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Exercise Sheet 5

The complex Klein-Gordon field

Consider a complex-valued Lorentz-scalar field. The Lagrangian, defining the free field equations is given by

$$\mathscr{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi. \tag{1}$$

Obviously the Lagrangian is a Lorentz scalar and thus the action too. The Lagrangian is not explicitly dependent on the space-time coordinates $\underline{x} = (x^\mu)$, with the fields transforming under both translations and Lorentz transformations as a scalar field.

1. Derive the equations of motion from the Lagrange equations. The complex scalar field has to be interpreted as two real field-degrees of freedom, i.e., you can vary Φ and Φ^* as independent fields, and thus you have the Euler-Lagrange equations for these two fields,

$$\Pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}, \quad \Pi^{*\mu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^*)}, \quad \partial_\mu \Pi^\mu = \frac{\partial \mathscr{L}}{\partial \Phi}, \quad \partial_\mu \Pi^{*\mu} = \frac{\partial \mathscr{L}}{\partial \Phi^*}. \tag{2}$$

   Show that the result is the **Klein-Gordon equation** for both Φ and Φ^*.

2. Use the Fourier ansatz for the field Φ

$$\Phi(\underline{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} A(t,\vec{p}) \exp(\mathrm{i}\vec{p}\cdot\vec{x}) \tag{3}$$

   to show that the general solution of the Klein-Gordon equation can be written in the form

$$\Phi(\vec{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} [a(\vec{p}) u_{\vec{p}}(\underline{x}) + b^*(\vec{p}) u^*_{\vec{p}}(\underline{x})] \tag{4}$$

   with the “relativistic plane-wave mode functions”

$$u_{\vec{p}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} \exp(-\mathrm{i}\underline{x}\cdot\underline{p})|_{p^0=E_p}, \quad E_p = \sqrt{m^2+\vec{p}^2} \tag{5}$$

   and arbitrary (square-integrable) $\mathbb{C}$-valued functions $a(\vec{p})$ and $b(\vec{p})$

3. For two scalar fields Φ_1 and Φ_2 we define

$$\Phi_1 \overleftrightarrow{\partial_\mu} \Phi_2 = \Phi_1 \partial_\mu \Phi_2 - (\partial_\mu \Phi_1)\Phi_2 \tag{6}$$

   and the *non-definite* bilinear form

$$(\Phi_1, \Phi_2) = \mathrm{i} \int_{\mathbb{R}^3} \mathrm{d}^3 x \Phi_1 \overleftrightarrow{\partial_t} \Phi_2. \tag{7}$$

   Show that for the mode functions (5) and $\vec{p}, \vec{q} \in \mathbb{R}^3$

$$(u_{\vec{p}}, u_{\vec{q}}) = (u^*_{\vec{p}}, u^*_{\vec{q}}) = 0, \quad (u^*_{\vec{p}}, u_{\vec{q}}) = -(u_{\vec{q}}, u^*_{\vec{p}}) = \delta^{(3)}(\vec{p}-\vec{q}). \tag{8}$$

4. Calculate the canonical energy-momentum tensor,

$$\Theta^{\mu\nu} = \Pi^{\mu}\partial^{\nu}\Phi + \Pi^{*\mu}\partial^{\nu}\Phi^{*} - \mathscr{L}\eta^{\mu\nu}$$

and express the total energy and momentum

$$P^{\nu} = \int_{\mathbb{R}^3} \mathrm{d}^3 x \Theta^{0\nu} \tag{9}$$

in terms of the Fourier components $a(\vec{p})$ and $b(\vec{p})$ defined in (4).

5. From the obvious invariance of the Lagrangian under the phase transformation

$$\Phi'(\underline{x}) = \exp(-\mathrm{i}q\alpha)\Phi(\underline{x}), \quad \Phi^{*\prime}(\underline{x}) = \exp(+\mathrm{i}q\alpha)\Phi^{*}(\underline{x}), \tag{10}$$

where $\alpha \in \mathbb{R}$ is the group parameter[1] the corresponding Noether current is given by

$$j_{\mu} = \mathrm{i}q\Phi^{*}\overleftrightarrow{\partial_{\mu}}\Phi. \tag{11}$$

Show that indeed the continuity equation,

$$\partial_{\mu} j^{\mu} = 0 \tag{12}$$

holds if Φ fulfills the Klein-Gordon equation and calculate the total charge

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3 x j^{0}(\underline{x}) \tag{13}$$

in terms of the Fourier components $a(\vec{p})$ and $b(\vec{p})$.

Extra task: Derive the form of this Noether current (12) from the Noether formalism as shown in the presentation slides of Lect. 5.

[1]The group is the U(1), i.e., the multiplication of complex numbers $z = x + \mathrm{i}y$ ($x, y \in \mathbb{R}$ with a phase factor, keeping the absolute value $|z|$ invariant, which is equivalent to the rotations of vectors $(x, y)^{\mathrm{T}} \in \mathbb{R}^2$, i.e., SO(2).