# Symmetries

# Ward Takahashi identities and all that

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*Content*

- 1PI- and 2PI-Functionals of quantum field theory
- Ward Takahashi identities
- Example: Goldstone's Theorem
- The $\Phi$-derivable scheme and symmetries
- Restoration of symmetries

# *1PI-Functionals I*



- Start with a local (O($N$) symmetric!) classical action functional

$$S[\vec{\phi}] = \left\{ \frac{1}{2}(\partial_\mu \vec{\phi}(1))(\partial^\mu \vec{\phi}(1)) - \frac{m^2}{2}\vec{\phi}^2(1) - \frac{\lambda}{8}(\vec{\phi}^2(1))^2 \right\}_1$$

- Generating functional for Green's functions:

$$Z[\vec{J}] = \int \mathrm{D}\vec{\phi} \exp\left[\mathrm{i}\left(S[\vec{\phi}] + \left\{\vec{J}(1)\vec{\phi}(1)\right\}_1\right)\right]$$

- Generates Green's functions:

$$\mathrm{i}G^{(n)}(1j_1, \ldots, nj_n) = \left\langle T_{\mathcal{C}} \phi^{j_1}(1) \cdots \phi^{j_n}(n) \right\rangle = \frac{(-\mathrm{i})^n}{Z[0]} \left. \frac{\delta^n Z[J]}{\delta J_{j_1}(1) \cdots \delta J_{j_n}(n)} \right|_{J=0}$$

- Generating functional for connected Green's functions:

$$Z[J] = \exp(\mathrm{i}W[J]), \quad G_c^{(n)}(1j_1, \ldots, nj_n) = (-\mathrm{i})^n \left. \frac{\delta^n W[J]}{J_{j_1}(1) \cdots J_{j_n}(n)} \right|_{J=0}$$

# 1PI-Functionals II



- Mean field:

$$\varphi^j(1) = \frac{\delta W[J]}{\delta J_j(1)} = \left\langle \phi^j(1) \right\rangle_J$$

- Generating functional for proper vertex functions
  1-particle irreducible (1PI) truncated Green's functions:

$$\Gamma[\vec{\varphi}] = W[\vec{J}] - \left\{ \vec{J}(1)\vec{\varphi}(1) \right\} \Leftrightarrow \vec{J}(1) = -\frac{\delta\Gamma[\vec{\varphi}]}{\delta\vec{\varphi}(1)}.$$

$$-\mathrm{i}\Gamma^{(n)}(1j_1, \ldots, nj_n) = \left. \frac{\delta^n \Gamma[\vec{\varphi}]}{\delta\varphi(1j_1)\cdots\delta\varphi(n, j_n)} \right|_{\frac{\delta\Gamma}{\delta\vec{\varphi}} = -\vec{J} = 0}$$

- Relation for the connected 2-point Green's function

$$\Gamma^{(2)}(1j_1, 2j_2) = \mathrm{i}(G_c^{(2)})^{-1}(1j_1, 2j_2)$$

- All connected (and disconnected) Green's functions can be expressed as tree diagrams with the proper vertex functions as vertices and $G_c^{(2)}$ as propagator lines
- Only proper vertex functions need to be renormalized!

# *Symmetries of the class. action: Noether's theorem*



- In our case $S[\vec{\varphi}]$ is symmetric under O($N$)-transformations of the fields:

$$\forall \delta\eta_a : \left\{ \frac{\delta S[\vec{\varphi}]}{\delta\vec{\varphi}(x_1)} \underbrace{\delta\eta_a \hat{\tau}^a \vec{\varphi}(x_1)}_{\delta\vec{\varphi}(x_1)} \right\}_1 = 0$$

- Must hold for any field configuration:

$$\exists j_\mu^a : \frac{\delta S[\vec{\varphi}]}{\delta\vec{\varphi}(x)} \hat{\tau}^a \vec{\varphi}(x) = \partial^\mu j_\mu^a$$

- Equations of motion

$$\frac{\delta S[\vec{\phi}]}{\delta\vec{\phi}} = 0$$

- For each independent global symmetry: conserved Noether current

$$\partial^\mu j_\mu^a = 0$$

# *Symmetries of the quantized theory*



- "Field-translation" invariance of path-integral measure:

$$Z[\vec{J}] = \int \mathrm{D}\vec{\phi} \exp\left[\mathrm{i}S[\vec{\phi}+\delta\vec{\phi}] + \mathrm{i}\left\{\vec{J}(x)(\vec{\phi}(x)+\delta\vec{\phi}(x))\right\}_x\right]$$

- In this general form: Ehrenfest's theorem
- For infinitesimal $\delta\vec{\varphi}$:

$$\left\{\int \mathrm{D}\vec{\phi}\left[\frac{\delta S[\vec{\phi}]}{\delta\vec{\phi}(x')} + \vec{J}(x')\right]\delta\vec{\phi}(x') \exp\left[\mathrm{i}S[\vec{\phi}] + \mathrm{i}\left\{\vec{J}(x)\vec{\phi}(x)\right\}_x\right]\right\}_{x'} = 0$$

- For infinitesimal Symmetry transformations green term vanish identically:

$$\delta\vec{\phi}(x) = \delta\eta_a \hat{\tau}^a \vec{\phi}(x) :\Rightarrow \forall\delta\eta_a : \ \delta\eta_a \left\{\vec{J}(x')\hat{\tau}^a \frac{\delta}{\delta \mathrm{i}\vec{J}(x')} Z[J]\right\}_{x'} = 0$$

- Linear in $\frac{\delta}{\delta\vec{J}}$: The same for $W \Rightarrow$

$$\delta\eta_a \left\{\frac{\delta\Gamma[\vec{\varphi}]}{\delta\varphi(\vec{x})}\hat{\tau}^a \varphi(\vec{x})\right\}_x = 0$$

# *Perturbative renormalizability*



- Linear symmetry operations + path-integral measure invariant + Existence of symmetry consistent regularization

☞ $\Gamma[\vec{\varphi}]$ has the same symmetry as the classical action

- Contains the full set of Ward Takahashi identities for proper Green's functions
- For (perturbative renormalization): If Lagrangian contains all monomials of order 4 or less allowed by symmetries also the counter terms are of the same form: Theory renormalizable to any order of $\hbar$ or $\lambda$
- $\hbar$: Overall factor in exponential of path integral; Order by order symmetric
- $\lambda$: quadratic part and "interaction part" of action are separately invariant
- Conclusion: Renormalized action is symmetric under $\mathrm{O}(N)$ order by order in the loop $(\hbar)$ or Coupling constant expansion
- Remark: Holds also true for perturbative large N-expansion

# “Hidden Symmetries” and Goldstone’s theorem



- Hidden symmetry: Solution $\frac{\delta\Gamma}{\delta\vec{\varphi}} = 0$ not invariant, i.e., solution $\vec{\varphi}_0 \neq 0$
- Inverse Green’s function:

$$G^{-1}(x_1 j_1, x_2 j_2) = \left. \frac{\delta^2 \Gamma[\vec{\varphi}]}{\delta\varphi_{j_1}(x_1)\delta\varphi_{j_2}(x_2)} \right|_{\vec{\varphi}=\vec{\varphi}_0}$$

- Taking derivative of WTI for $\Gamma$ at $\vec{\varphi}_0$

$$\delta\eta_a \left\{ G^{-1}(x'_1 j'_1, x_2 j_2)(\tau^a)_{j'_1 j_1} \varphi_{0 j_1}(x') \right\}_{x'} = 0$$

- If theory translation invariant (e.g., vacuum or thermal equilibrium)

$$\delta\eta_a (G^{-1})_{j'_1 j_2}(p=0)(\tau^a)_{j'_1 j_1} \varphi_{0 j_1} = 0$$

- $\hat{\tau}^a$ generators of $\mathrm{O}(N)$ and $\vec{\varphi}_0 \neq 0 \Rightarrow$
- If $G$ is symmetry group of $\Gamma$ and $H$ is the subgroup which leaves $\vec{\varphi}_0$ invariant $\Rightarrow$

$$N_{\mathrm{NG}} = \dim G - \dim H (= N-1 \text{ for } \sigma\text{-model})$$

  massless field degrees of freedom: Nambu-Goldstone modes
- Nambu-Goldstone phase can be renormalized with symmetric counter terms
- Need to introduce mass renormalization scale

# *The Φ-Functional*



- Generating functional

$$Z[J,K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{J_1\phi_1\right\}_1 + \left\{\frac{\mathrm{i}}{2}K_{12}\phi_1\phi_2\right\}_{12}\right], \quad Z[J,K] = \exp(\mathrm{i}W[J,K])$$

- The mean field and the connected Green's function

$$\varphi_1 = \frac{\delta W}{\delta J_1}, \; G_{12} = -\frac{\delta^2 W}{\delta J_1 \delta J_2} \Rightarrow \frac{\delta W}{\delta K_{12}} = \frac{1}{2}[\varphi_1\varphi_2 + \mathrm{i}G_{12}]$$

- Legendre transformation for $\varphi$ and $G$:

$$\Gamma[\varphi, G] = W[J,K] - \{\varphi_1 J_1\}_1 - \frac{1}{2}\{(\varphi_1\varphi_2 + \mathrm{i}G_{12})K_{12}\}_{12}$$

- Exact closed form:

$$\Gamma[\varphi, G] = S_0[\varphi] + \frac{\mathrm{i}}{2}\operatorname{Tr}\ln(M^2 G^{-1}) + \frac{\mathrm{i}}{2}\left\{D_{12}^{-1}(G_{12} - D_{12})\right\}_{12}$$
$$+ \Phi[\varphi, G] \Leftarrow \text{all closed 2PI interaction diagrams}$$
$$D_{12} = \left(-\Box - m^2\right)^{-1}$$

# *Equations of Motion*



- External sources should vanish ⇒ Equations of motion:

$$\frac{\delta \Gamma}{\delta \varphi_1} = -J_1 - \{K_{12}\varphi_2\}_2 \overset{!}{=} 0$$

$$\frac{\delta \Gamma}{\delta G_{12}} = -\frac{\mathrm{i}}{2} K_{12} \overset{!}{=} 0$$

- Equation of motion for the mean field $\varphi$

$$-\Box\varphi - m^2\varphi := j = -\frac{\delta \Phi}{\delta \varphi}$$

- for the "full" propagator $G$ ⇒ Dyson's equation:

$$-\mathrm{i}(D_{12}^{-1} - {G_{12}}^{-1}) := -\mathrm{i}\Sigma = 2\frac{\delta \Phi}{\delta G_{21}}$$

- Integral form of Dyson's equation:

$$G_{12} = D_{12} + \{D_{11'}\Sigma_{1'2'}G_{2'2}\}_{1'2'}$$

- Closed set of equations of for $\varphi$ and $G$

# *Properties of the Φ-derivable Approximations*



- Same technique as for 1PI-functional ⇒ Generalized WTI:

$$\delta\eta_a \left( \left\{ \frac{\delta \Gamma[\vec{\varphi}, G]}{\delta \vec{\varphi}_1} \hat{\tau}^a \vec{\varphi}_1 \right\}_1 + \left\{ \frac{\delta \Gamma[\vec{\varphi}, G]}{\delta G_{12}^{jk}} \left[ (\tau^a)_{jj'} G_{12}^{j'k} + (\tau^a)_{kk'} G_{12}^{jk'} \right] \right\}_{12} \right) = 0$$

☞ $\Gamma$ invariant under $O(N)$ with $\vec{\varphi}$ transforming as a vector, $G$ transforming as a 2nd-rank tensor

- Truncation ($\hbar$-expansion, $\lambda$-expansion, ...) of the Series of diagrams for $\Phi$

☞ Expectation values for Noether currents are exactly conserved

- In equilibrium $\mathrm{i}\Gamma[\varphi, G] = \ln Z(\beta)$ ⇒ thermodynamical potential
- consistent treatment of Dynamical quantities and thermodynamical bulk properties like energy, pressure, entropy
- Problem: Equations of motion ⇒ partial resummation of infinite series of pert. diagrams. No systematic expansion parameter for solutions

☞ Crossing symmetry violated

☞ Although functional is symmetric $\Sigma$ and higher $n$-point functions do not fulfill usual 1PI WTIs!

☞ Especially: In general Goldstone's theorem violated!

# *Repairing symmetries*



- First aim: Repair crossing symmetry ⇒ Look for non-perturbative 1PI-effective action:

$$\tilde{\Gamma}[\vec{\varphi}] = \mathbb{\Gamma}[\vec{\varphi}, \tilde{G}[\vec{\varphi}]] \text{ with } \left.\frac{\delta \mathbb{\Gamma}[\vec{\varphi}, G]}{\delta G}\right|_{G=\tilde{G}[\vec{\varphi}]} \stackrel{!}{=} 0$$

- Solutions of 2PI equations of motion given by

$$\left.\frac{\delta \tilde{\Gamma}[\vec{\varphi}]}{\delta \vec{\varphi}}\right|_{\vec{\varphi}=\vec{\varphi}_0} = 0, \quad G = \tilde{G}[\vec{\varphi}_0]$$

- Define 1PI effective proper vertex functions as usual

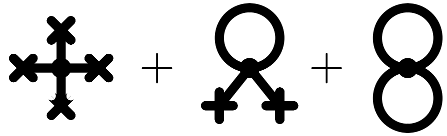
$$\tilde{\Gamma}^{(n)}(x_1 j_1, \ldots, x_n j_n) = \mathrm{i} \left.\frac{\delta^n \tilde{\Gamma}[\vec{\varphi}]}{\delta \varphi_{j_1}(x_1) \cdots \delta \varphi_{j_n}(x_n)}\right|_{\vec{\varphi}=\vec{\varphi}_0}$$

- Can be expressed with self-consistent propagators as internal lines and mean fields
- Crossing symmetric, fulfill 1PI-WTIs
- Remainders of symmetry violations: Internal lines do not fulfill Goldstone's theorem; wrong thresholds
- Wrong phase transition behaviour

# *Example: Hartree approximation*

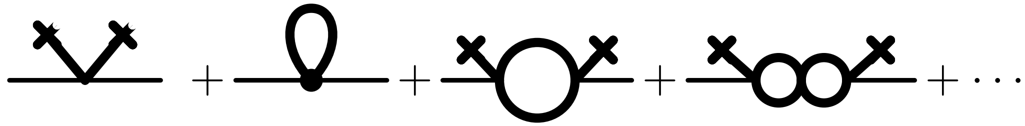


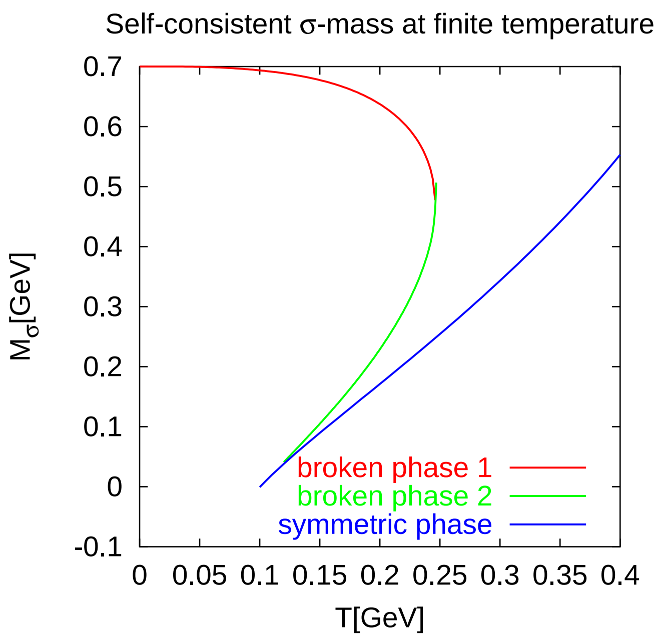
- Hatree approximation:

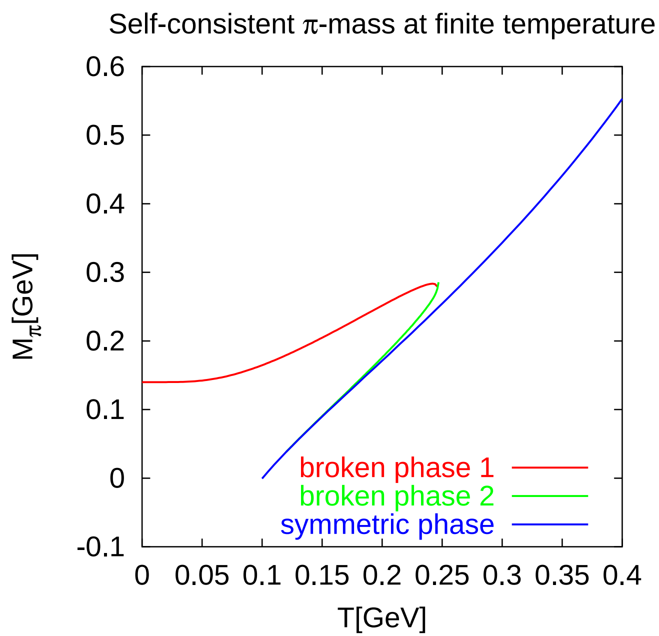
$i\Phi =$ 

- 1PI self–energy defined on top of Hartree approximation

☞ Random phase approximation (RPA):

$-i\tilde{\Sigma} =$ 

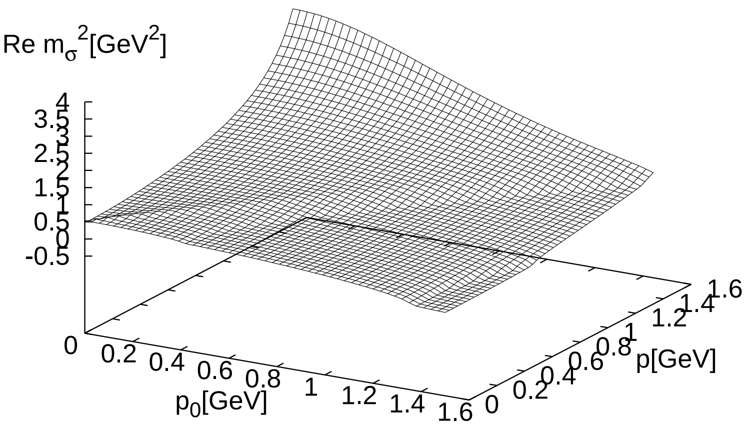


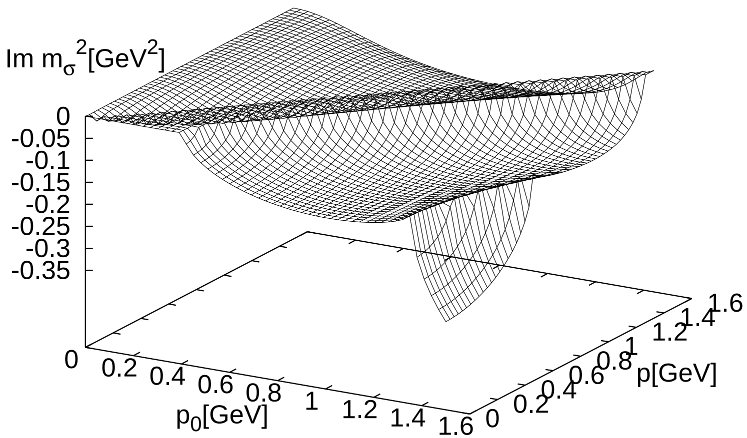
# *RPA-resummation*



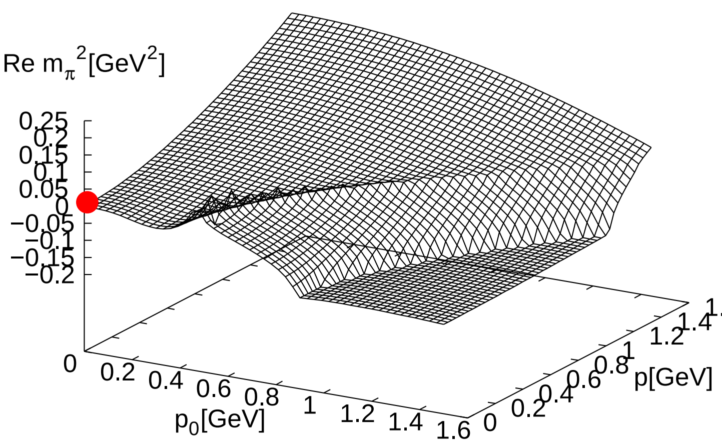
External σ-mass at T=150 MeV (stable solution)



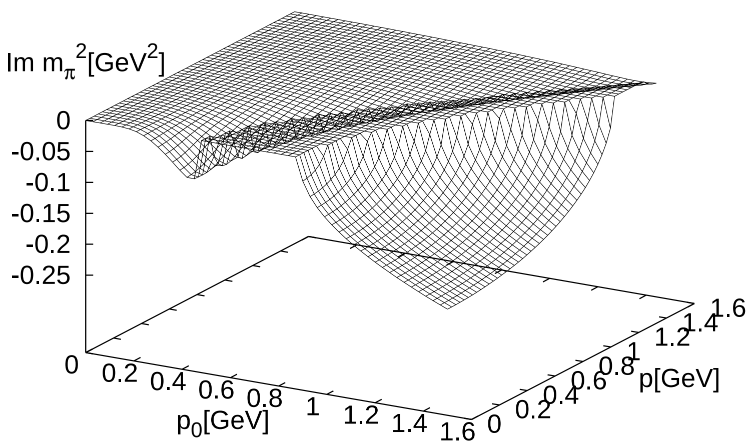

External σ-mass at T=150 MeV (stable solution)




External π−mass at T=150 MeV (stable solution)



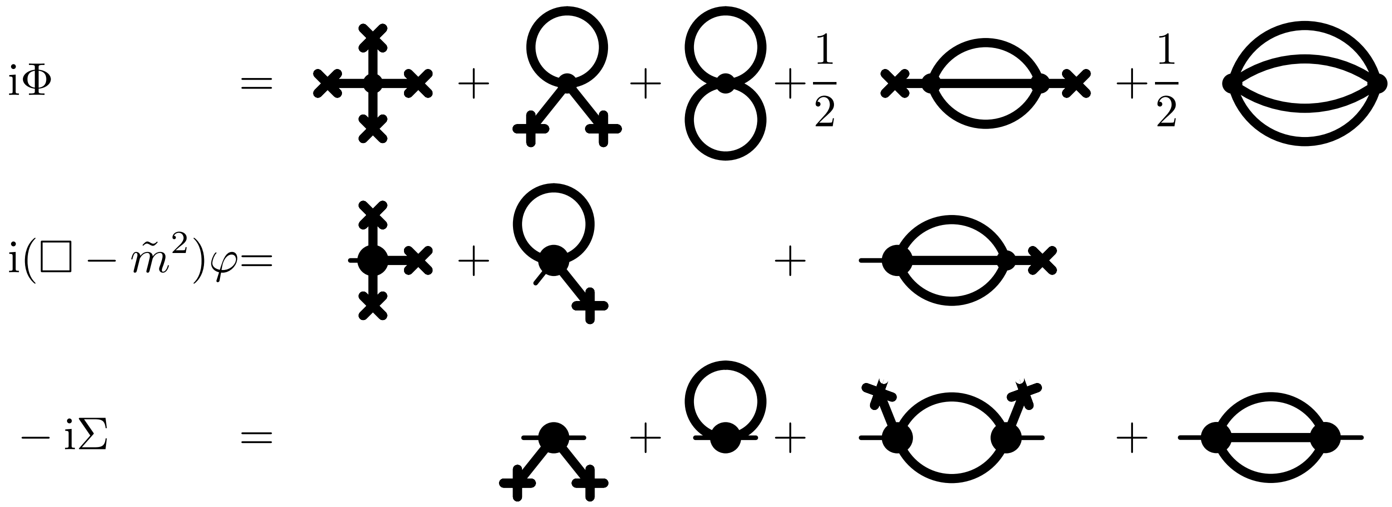

External σ-mass at T=150 MeV (stable solution)



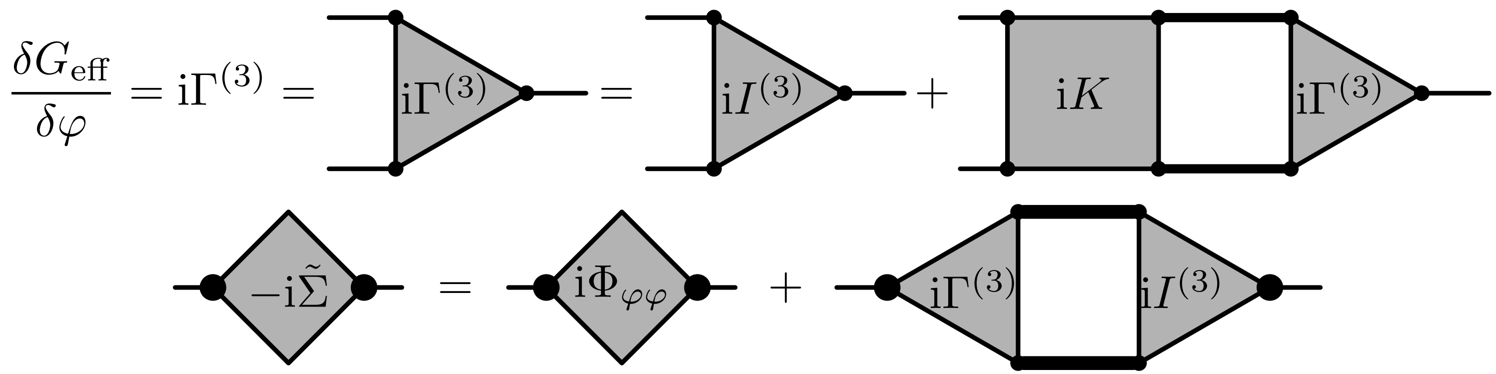
# *General scheme for* $\tilde{\Sigma}$



- 1st step: define $\Phi$ and internal propagator




- $\Phi$ defines kernels for Bethe-Salpeter equation

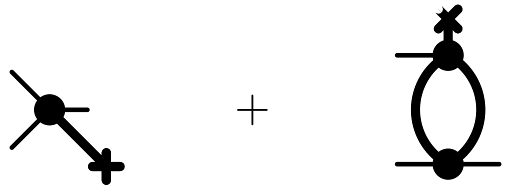


# *Definition of Bethe-Salpeter ingredients*

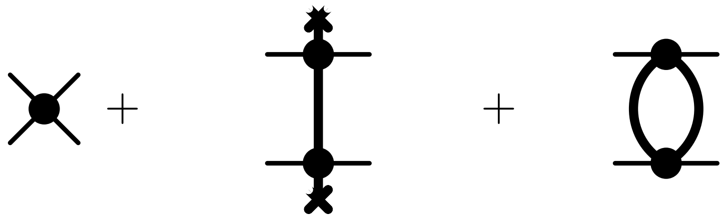


$\mathrm{i}\Phi_{\varphi,\varphi} =$

$\mathrm{i}I^{(3)} = \mathrm{i}\Phi_{\mathrm{i}G,\varphi} =$



$\mathrm{i}K = \mathrm{i}\Phi_{\mathrm{i}G,\mathrm{i}G} =$



- Green's function lines and mean fields fixed from self–consistent Φ–Functional solution

# *Conclusions and outlook*



- Reminder of usual 1PI functional formalism
- Self–consistent Φ–derivable schemes
- Symmetry analysis
- Violations of symmetries by solutions
- Reparation of symmetries for external vertices
- Remainder of symmetry violations: Wrong dynamics in internal lines!
- "Toolbox" for application to realistic models

- Perspectives for self–consistent treatment of gauge theories
- But Symmetry violations for internal lines worse for local gauge symmetries: Internal lines contain unphysical degrees of freedom
- QCD e.g. beyond HTL?
- Transport equations for particles with finite width

??? Wanted: selfconsistent and symmetry conserving scheme beyond mean field approximation for vector fields! Still not in sight :-(