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Exercise Sheet 6

1. The quantized Klein-Gordon field

In this exercise we consider the quantized charged Klein-Gordon field.

(a) The charge operator is given by

$$\mathbf{Q} = \mathrm{i}q \int_{\mathbb{R}^3} \mathrm{d}^3\vec{y} : \mathbf{\Phi}^\dagger(\underline{y}) \overleftrightarrow{\partial_{t_y}} \mathbf{\Phi}(\underline{y}). \tag{1}$$

Show that it generates the phase transformation, which is the symmetry corresponding to this conserved charge via Noether's theorem,

$$\exp(\mathrm{i}\mathbf{Q}\alpha)\mathbf{\Phi}(\underline{x})\exp(-\mathrm{i}\mathbf{Q}) = \exp(-\mathrm{i}\alpha q)\mathbf{\Phi}(\underline{x}). \tag{2}$$

To this end calculate

$$\mathbf{\Phi}_\alpha(\underline{x}) = \exp(\mathrm{i}\mathbf{Q}\alpha)\mathbf{\Phi}(\underline{x})\exp(-\mathrm{i}\mathbf{Q}). \tag{3}$$

Hint: Take the derivative of this wrt. α and derive a differential equation for $\mathbf{\Phi}_\alpha(\underline{x})$ and then solve it with the initial condition $\mathbf{\Phi}_{\alpha=0}(\underline{x}) = \mathbf{\Phi}(\underline{x})$.

(b) Calculate the commutator function

$$\mathrm{i}\Delta(\underline{x}-\underline{y}) = \left[\mathbf{\Phi}(\underline{x}), \mathbf{\Phi}^\dagger(\underline{y})\right] \tag{4}$$

by using the mode decomposition of the field,

$$\mathbf{\Phi}(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\left[\mathbf{a}(\vec{p})u_{\vec{p}}(\underline{x}) + \mathbf{b}^\dagger(\vec{p})u^*_{\vec{p}}(\vec{p})\right]. \tag{5}$$

(c) Show that $\Delta(\underline{x}-\underline{y})$ is a scalar field under proper orthochronous Lorentz transformations.

(d) From the equal-time commutation relations of the field operators, it follows that $\Delta(x-y)|_{t_x=t_y} = 0$. Use this to show the **micro-causality property**

$$\Delta(\underline{z}) = 0 \quad \text{if} \quad \underline{z}\cdot\underline{z} < 0. \tag{6}$$

To this end show that you can always find an η in the Lorentz boost $\hat{\Lambda}(\vec{z}/|\vec{z}|, \eta)$ with

$$\underline{z}' = \hat{\Lambda}(\vec{n},\eta)\underline{z} = \begin{pmatrix} 0 \\ \vec{z}' \end{pmatrix} \Rightarrow \Delta(\underline{z}) = \Delta'(\underline{z}') = 0. \tag{7}$$

2. Ideal relativistic Bose gas

In this exercise we investigate an ideal relativistic Bose gas in the grand-canonical ensemble of quantum-statistical physics. To this end consider the finite-volume box regularization for the quantized charged Klein-Gordon field discussed in the lecture.

The grand-canonical statistical operator for the ideal gas is given by

$$\hat{\rho} = \frac{1}{Z}\exp(-\beta\mathbf{H} - \alpha\mathbf{Q}), \quad Z = \mathrm{Tr}\exp(-\beta\mathbf{H} - \alpha\mathbf{Q}). \tag{8}$$

The Hamilton operator and conserved charge (with $q=1$, i.e., the "net-particle number" $N_a - N_b$) is given in terms of the number operators

$$\begin{aligned} \mathbf{H} &= \sum_{\vec{p}} E_{\vec{p}} \left[\mathbf{N}_a(\vec{p}) + \mathbf{N}_b(\vec{p})\right], \\ \mathbf{Q} &= \sum_{\vec{p}} \left[\mathbf{N}_a(\vec{p}) - \mathbf{N}_b(\vec{p})\right]. \end{aligned} \tag{9}$$

The momenta run over the momenta $\vec{p} \in 2\pi\mathbb{Z}^3/L$. Note that the momenta depend on $L = V^{1/3}$, where V is the volume of the cubic box.

(a) Calculate the partition sum, Z, as defined in (8). Use the occupation-number basis to take the trace, i.e., for an operator $\mathbf{A}$

$$\operatorname{Tr}\mathbf{A} = \prod_{\vec{p}} \sum_{N_a(\vec{p})=0}^{\infty} \sum_{N_b(\vec{p})=0}^{\infty} \left\langle \{N_a(\vec{p}), N_b(\vec{p})\}_{\vec{p}} \middle| \mathbf{A} \middle| \{N_a(\vec{p}), N_b(\vec{p})\}_{\vec{p}} \right\rangle. \tag{10}$$

What are the physically meaningful ranges for the parameters β and α?

(b) Calculate the internal energy U and mean net-particle number Q by showing that with $\Omega(\beta, V, \alpha)$[1]

$$U = -\partial_\beta \Omega(\beta, V, \alpha), \quad Q = -\partial_\alpha \Omega(\beta, V, \alpha). \tag{11}$$

(c) Prove that the entropy (with $k_{\mathrm{B}} = 1$) is given by

$$S = -\operatorname{Tr}(\boldsymbol{\rho} \ln \boldsymbol{\rho}) = \Omega + \beta U + \alpha Q \tag{12}$$

(d) Prove that the total differential of S is given by

$$\mathrm{d}S = \beta \mathrm{d}U + \mathrm{d}V \partial_V \Omega(\beta, V, \alpha) + \alpha \mathrm{d}Q. \tag{13}$$

(e) Identify the quantities β, $\partial_V \Omega$, and α with the usual thermodynamic variables by comparing

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \mu \mathrm{d}Q. \tag{14}$$

(f) Calculate the pressure.

Extra (pretty hard!) puzzles:

- Take the "thermodynamic limit", i.e., $V \to \infty$. Consider without restriction of generality the case $\mu > 0$ (and thus $Q > 0$). Consider the limit at fixed $\beta = 1/T$ and be aware that you need to treat the contribution from the single-particle ground state $\vec{p} = 0$ separtely. Show that taking the naive prescription

$$\sum_{\vec{p}} \xrightarrow[V\to\infty]{} V \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tag{15}$$

  provides only the contributions to the thermodynamic quantities for the "excited states" $\vec{p} \neq 0$ in the finite-box description.

- for the ground-state contribution you have to take the limit such that the charge density $q = Q/V$ stays kept fixed.

- In the so established "thermodynamic limit". Discuss that for $T \to 0^+$ keeping the total charge density fixed, you always get Bose-Einstein condensation, i.e., all particles occupy the single-particle ground state. Which limit is implies for μ for $T \to 0$?

- How is the critical temperature determined, i.e., what's the temperature T_{c} such that a finite density of particles occupying the single-particle ground state $q_0 \neq 0$ is necessary for $T < T_{\mathrm{c}}$.

[1] Note that, of course, you cannot solve the sums over $\vec{p}$ in closed form. So just leave the results in terms of such sums!