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Exercise Sheet 2

Exercise 2.1: Fermi Gas Model

Assuming $Z/A \sim 1/2$ and $R_0 = 1.2$ fm, determine the average momentum and average energy of a nucleon in a nucleus using the Fermi Gas Model. How can this information be measured experimentally? Read and summarize the following paper **Phys. Rev. Lett. 26, 445**, which can be found in the Olat directory.

Exercise 2.2: White Dwarfs

a) Why are there no nuclei composed only of neutrons? Then, how can neutron stars exist?

b) A white dwarf consists of helium nuclei with a temperature of $T \sim 10^7\text{K} \sim \mathcal{O}(100)$ eV. Since the ionization energy of the electrons is significantly lower ($\mathcal{O}(1)$ eV), a white dwarf can be greatly simplified as a gas of α-particles and a relativistic gas of electrons. Let N be the number of electrons in the star and $\rho_s = 3.8 \cdot 10^9\,\text{kg/m}^3$ the total density. Determine the electron density and their Fermi momentum.

c) From special relativity, we know that the energy per particle is given by

$$\varepsilon = \sqrt{(pc)^2 + (mc^2)^2} \tag{1}$$

Why must relativistic calculations be used in this case at all?

d) Use the formula for ε_F and ε to calculate the pressure of the Fermi gas

$$P_0 = \frac{8\pi c}{3(2\pi\hbar)^3} \int_0^{p_\text{F}} \mathrm{d}p \frac{p^4}{\sqrt{m^2c^2 + p^2}} \tag{2}$$

[Hint: Introduce the variable $x_F = \frac{p}{mc}$.]

This pressure is not negligible. In a white dwarf, this is simplified to be balanced by the gravity of the α-particles. Show that the gravitational binding energy of a homogeneous sphere of mass M and radius R is given by

$$U_\text{grav} = -\frac{3}{5}\frac{GM^2}{R}\,. \tag{3}$$

In the ultrarelativistic limit, where one can assume $m = 0$, the energy density is $\epsilon_0 = 3P_0$ and thus the total internal energy of the gas

$$U_\text{gas} = \frac{4\pi}{3}\frac{4\pi}{3}R^3\epsilon_0 = 4\pi R^3 P_0 \tag{4}$$

The total energy should be < 0 for the star to be stable, i.e., the maximum pressure is determined by $U_\text{gas} = |U_\text{grav}|$, i.e.,

$$P_0 = \frac{3}{20\pi}\frac{GM^2}{R^4}. \tag{5}$$

e) Determine the relationship between the radius R and the mass M in the limit $x_F \gg 1$ and calculate a critical mass M_0 for which a white dwarf is stable in this simplified model.