equation,

$$(\mathrm{i}\not{\partial} - m)\mathbf{\Psi} = 0, \quad \overline{\mathbf{\Psi}}(\mathrm{i}\overleftarrow{\not{\partial}} + m) = 0. \tag{4}$$

The canonical equal-time anticommutator relations read

$$\{\mathbf{\Psi}_a(t,\vec{x}), \mathbf{\Psi}_b(t,\vec{y})\} = 0, \quad \left\{\mathbf{\Psi}_a^\dagger(t,\vec{x}), \mathbf{\Psi}_b(t,\vec{y})\right\} = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}), \tag{5}$$

and $a, b \in \{1,2,3,4\}$ are labeling the Dirac-spinor components.

(a) Show that $\mathbf{P}_\nu$, $\vec{\mathbf{J}}$, and $\vec{\mathbf{K}}$ are all self-adjoint operators.

Hint: You need to do integration by parts wrt. to the integral over $\vec{x}$. For the time derivatives use the Dirac-equation to express it in terms of spatial derivatives, $\vec{\nabla}$!

(b) The sign conventions for the translation and Lorentz-transformation operators are as follows

$$\begin{aligned}
\mathbf{U}_{\mathrm{transl}}(\underline{a}) &= \exp(\mathrm{i}\underline{a} \cdot \underline{\mathbf{P}}) = \exp(\mathrm{i}a^0\mathbf{P}^0 - \mathrm{i}\vec{a} \cdot \vec{\mathbf{P}}), \\
\mathbf{U}_{\mathrm{boost}}(\eta, \vec{n}) &= \exp(+\mathrm{i}\eta\vec{n} \cdot \vec{\mathbf{K}}), \\
\mathbf{U}_{\mathrm{rot}}(\varphi, \vec{n}) &= \exp(-\mathrm{i}\varphi\vec{n} \cdot \vec{\mathbf{J}}).
\end{aligned} \tag{6}$$

[1]Note that when calculating commutators, you don't need to worry about normal ordering, because normal ordering of bilinear operator products only leads to additive (usually diverging) c-number contributions. In this problem you don't need to explicitly consider the mode decomposition of the dirac field in annihilation and creation operators!

The corresponding transformations for the field operators are as for the corresponding classical fields, i.e.,

$$\begin{aligned}
&\mathbf{U}^{\dagger}_{\text{transl}}(\underline{a})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{transl}}(\underline{x}) = \mathbf{\Psi}(\underline{x}-\underline{a}),\\
&\mathbf{U}^{\dagger}_{\text{boost}}(\eta,\vec{n})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{boost}}(\eta,\vec{n}) = \exp(+\mathrm{i}\eta\vec{n}\cdot\hat{\vec{k}}_{\mathrm{D}})\mathbf{\Psi}\left(\hat{\Lambda}_{\mathrm{B}}^{-1}(\eta,\vec{n})\underline{x}\right),\\
&\mathbf{U}^{\dagger}_{\text{rot}}(\varphi,\vec{n})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{rot}}(\mathrm{D})\mathbf{\Psi}\left(\hat{\Lambda}_{\mathrm{R}}^{-1}(\varphi,\vec{n})\underline{x}\right)
\end{aligned} \tag{7}$$

with

$$\hat{\Lambda}_{\mathrm{B}}^{-1}(\eta,\vec{n}) = \begin{pmatrix} \cosh\eta & -\sinh\eta\vec{n}^{\mathrm{T}} \\ -\sinh\eta\vec{n} & (\cosh\eta-1)\vec{n}\vec{n}^{\mathrm{T}}+\mathbb{1}_3 \end{pmatrix} \tag{8}$$

Expand the both sides of these equations for "infinitesimal" $\delta\underline{a}$, $\delta\eta$, and $\delta\varphi$ to first order in these quantities.

(c) Show that the commutators of the unitary generators (3) resulting from this expansion on the left-hand of the equations (7) side match with what you get on the right-hand side of these equations, i.e., that the self-adjoint operators (3) really are the generators for the corresponding Poincaré transformations.

Hint: For this purpose use the equal-time anticommutator relations for the Dirac field. For anticommutators involving time derivatives of field operators use the Dirac equation to express the time derivatives in terms of spatial derivatives, $\vec{\nabla}$.