Contents

Selfconsistent Conserved Approximation for π- and ρ-Mesons

Hendrik van Hees, Jörn Knoll

March 1999

▶ The investigated model for the ρ and π mesons

▶ Fit to data

▶ Selfconsistent approximations and renormalization

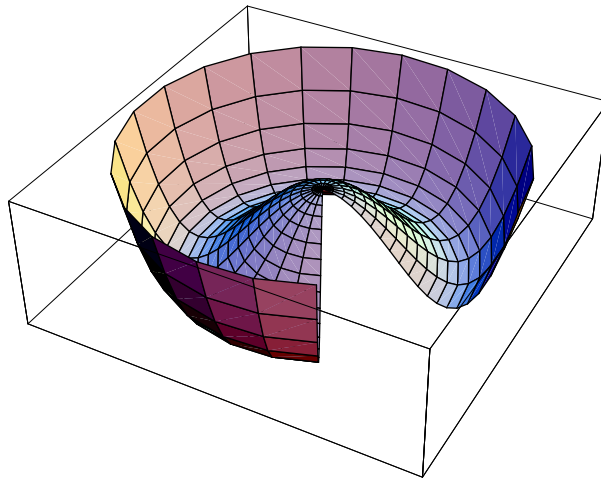▶ Numerical Results in the vacuum and at finite Temperature

▶ Outlook

The Model

► Renormalizable model for massive ρ-mesons $\Rightarrow$ Higgs-Kibble-formalism for Gauge theories

► Start with a $SU(2)$ duplett with gauged symmetry group

$$\mathscr{L}_1 = -\frac{1}{2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}(D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi)$$

► Mexican hat potential $V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2$

► Physical gauge (around the stable vacuum):

   ► ρ-fields become massive $m_\rho^2 = g^2\mu^2/(4\lambda)$

   ► Three Φ-degrees of freedom become ρ degrees of freedom

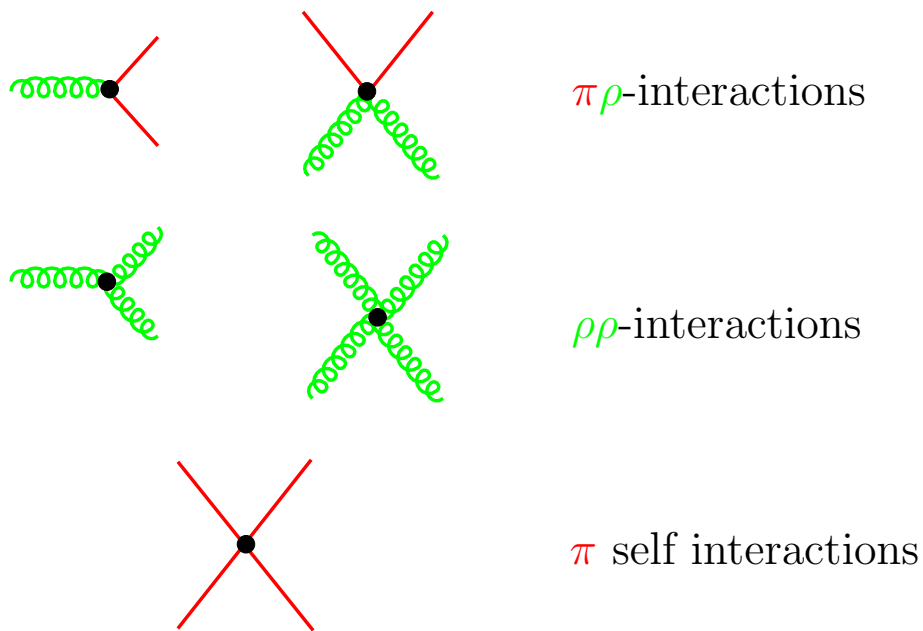   ► One Φ-degree of freedom gives a massive "Higgs"-particle

The Model

▶ Introduce Pions as adjoint representation, i.e., SO(3)-triplett

$$\mathscr{L}_2 = \frac{1}{2}(D_\mu \vec{\pi}) \cdot (D^\mu \vec{\pi}) - \frac{\lambda_2}{8}(\vec{\pi}^2)^2 - \frac{\lambda_3}{4}\vec{\pi}^2 \Phi^\dagger \Phi$$
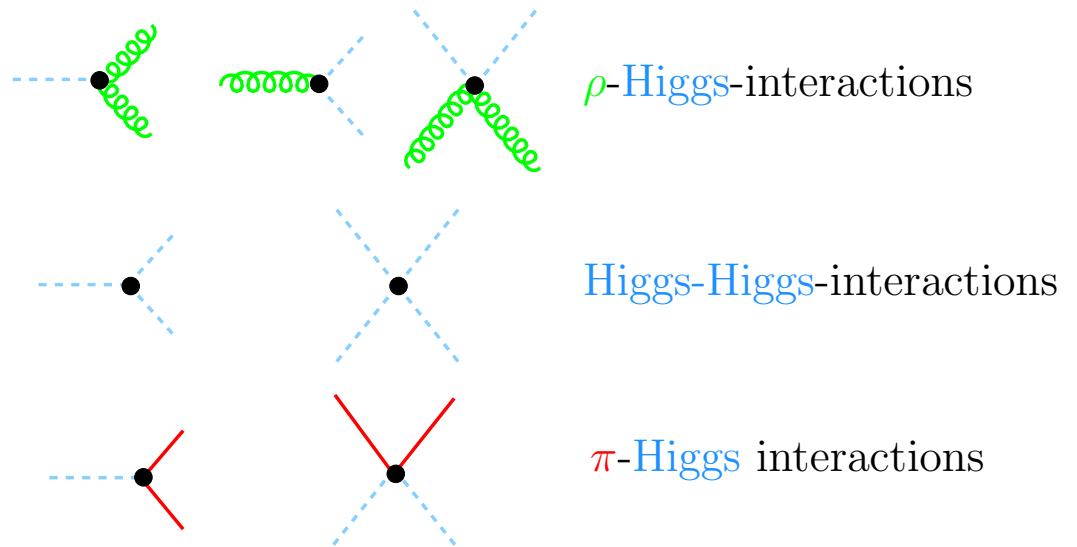
▶ Consistency condition:

$$m_\pi^2 = \frac{2m_\rho^2}{g}\lambda_3$$

Unitary Gauge - Physical Vertices I

$\pi\rho$-interactions

$\rho\rho$-interactions

π self interactions

The Model

ρ-Higgs-interactions

Higgs-Higgs-interactions

π-Higgs interactions

Remarks about Quantization

▶ Unitary gauge contains only physical dof. $\Rightarrow$ manifestly unitary

▶ To get renormalizable gauge $\Rightarrow$ Introducing R_ξ-gauges ('t Hooft)

▶ R_ξ-gauge: manifestly renormalizble

▶ R_ξ-gauge: Faddeev-Popov-ghosts

▶ BRST-invariance $\Rightarrow$ S-Matrix gauge invariant

▶ R_ξ-gauge has unitary gauge as limit $\Rightarrow$ Renormalized theory also unitary

The Model

▶ Extending the gauge group to $U(1) \times SU(2)$

▶ $U(1)$ unbroken $\Rightarrow$ One of the four gauge bosons remains massless $\Rightarrow$ photon

▶ Equations of Motion $\Rightarrow$ Pions couple to photons only through $\rho \Rightarrow$ Vector-Meson-Dominance

The Form Factor

▶ Electromagnetic Form Factor of the Pion:

$$F(k^2) = \frac{\text{[diagram: } \pi^-, \pi^+ \to \rho \to \gamma \to e, e\text{]}}{\text{[diagram: } \pi^-, \pi^+ \to \gamma \to e, e\text{]}}$$
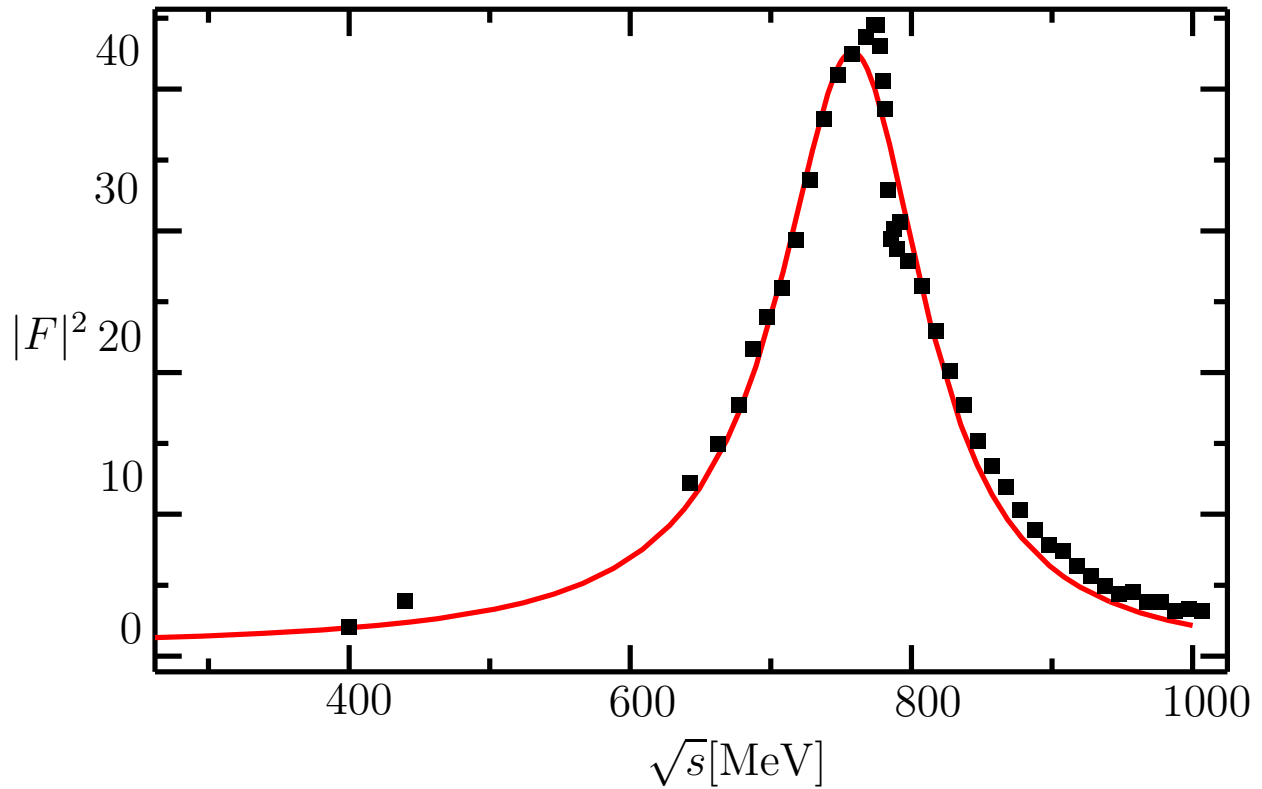

▶ Feynman rules: $\Gamma_{\rho\gamma} = i\delta^{a3} M_\rho^2 e/g \Rightarrow$

$$|F(s)|^2 = \frac{m_\rho^4}{[s - m_\rho^2 - \operatorname{Re}\Pi_\rho(s)]^2 + [\operatorname{Im}\Pi_\rho(s)]^2}$$

Fit of the parameters

▶ Using dimensional regularization and renormalization of the
one-loop-self-energy diagrams

$$-\mathrm{i}\Sigma_\rho =$$
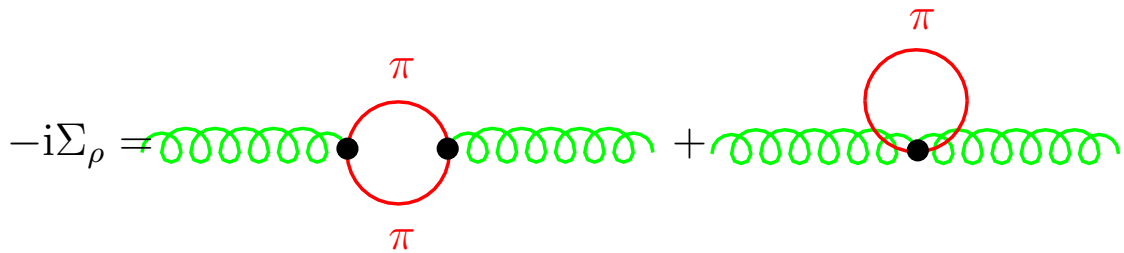
Data: Amendolia et al. Phys. Lett. **138B** (1984) 454
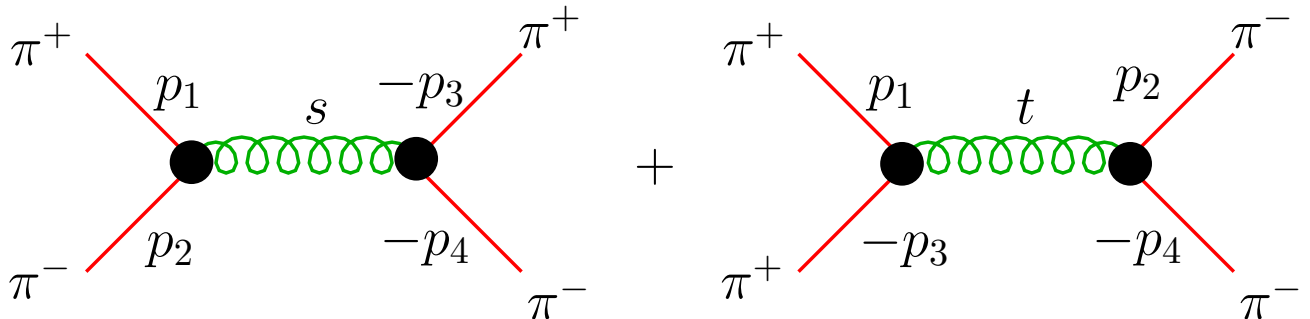Barkov et al. Nucl. Phys. **B256** (1985) 365

Fit of the parameters

Total $\pi^+ \pi^-$ elastic cross-section

▶ Four π-vertex

$$\Gamma^{abcd}(p_1, \ldots, p_4) = \begin{cases} & A(s,t,u)\delta_{ab}\delta_{cd} + \\ + & A(t,s,u)\delta_{ac}\delta_{bd} + \\ + & A(u,t,s)\delta_{ad}\delta_{bc} \end{cases}$$

▶ With the invariants $s = (p_1 + p_s)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^4$

▶ Feynman rules $\Rightarrow$ invariant transition amplitude:
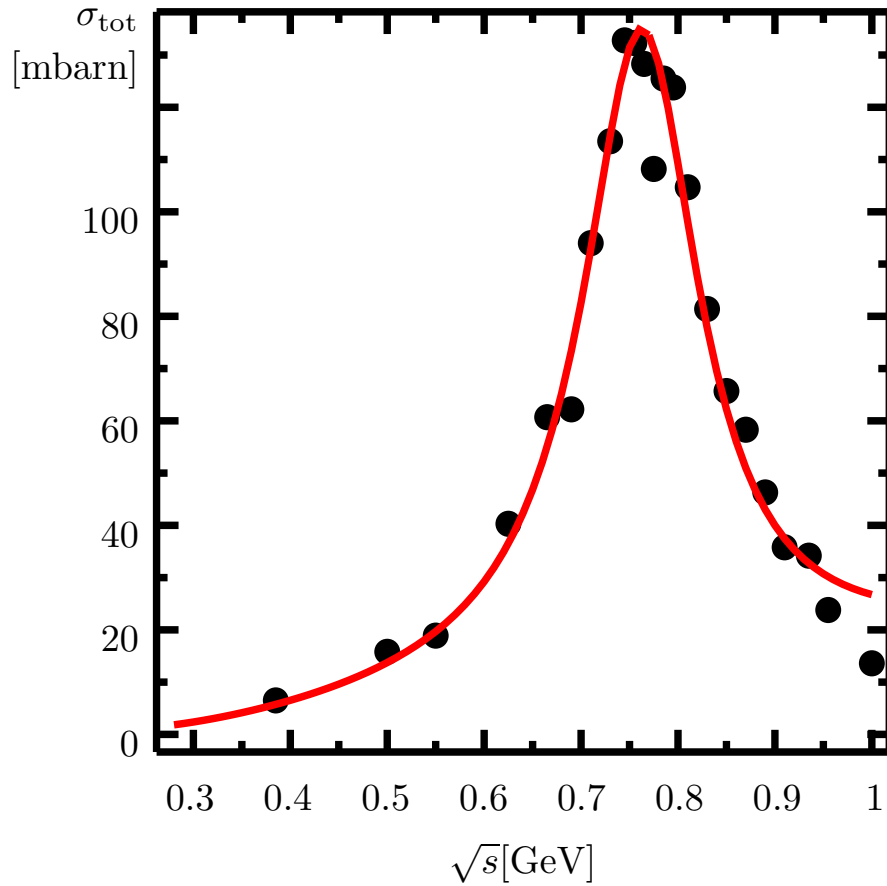
$$M_{fi}(s,t) = A(s,t,u) + A(t,s,u)|_{u=4m_\pi^2 - s - t}$$
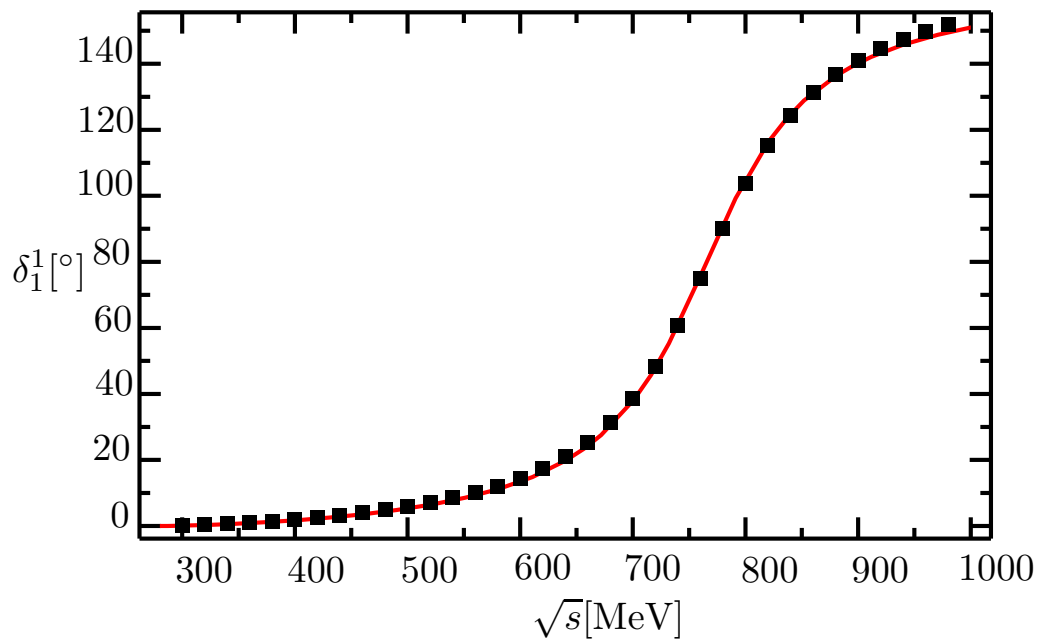
▶ Total cross section:

$$\sigma_{\text{tot}} = \frac{1}{64\pi} \frac{1}{s(s - 4m_\pi)} \int_{4m_\pi^2 - s}^{0} |M_{fi}(s,t)|^2$$

Fit of the parameters

▶ With the parameters from the fitting to phase-shift and form-factor:

▶ The δ_1^1-phase-shift

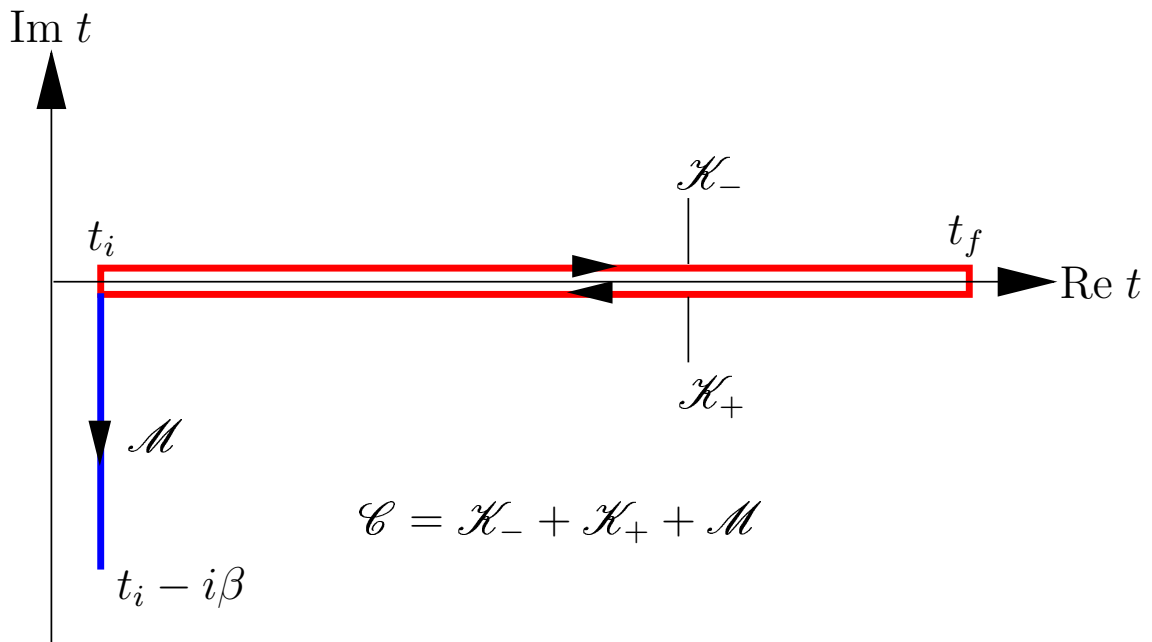▶ Data from: Forgatt, Petersen, Nucl. Phys. **B129** (1977) 89

Selfconsistent approximations

▶ Introduce a bilocal source term in addition to the local source term into the Z-functional:

$$Z[J,K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\langle J_1\phi_1\rangle_1 + \frac{\mathrm{i}}{2}\langle K_{12}\phi_1\phi_2\rangle_{12}\right], \; W = -\mathrm{i}\ln Z$$

▶ S is the classical action of the field theory along the time axis (vacuum) or the Schwinger-Keldysh contour (thermal FT)

$$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+ + \mathscr{M}$$

▶ Functional Legendre transformation wrt. both J and K:

$$\Gamma[\varphi, G] = W[J,K] - \langle\varphi_1 J_1\rangle_1 - \frac{1}{2}\langle(\varphi_1\varphi_2 + \mathrm{i}G_{12})K_{12}\rangle_{12}$$

$$\text{with } \varphi_1 = \frac{\delta W[J,K]}{\delta J_1} \text{ and}$$

$$G_{12} = -\frac{\delta^2 W[J,K]}{\delta J_1 \delta J_2} = -2\mathrm{i}\left(\frac{\delta W[J,K]}{\delta K_{12}} - \frac{1}{2}\varphi_1\varphi_2\right)$$

▶ Define $\Phi = \Gamma_2$ to be then 2PI vacuum diagrams with at least 1 loop:

$$\Gamma[\varphi, G] = S[\varphi] + \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(DG^{-1}) + \frac{\mathrm{i}}{2}\langle\mathscr{D}_{12}^{-1}(G_{12} - \mathscr{D}_{12})\rangle_{12} + \Phi[\varphi, G]$$

Selfconsistent approximations

▶ Equations of motion: $J = K = 0$

$$\frac{\delta\Gamma[\varphi, G]}{\delta\varphi} = 0 \Leftrightarrow (-\Box - m^2)\varphi + \frac{\delta S_I}{\delta\varphi} + \frac{\mathrm{i}}{2}\left\langle \frac{\delta\mathscr{D}_{12}^{-1}}{\delta\varphi} G_{12}\right\rangle_{12} + \frac{\delta\Phi[\varphi, G]}{\delta\varphi} = 0$$

$$\frac{\delta\Gamma[\varphi, G]}{\delta G} = 0 \Leftrightarrow -\mathrm{i}\Sigma_{12} := -\mathrm{i}(\mathscr{D}_{12}^{-1} - G_{12}^{-1}) = 2\frac{\delta(\mathrm{i}\Phi[\varphi, G])}{\delta G}$$

▶ Simple example: ϕ^4-theory:

$$\mathrm{i}\Phi = \qquad \frac{1}{8} \qquad + \qquad \frac{1}{2\cdot 3!} \qquad + \qquad \frac{1}{2\cdot 4!} \qquad + \cdots$$

with $\underset{x}{\bigotimes} = \varphi(x)$ $\qquad \underset{x \qquad y}{\rule{2cm}{0.4pt}} = \mathrm{i}G(x, y)$

$$-\mathrm{i}j(x) = \quad \underset{\frac{1}{3!}}{\underset{x}{}} \quad + \quad \underset{\frac{1}{2!}}{} \quad + \quad \underset{\frac{1}{3!}}{} \quad + \cdots$$

$$-(\Box + m^2)\varphi = j$$
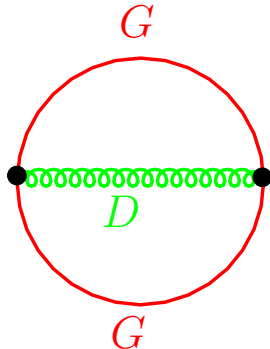
$$-i\Sigma_{12} = \quad \underset{\substack{x_1 \\ x_2 \\ \frac{1}{2!}}}{} \quad + \quad \underset{\substack{x_1 \qquad x_2 \\ \frac{1}{3!}}}{} \quad + \quad \underset{\substack{x_1 \qquad x_2 \\ \frac{1}{2!}}}{} \quad + \cdots$$

Selfconsistent approximations

Generating functional

▶ $\Phi[G, D]$: sum over all 2PI closed diagrams with at least two loops

$$\mathrm{i}\Phi[G, D] = \quad + \cdots$$

▶ Variation with respect to Green's functions $\Rightarrow$ self energies fulfilling Dyson's equations

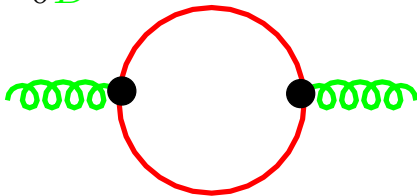$$\frac{\delta \mathrm{i}\Phi}{\delta D} = -\mathrm{i}\Pi_\rho =$$

$$\frac{\delta \mathrm{i}\Phi}{\delta G} = -\mathrm{i}\Sigma_\pi =$$

$$D = D_0 + D_0 \Pi_\rho D$$
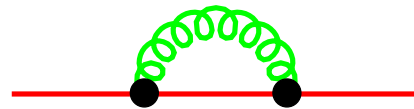
$$G = G_0 + G_0 \Sigma_\pi G$$

▶ Sum up to a certain loop order $\Rightarrow$ Selfconsistent effective approximation

▶ Respects all conservation laws basing on global symmetries

▶ In thermal field theory: Thermodynamically consistent approximation

Renormalization

Renormalizing the selfconsistent approximation

▶ Can be seen as resummation of all self energy insertions ⇒ Infinities to all orders

▶ Renormalizable theory ⇒ finite by renormalizing parameters already present in Lagrangian

▶ Physical renormalization conditions

$$\Sigma_\pi(m_\pi^2) = \partial_s\Sigma_\pi(m_\pi^2) = 0, \ \Pi_\rho(0) = \partial_s\Pi_\rho(0) = 0$$

▶ Analytical properties of Green's functions

$$G(s) = \frac{1}{\pi}\int_0^\infty dm^2\Delta(m^2,s)A(m^2) \text{ with } A(s) = -\operatorname{Im}G(s)$$

▶ $\Delta(m^2,s)$: Feynman-propagator ⇒ integral kernels ⇒ can be renormalized using standard techniques

▶ self consistent finite set of coupled integral equations solvable numerically by iteration

▶ Tadpole in vacuum absorbed into mass renormalization

Results in vacuum

The π-Self-Energy

$$-i\Sigma_\pi =$$

Result in vacuum

The ρ-Self-Energy

$-i\Pi_\rho =$

Imag. Part

$s[\text{GeV}^2]$

$\Pi_\rho[\text{GeV}^2]$

Real Part

$s[\text{GeV}^2]$

Re $\Pi_\rho[\text{GeV}^2]$
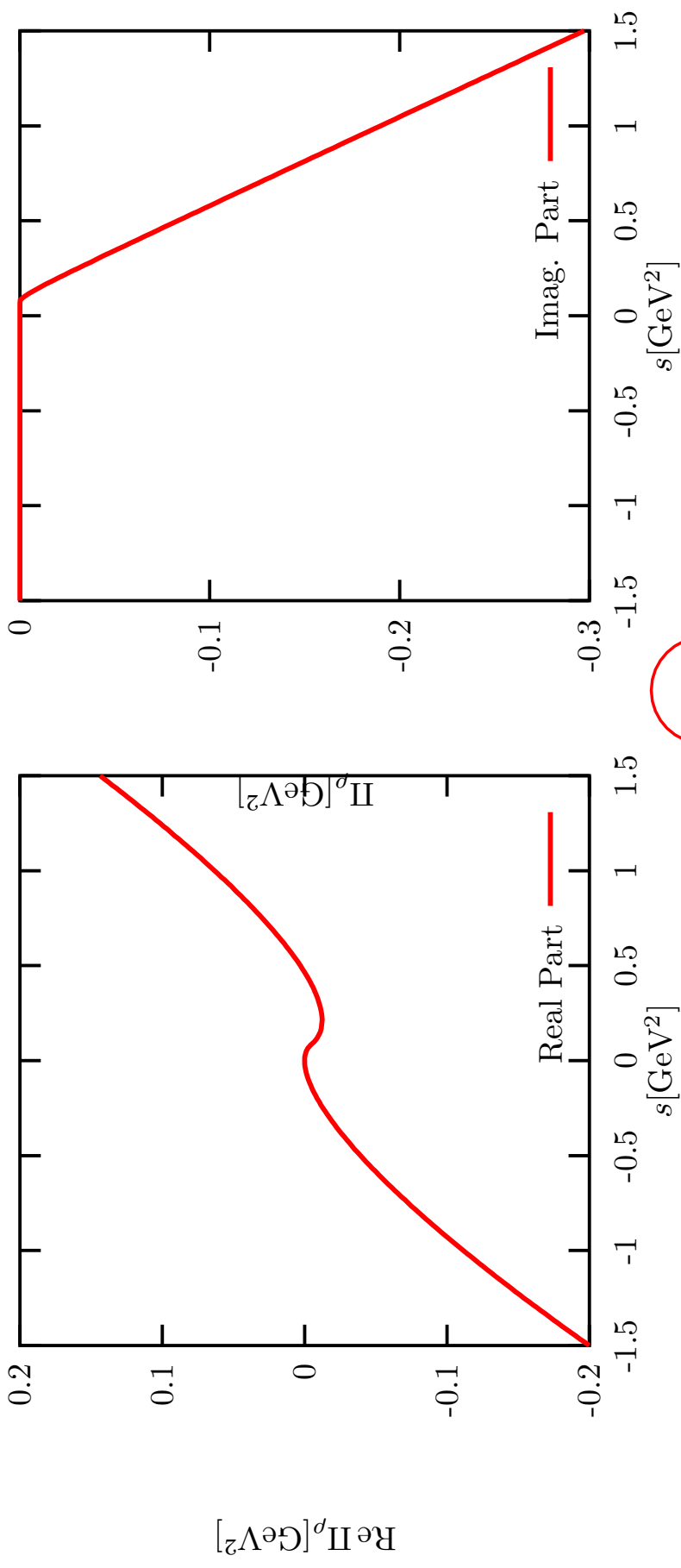
Finite Temperature

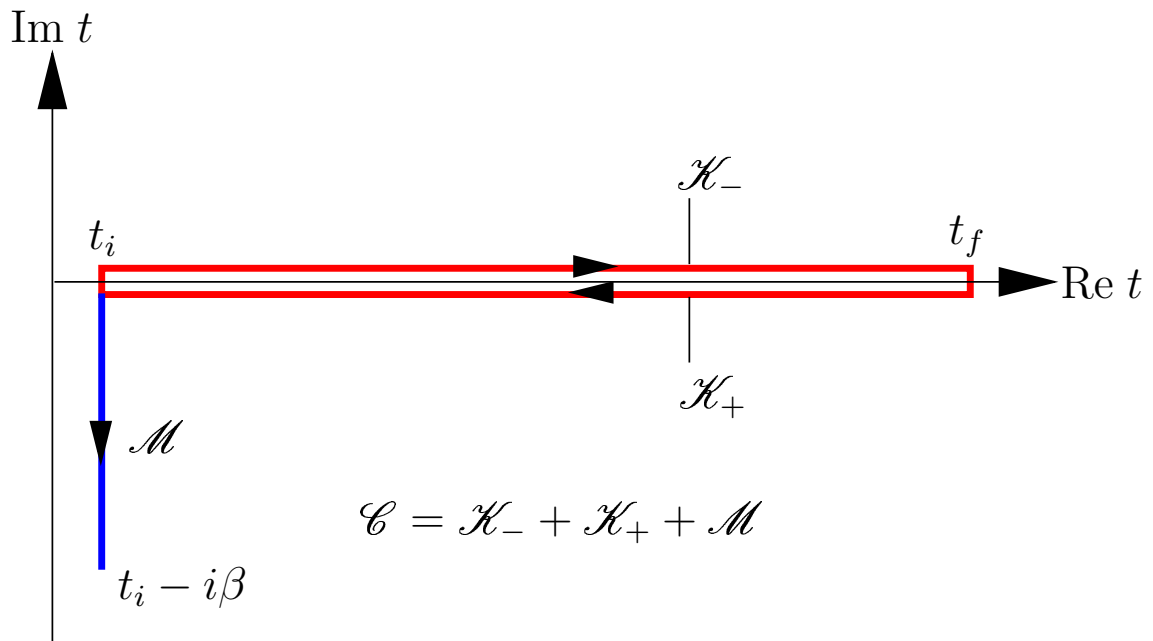<div style="border:1px solid;background:#5b9bd5;text-align:center;font-weight:bold">Quantum Field Theory at finite Temperature</div>

▶ Using the modified Schwinger Keldysh contour for equilibrium

Im t

$\mathscr{K}_-$

t_i　　　　　　　　　　　　t_f

Re t

$\mathscr{K}_+$

$\mathscr{M}$

$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+ + \mathscr{M}$

$t_i - i\beta$

▶ Timeordering in vacuum $\rightarrow$ contour ordering

▶ Path integral formalism: Generating functional Z factorizes in real time and imaginary time part.

▶ Calculate $\langle O \rangle = \mathrm{Tr}[\exp(-\beta \underset{\sim}{H}) \underset{\sim}{Q}]$

▶ Wick's theorem $\Rightarrow$ Path ordered Green's functions $\rightarrow$ Matrix Formalism

▶ Trace $\Rightarrow$ (Anti-)Periodicity of fields $\rightarrow$ KMS condition

Finite Temperature

Analytic Properties of Green's Functions

▶ The 2-point Green's functions can be expressed in terms of the spectral function:

$$\mathrm{i}G^{--}(p) = \int_0^\infty \frac{\mathrm{d}k_0}{\pi} \frac{2\mathrm{i}k_0 A(p_0, \vec{p})}{p_0^2 - k_0^2 + \mathrm{i}\epsilon} + 2n(p_0)A(p^2, \vec{p}),$$

$$\mathrm{i}G^{++}(p) = -\int_0^\infty \frac{\mathrm{d}k_0}{\pi} \frac{2\mathrm{i}k_0 A(p_0, \vec{p})}{p_0^2 - k_0^2 - \mathrm{i}\epsilon} + 2n(p_0)A(p^2, \vec{p}),$$

$$\mathrm{i}G^{+-}(p) = 2[\Theta(p_0) + n(p_0)]A(p_0, \vec{p}) = 2[1 + f(l_0)]\tilde{A}(p),$$

$$\mathrm{i}G^{-+}(p) = 2[\Theta(-p_0) + n(p_0)]A(p_0, \vec{p}) = 2f(l_0)\tilde{A}(p).$$

with $\tilde{A}(p) = -\operatorname{Im} G_R(p) = \operatorname{sign} p_0 A(p)$

and $f(x) = \dfrac{1}{\exp(\beta x) - 1}$ $n(x) = f(|x|)$

▶ Feynman rules for imaginary part:

$$\operatorname{Im} G_R = \frac{G^{+-} - G^{-+}}{2\mathrm{i}} \tag{1}$$

▶ Self energies and Dyson equation

$$G_R(p) = \frac{1}{p^2 - m^2 - \Sigma_R}, \quad \operatorname{Im}\Sigma_R = \frac{\Sigma^{-+} - \Sigma^{+-}}{2\mathrm{i}}$$

▶ Causality $\Rightarrow$ Kramers-Kronig-Relations (Dispersion relations):

$$f(z) = \int_{-\infty}^\infty \frac{\mathrm{d}z'}{\pi} \frac{\operatorname{Im} f(z')}{(z' - z)(z' - z_r)^n} + \sum_{k=0}^{n=1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

▶ Crucial: Subtractions ONLY in vacuum parts of the self energies!

Π and ρ at finite Tempererature

The selfconsistent equations

► Breaking of Lorentz invariance due to temperature:

$$\Pi_{\mu\nu} = -\Pi_T \Theta^T_{\mu\nu} - \Pi_L \Theta^L_{\mu\nu} \text{ with:}$$

$$\Theta_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2},$$

$$\Theta^T_{\mu\nu} = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0 \\ -\delta_{ij} + \frac{p_i p_j}{\vec{p}^2} & \text{for } \mu, \nu \in \{1, 2, 3\} \end{cases},$$

$$\Theta^L_{\mu\nu} = \Theta_{\mu\nu} - \Theta^T_{\mu\nu}$$

► Dyson equation for transverse gauge (Landau gauge):

$$G^\rho_{\mu\nu} = -\frac{\Theta^L_{\mu\nu}}{p^2 - m^2 - \Pi_L} - \frac{\Theta^T_{\mu\nu}}{p^2 - m^2 - \Pi_T}$$

► Calculate iteratively: Imaginary part of self energies (finite!):

$$\operatorname{Im} \Pi^L_{\mu\nu}(p) = -\frac{g^2}{2\pi^4} \int d^4l \frac{[p_0(\vec{l}\vec{p}) - l_0\vec{p}^2]^2}{\vec{p}^2 p^2} [f(l_0) - f(l_0 + p_0)] A_\pi(l + p) A_\pi(l)$$

$$\operatorname{Im} \Pi^T_{\mu\nu}(p) = -\frac{g^2}{4\pi^4} \int d^4l \frac{\vec{l}^2\vec{p}^2 - (\vec{p}\vec{l})^2}{\vec{p}^2} [f(l_0) - f(l_0 + p_0)] A_\pi(l + p) A_\pi(l)$$

$$\operatorname{Im} \Sigma(p)_\pi = -\frac{g^2}{\pi^4} \int d^4l [f(l_0) - f(l_0 + p_0)](2p_\mu + l_\mu)(2p_\nu + l_\nu) \times$$
$$\times \left[\Theta^{\mu\nu}_L(l) A_{\rho L}(l) + \Theta^{\mu\nu}_T(l) A_{\rho T}(l)\right] A_\pi(l + p)$$

► Calculate real parts for the temperature part with a dispersion relation without subtractions
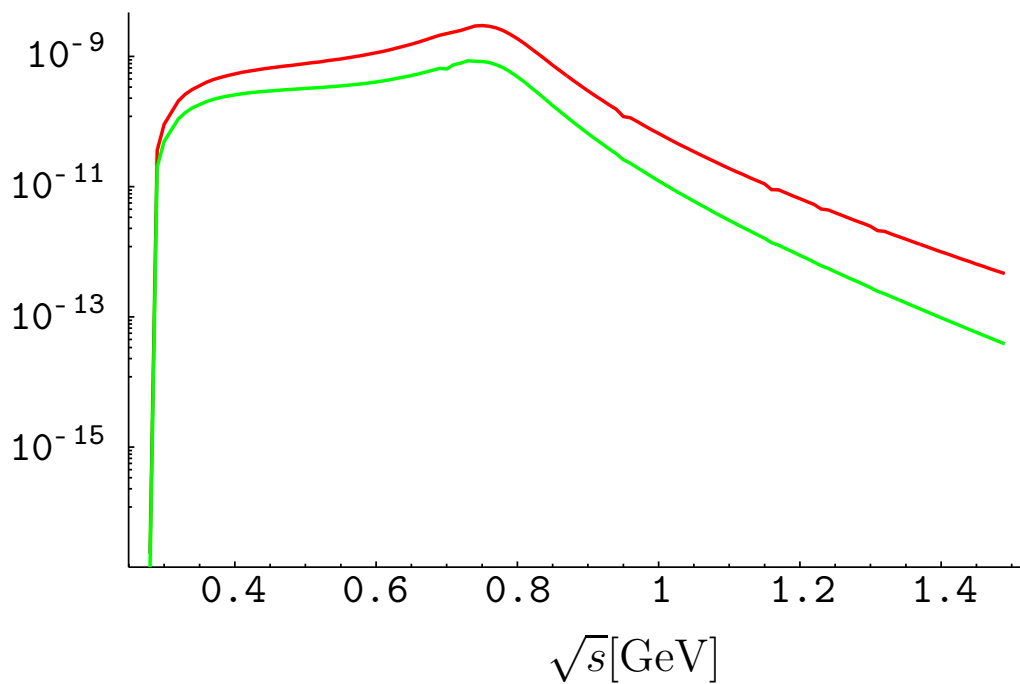
Dilepton Rate

▶ Kadanoff-Baym-Equations: Exact result for strong coupling:

$$\frac{\mathrm{d}^4 R}{\mathrm{d}\sqrt{s}\mathrm{d}P^3}\bigg|_{\vec{P}=0} = \frac{2\alpha^2}{(2\pi)^3}\frac{m_\rho^2}{g^2}\frac{1}{s}A_\rho(\sqrt{s},0)f_B(\sqrt{s})$$

▶ Dilepton Production Rate

$$\frac{\mathrm{d}^4 R}{\mathrm{d}\sqrt{s}\mathrm{d}^3\vec{P}}[\mathrm{GeV}^{-3}]$$

▶ $T = 150\mathrm{MeV}$, $200\mathrm{MeV}$

Conclusions and Outlook

Conclusions

▶ Selfconsistent treatment of all particles with width is possible

▶ Medium modification of the widths and masses

▶ At $T > 0$: contributions below light cone $\rightarrow$ "Screening"

Work to do

▶ To avoid trouble with unphysical states in vector-particle propagators:
Apply $\xi \gg 1$ ("Unitary Gauge")

▶ Other way: Give up the gauge model and generate the ρ from an
effective pion model which does not contain unphysical states at all

▶ Exploit non-abelian part of the ρ-interaction

▶ Include other particles $(A_1, \omega, N, \Delta, \dots)$

▶ Question of theoretical interest: Is it possible to extend the
selfconsistent approximation scheme to a gauge invariant one?