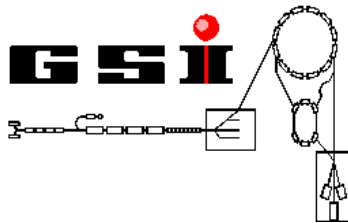# Self-consistent Conserving Approximations and Renormalization in Quantum Field Theory at Finite Temperature

Hendrik van Hees

in collaboration with Jörn Knoll

Contents

- Schwinger-Keldysh real-time formalism
- The Φ-derivable scheme
- Tadpole resummation for ϕ^4-theory
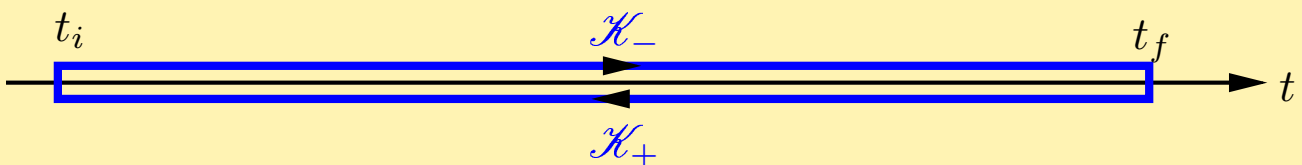- The self-consistent sunset diagram
- Thermodynamical quantities
- Outlook

Schwinger-Keldysh Formalism

- Initial statistical operator $\boldsymbol{\rho}_i$ at $t = t_i$
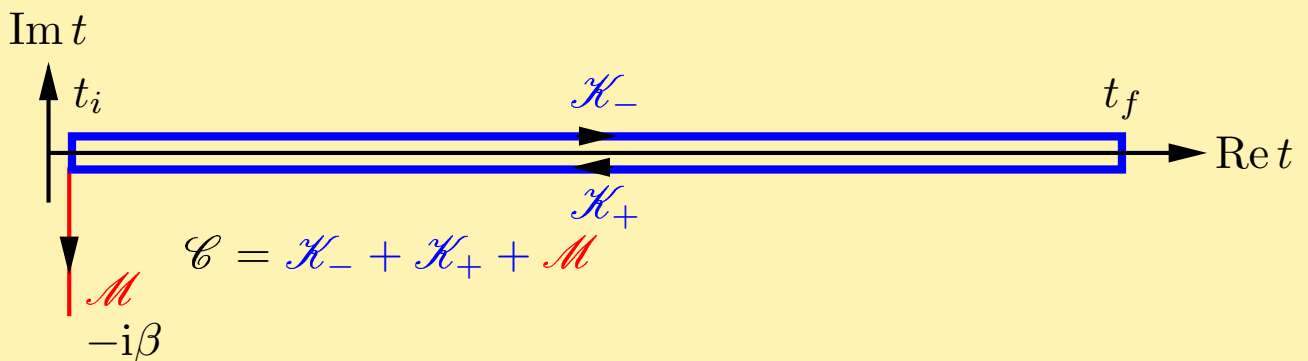- Time evolution of expectation values of observables:

$$\langle O \rangle = \mathrm{Tr}[\boldsymbol{\rho}(t)\mathbf{O}(t)]$$

- Feynman rules
- Difference to vacuum: Contour-ordered Green's functions

$$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+$$

- In equilibrium: $\boldsymbol{\rho} = \exp(-\beta\mathbf{H})/Z$ with $Z = \mathrm{Tr}\exp(-\beta\mathbf{H})$
- Imaginary part of the time contour

$$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+ + \mathscr{M}$$

- Fields periodic (bosons) or anti-periodic (fermions)

The Φ-Functional

- Introduce local and bilocal auxiliary sources
- Generating functional

$$Z[J, K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{ J_1\phi_1 \right\}_1 + \left\{ \frac{\mathrm{i}}{2} K_{12}\phi_1\phi_2 \right\}_{12} \right]$$

- Generating functional for connected diagrams

$$Z[J, K] = \exp(\mathrm{i}W[J, K])$$

- The mean field and the connected Green's function

$$\varphi_1 = \frac{\delta W}{\delta J_1}, \ G_{12} = -\frac{\delta^2 W}{\delta J_1 \delta J_2} \Rightarrow \frac{\delta W}{\delta K_{12}} = \frac{1}{2}[\varphi_1\varphi_2 + \mathrm{i}G_{12}]$$

- Legendre transformation for φ and G:

$$\Gamma[\varphi, G] = W[J, K] - \left\{ \varphi_1 J_1 \right\}_1 - \frac{1}{2}\left\{ (\varphi_1\varphi_2 + \mathrm{i}G_{12})K_{12} \right\}_{12}$$

- Exact saddle point expansion:

$$\Gamma[\varphi, G] = S[\varphi] + \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(-\mathrm{i}G^{-1}) + \frac{\mathrm{i}}{2}\left\{ \mathscr{D}_{12}^{-1}(G_{12} - \mathscr{D}_{12}) \right\}_{12}$$

$$+ \Phi[\varphi, G] \Leftarrow \text{all 2PI diagrams with at least 2 loops}$$

$$\mathscr{D}_{12} = \left(-\Box - m^2 - \frac{\lambda}{2}\varphi^2 \right)^{-1} \delta(x_1 - x_2)$$

Equations of Motion

- Physical solution defined by vanishing auxiliary sources:

$$\frac{\delta\Gamma}{\delta\varphi_1} = -J_1 - \{K_{12}\varphi_2\}_2 \overset{!}{=} 0$$

$$\frac{\delta\Gamma}{\delta G_{12}} = -\frac{i}{2}K_{12} \overset{!}{=} 0$$

- Equation of motion for the mean field φ

$$-\Box\varphi - m^2\varphi - \frac{\lambda}{3!}\varphi^3 - \frac{i}{2}\varphi\{G(x,x)\}_x + \frac{\delta\Phi}{\delta\varphi} = 0$$

- for the "full" propagator $G \Rightarrow$ Dyson's equation:

$$2\frac{\delta\Phi}{\delta G_{12}} = -i(\mathscr{D}_{12}^{-1} - G_{12}{}^{-1}) := -i\Sigma$$

- Closed set of equations of for φ and G

"Diagrammar"

- ϕ^4-theory

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- 2PI Generating Functional

$$\mathrm{i}\Phi = \quad \tfrac{1}{4!} \quad + \quad \tfrac{1}{4} \quad + \quad \tfrac{1}{8} \quad + \quad \tfrac{1}{2\cdot 3!} \quad + \quad \tfrac{1}{2\cdot 4!} \quad + \cdots$$

- Mean field equation of motion

$$\mathrm{i}(\Box + m^2)\varphi = \quad \tfrac{1}{3!} \quad + \quad \tfrac{1}{2!} \quad + \quad \tfrac{1}{3!} \quad + \cdots$$

- Self-energy

$$-\mathrm{i}\Sigma_{12} = \quad \tfrac{1}{2!} \quad + \quad \tfrac{1}{2!} \quad + \quad \tfrac{1}{3!} \quad + \cdots$$

Properties of the Φ-derivable Approximations

Why using the Φ-functional?

- Series of diagrams for Φ truncated at a certain loop order
☞ Linearly realized Noether symmetries are respected
☞ Conserving Approximations (mod. anomalies)

- In equilibrium $\mathrm{i}\Gamma[\varphi, G] = \ln Z(\beta)$ (thermodynamical potential)
- consistent treatment of Dynamical quantities (real time formalism) and thermodynamical bulk properties (imaginary time formalism) like energy, pressure, entropy

- Real- and Imaginary-Time quantities "glued" together by Analytic properties from (anti-)periodicity conditions of the fields (KMS-condition)

- Self-consistent set of equations for self-energies and mean fields

Problem of Renormalization

Why renormalization?

☞ Diagrams UV-divergent

☞ Control the physical parameters in the vacuum

☞ Temperature dependence from theory alone

How to renormalize self-consistent diagrams?

☞ In terms of perturbation theory: Resummation of all self-energy insertions in propagators

☞ Self-consistent diagrams with explicit nested and overlapping sub-divergences

☞ "Hidden" sub-divergences from self-consistency

How to manage it numerically?

☞ Power counting (Weinberg) valid for self-consistent diagrams

☞ At finite temperatures:
Self-consistent scheme rendered finite with local counterterms independent of temperature

☞ Analytical properties ⇒ subtracted dispersion relations

☞ BPHZ-renomralization ⇒ Subtracting the integrands

☞ Advantage: Clear scheme how to subtract temperature independent sub-divergences

☞ Φ-functional ⇒ consistency of counterterms

Self-Consistent Tadpole I

The tadpole approximation

$$\Phi = \bigcirc\!\!\!\bullet\!\!\!\bigcirc \qquad -\mathrm{i}\Sigma = \underset{\bullet}{\bigcirc}$$

- Here: Only time-ordered propagator needed
- The renormalized tadpole $d = 2\omega = 2(2 - \epsilon)$:

$$-\mathrm{i}\Sigma = -\frac{\mathrm{i}\lambda}{2}\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\mu^{2\epsilon}\mathrm{i}G(p) + \mathrm{CT}$$

- Self-energy constant in p

☞ temperature dependent effective mass

- Dyson's equation can be resummed:

$$\mathrm{i}G(p) = \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\eta} + 2\pi n(p_0)\delta(p^2 - M^2)$$

$$\text{with } M^2 = m^2 + \Sigma, \ n(p_0) = \frac{1}{\exp(\beta|p_0|) - 1}$$

- Use standard formulae for dimensional regularized Feynman integrals:

$$\Sigma_{\mathrm{inf}} = -\frac{\lambda}{32\pi^2}M^2\left[\frac{1}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{M^2}\right)\right] \tag{1}$$

- Does one need temperature dependent counter terms $(\propto \lambda M^2/\epsilon)$?

Self-Consistent Tadpole II

Counterterms

- Solution: We do not only need an overall but also a vertex counter-term for sub-divergence:

$$-\mathrm{i}\Sigma = \underbrace{\includegraphics{}}_{-\mathrm{i}\Sigma_{\mathrm{reg}}} + \underbrace{\includegraphics{}}_{\mathrm{i}\frac{\delta\lambda}{\lambda}\Sigma_{\mathrm{reg}}} + \underbrace{\includegraphics{}}_{-\mathrm{i}\delta m^2}$$

How to determine the vertex counterterm?

- Use BPHZ-Renormalization: Subtract the integrands

$$-\mathrm{i}\Sigma_{\mathrm{ren}}$$

$$-\mathrm{i}R_{\Sigma}(l) = \underbrace{\includegraphics{}}_{\frac{\lambda}{2}G(l)} - \underbrace{\includegraphics{}}_{\frac{\lambda}{2}G_v^2(l)\Sigma} - \underbrace{\includegraphics{}}_{\frac{\lambda}{2}G_v(l)}$$

- Renormalize the "Dinosaur diagram"

☞ vertex counterterm

$$-\mathrm{i}t_{\Gamma}I_{\Gamma^{(4)}} = \includegraphics{} = \frac{\lambda^2}{2}G_v^2(l)$$

The Finite Equation

The counterterms

- Renormalization conditions (physical scheme)

☞ $\Gamma^{(4)}_{\text{vac}}(0) = 0$, $\Sigma_{\text{vac}}(m^2) = 0$

- Counterterms:

$$\delta m^2 = \frac{\lambda}{32\pi^2} m^2 \left[\frac{1}{\epsilon} - \gamma + 1 + \ln \frac{4\pi\Lambda^2}{m^2} \right]$$

$$\delta\lambda = -\frac{\lambda^2}{32\pi^2} \left[\frac{1}{\epsilon} - \gamma + 1 + \ln \frac{4\pi\Lambda^2}{m^2} \right]$$
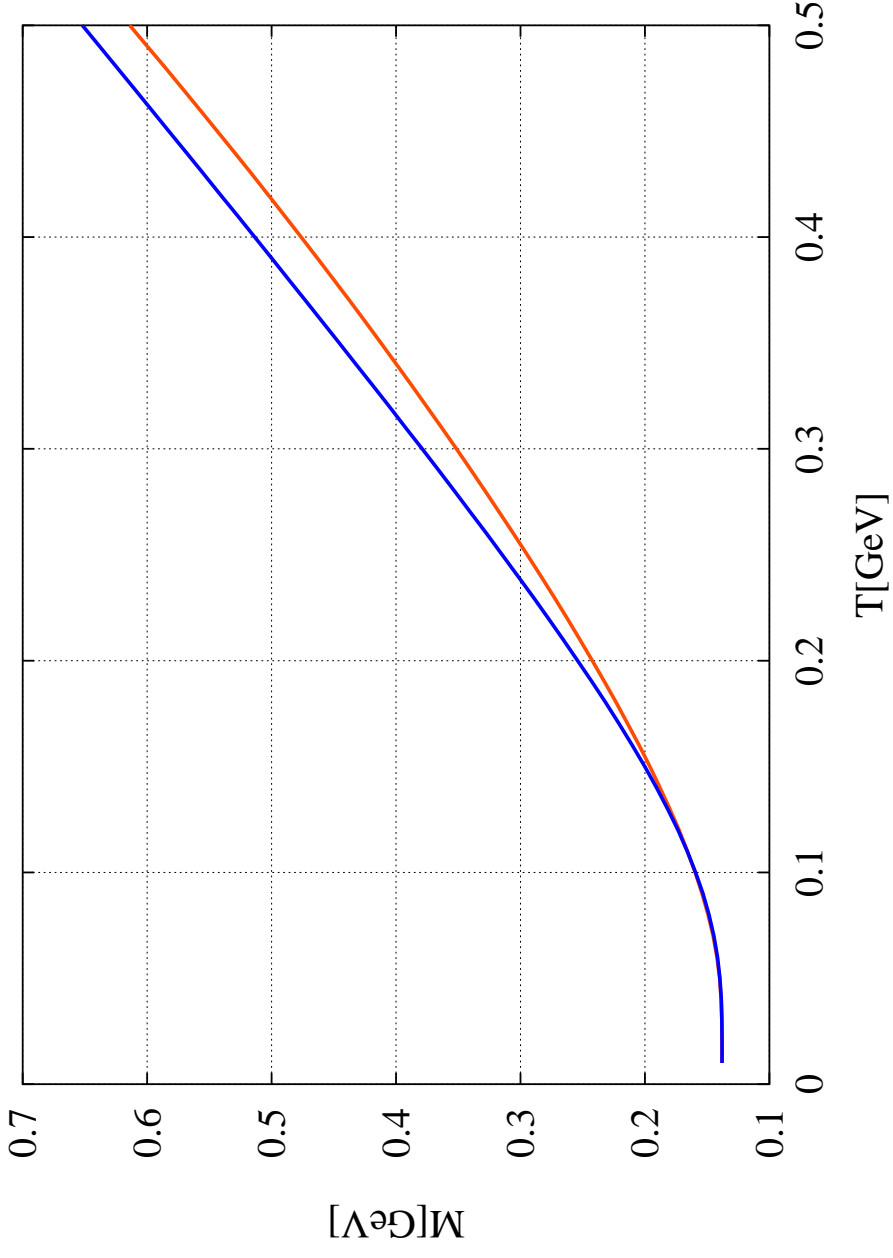
- Counterterms are independent of temperature and adjusted in vacuum

Self-consistent equation (gap equation)

$$M^2 = m^2 + \frac{\lambda}{32\pi^2} M^2 \ln \frac{M^2}{m^2} + \frac{\lambda}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi \delta(p^2 - M^2) n(p_0)$$

Numerical Results

Numerical solution of the self-consistent tadpole equation compared to the perturbative result for $m = 140 \mathrm{MeV}$ and $\lambda = 50$

Spontaneously Broken Symmetry

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Stable (tree-level) vacuum at $\phi = \varphi = \frac{6\mu^2}{\lambda}$. Particles have $m^2 = +2\mu^2 > 0$

- Have to take the Mean Field φ into account and solve the coupled equations for both $\varphi = \text{const}$ and $\Sigma = \text{const}$. The renormalization procedure is the same as in the unbroken case.

- Parameters from linear the σ-model: $\mu = 400\text{MeV}$, $\lambda = 100$
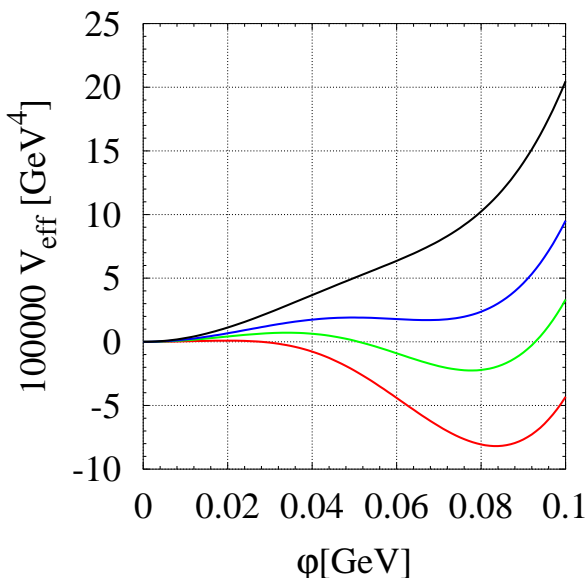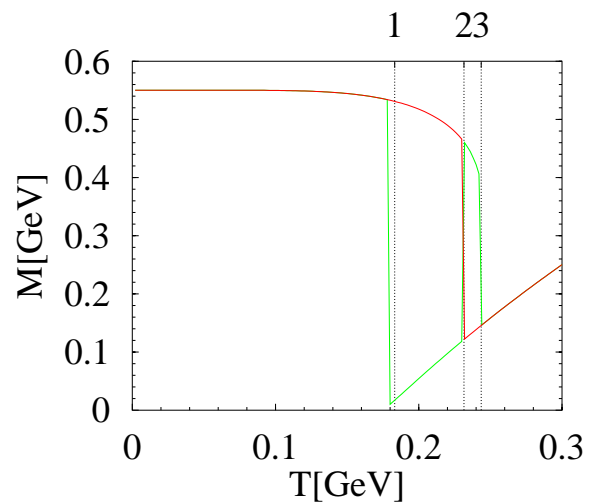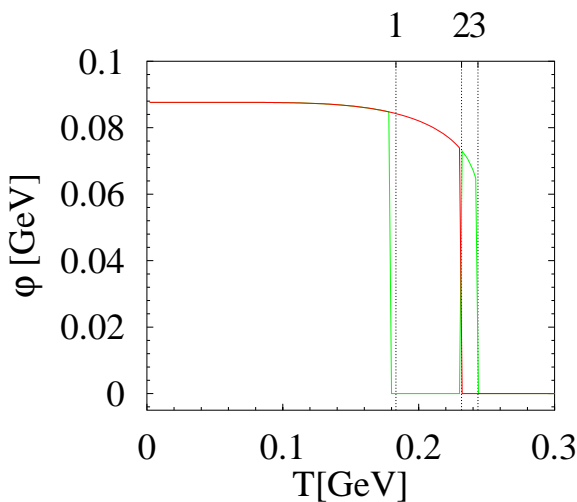
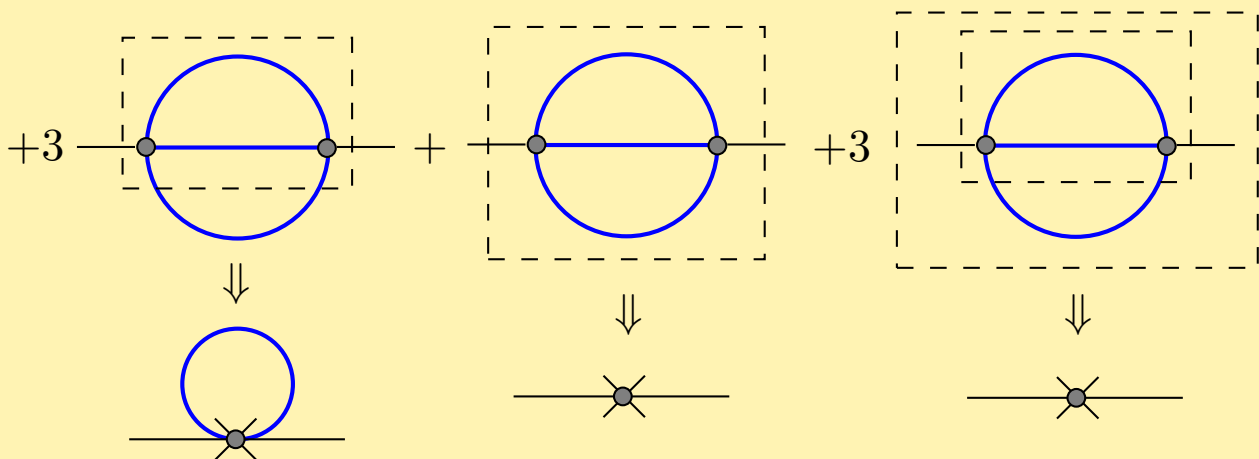The effective potential for $T = 180\text{MeV}$ to $T = 280\text{MeV}$

The Sunset Diagram

The vacuum part

$$\mathrm{i}\Phi = \qquad\qquad\qquad \Leftrightarrow \quad -\mathrm{i}\Sigma = $$

$$\frac{1}{2\cdot 4!} \qquad\qquad\qquad\qquad\qquad \frac{1}{3!}$$

- Overall and sub-divergences to all orders perturbation theory
☞ Subtracted dispersion relations for vacuum divergences

$$+3 \qquad\qquad + \qquad\qquad +3$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

- Counterterms of tadpole-type $\rightarrow$ const. and the overall subtraction $\rightarrow O(p^2)$ (mass- and wave function renormalization)
☞ Dispersion relation with two subtractions

Algorithm

Strategy for the vacuum

- Starting with perturbative propagators calculate imaginary part of the retarded self-energy (finite!):

$$\text{Im}\,\Sigma_R = \frac{\Sigma^{-+} - \Sigma^{+-}}{2i}$$

- Real part from twice subtracted dispersion relation:

$$\Sigma_R(s) = (s - m_{\text{ren}}^2)^2 \int_0^\infty \frac{\mathrm{d}z}{\pi} \frac{\text{Im}\,\Sigma_R(z)}{(z - s + i\epsilon\sigma(p_0))(z - m_{\text{ren}}^2)^2} +$$

$$+ \Sigma_R'(m_{\text{ren}}^2)(s - m_{\text{ren}}^2) + \Sigma_R(m_{\text{ren}}^2)$$

- Dyson's equation for the retarded propagator

$$G_R = \frac{1}{p^2 - m^2 - \Sigma_R + i\epsilon\sigma(p_0)}$$

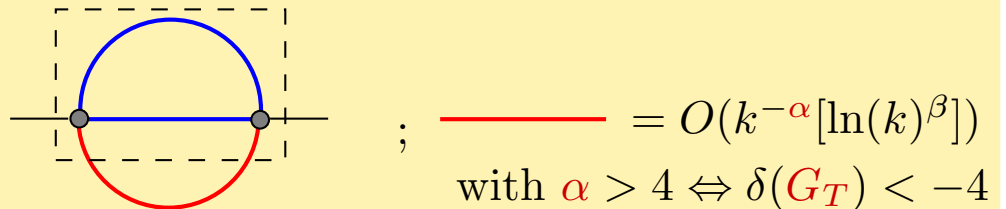 and plug it into the next calculation for $\text{Im}\,\Sigma_R$.

- Iterate this procedure until the Σ_R does not change anymore

The Sunset Diagram for $T > 0$

Renormalization of vacuum sub-divergences

; $\underline{\hspace{2cm}}$ $= O(k^{-\alpha}[\ln(k)^{\beta}])$

with $\alpha > 4 \Leftrightarrow \delta(G_T) < -4$

Algorithm

- Calculate renormalized $\Gamma^{(4)}_{\text{vac}}$ with already given self-consistent vacuum propagator (blue)

- Renormalization condition: $\Gamma^{(4)}_{\text{vac}}(s = 0) = 0$.

- Calculate $\text{Im}\,\Sigma_R$ with the full thermal propagator

- Subtract the vacuum part and $3\times$ the "bad diagram"

- Dispersion relation without subtraction for the rest

☞ Only pure vacuum subtractions in this part:

$$\Sigma^{--}_{vvT}(p) \propto \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \Gamma^{--}_v(l) G^{--}_T(l + p)$$

- Full retarded "bad diagram" with vacuum subtracted $\Gamma^{(4)}$-insertion

☞ finite

☞ Only vacuum sub-divergence subtracted

- Result: explicit overlapping divergences of the sunset diagram are completely subtracted with pure vacuum counter terms

Sunset+Tadpole I

New divergences

- $\delta(G - G_v) = -4$: Additional nested and overlaping vacuum subdivergences in sunset; subdivergences in tadpole.
- ☞ Expansion of propagator wrt. tadpole:

$$G = \sum_{k=0}^{\infty} G_0^{(k)} T^k, \quad \text{with } G_0^{(k)} = \frac{1}{k!} \left. \frac{\partial^k G}{\partial T^k} \right|_{T=0}$$

☞ $\delta(G_0^{(k)}) = -2(k+1)$

☞ Need only counter terms for the integrands!

Renormalization of the tadpole

- From Dyson's equation (Matrix form!):
  $G_0^{(1)} = G_0(\sigma_3 + \Sigma_{01})G_0$
- Tadpole: first divergent contribution:

$$-\mathrm{i}T_0^{(0)} = \underset{}{\underbrace{}} - \underset{}{\underbrace{}}$$

$\delta(G_0 - G_v) < 4 \Rightarrow$ finite

- Tadpole: second divergent contribution:

$$-\mathrm{i}T_0^{(1)} = \boxed{\sigma_3 + \Sigma_0^{(1)}} - \boxed{\sigma_3 + \Sigma_v^{(1)}}$$
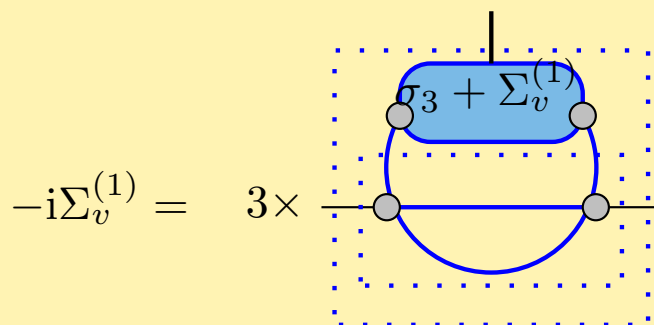
Sunset+Tadpole II

Renormalization of the sunset

- First contribution $\Sigma_0^{(0)}$: Already done!

- Second contribution $\Sigma_0^{(1)}$:

$$-i\Sigma_0^{(1)} = - $$
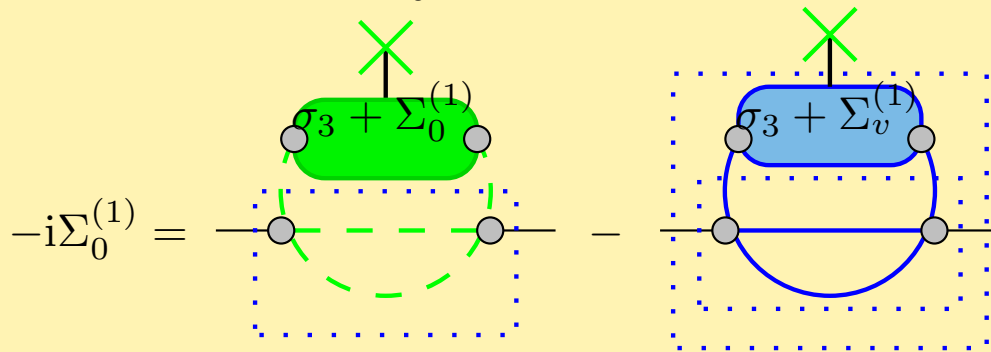
Calculation of $\Sigma_v^{(1)}$

- Need to calculate only $---$-part $\Sigma_v^{(1)}$ as additional building block
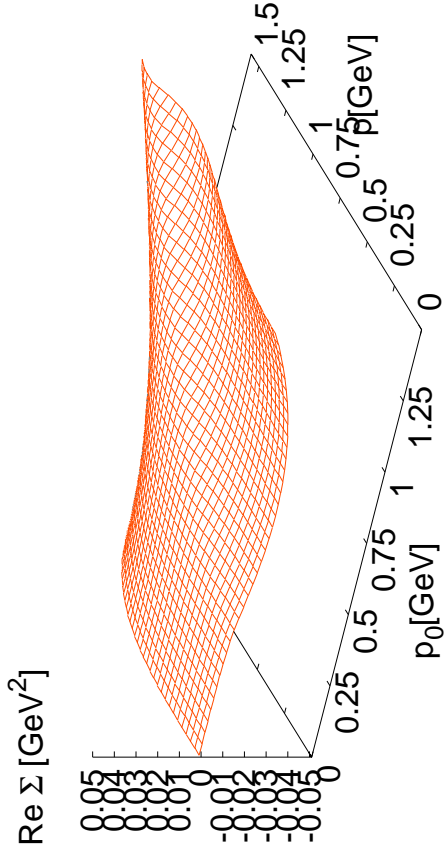- Derivate sunset diagram wrt. to T and take only the pure vacuum part:

$$-i\Sigma_v^{(1)} = 3\times$$

- Self-consistent equation: can be solved iteratively.

☞ Also sunset+tadpole can be renormalized with vacuum counter terms

Results for the Vacuum Sunset Diagram
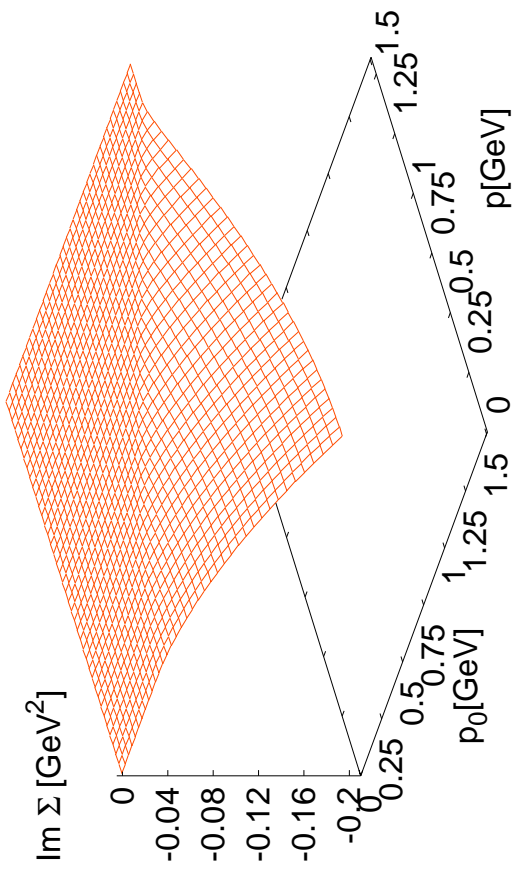
ReΣ for T=0

Re Σ [GeV^2]

ImΣ for T=0MeV

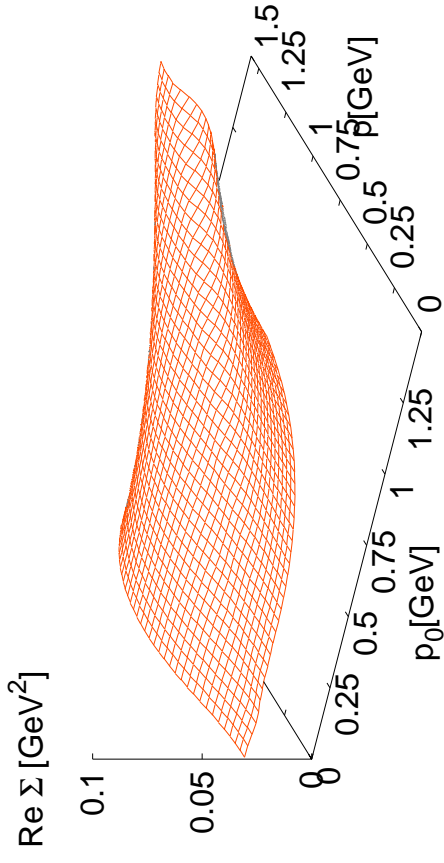Im Σ [GeV^2]

p$_0$[GeV]
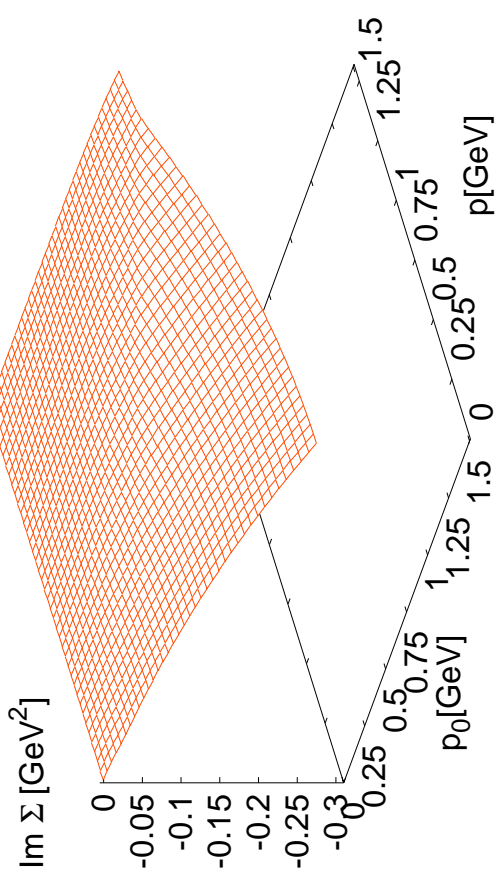
p[GeV]

Vacuum: $m = 140\text{MeV}, \lambda = 65$

Perturbative Result for the Sunset Diagram at $T > 0$

ReΣ for T=100MeV

ImΣ for T=100MeV
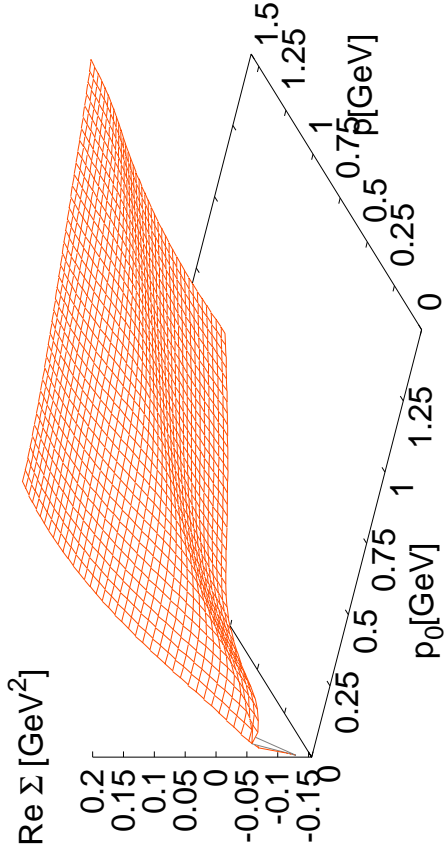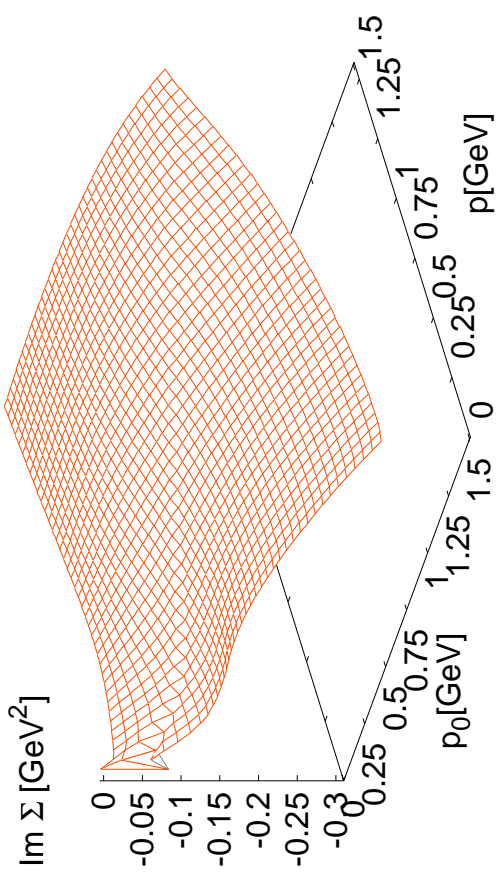
$T = 100\text{MeV}$

Self-consistent Result for the Sunset Diagram at $T > 0$

ReΣ for T=100MeV

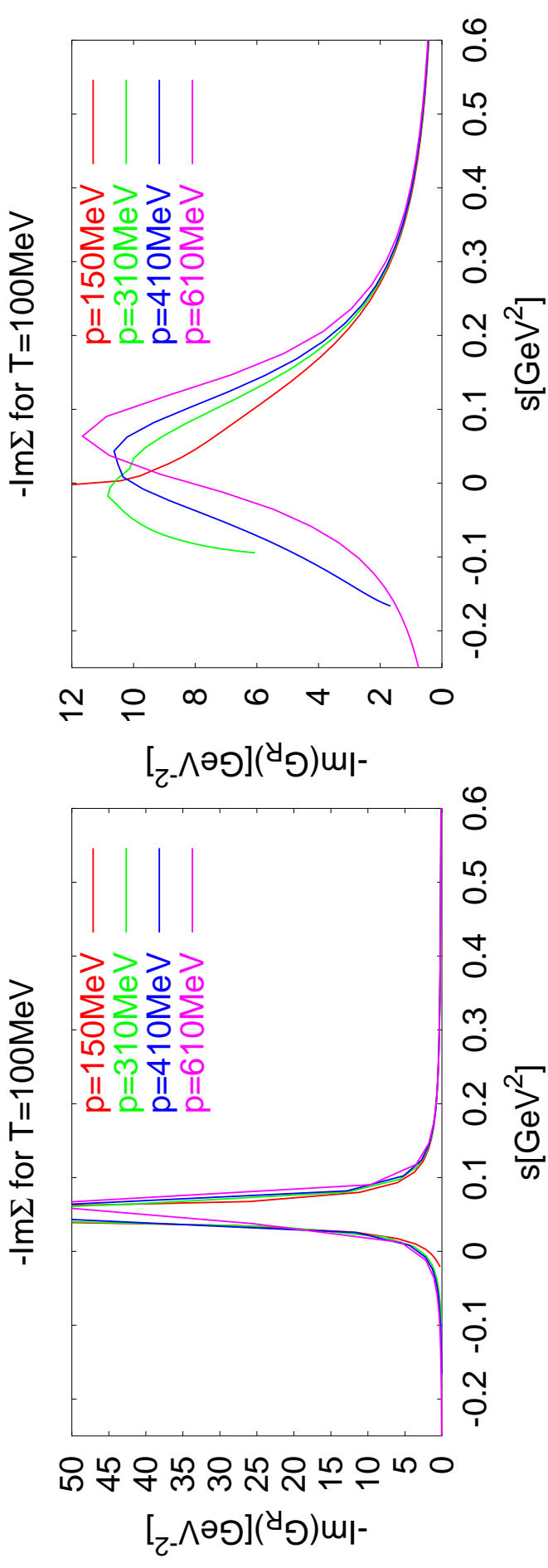ImΣ for T=100MeV

$T = 100\text{MeV}$

Spectral function of the "Meson"

$T = 100\text{MeV}$:Perturbative (left) and self-consistent (right) calculation

The Analytic Green's Function

The imaginary part of the contour

- So far real-time formalism

- Entropy, pressure, mean energy,···

- Analytic propagator

- Branch of analytic continuation of $G_{\text{Matsubara}}$

$$G_C(p_0, \vec{p}) = \int \frac{\mathrm{d}z'}{\pi} \frac{\rho(z', \vec{p})}{z' - p_0} \text{ with}$$

$$\forall z \in \mathbb{R} : \mathbb{R} \ni \rho(z, \vec{p}) = -\rho(-z, \vec{p}) = -\operatorname{Im} G_R(z, \vec{p})$$

- Causality structure of G_R and G_A

$$G_C(p_0 \pm \mathrm{i}0) = -G_{R/A}(p) \text{ for } p_0 \in \mathbb{R}$$

- Matsubara-propagator

$$G_M(\mathrm{i}\omega_n, \vec{p}) = G_C(\mathrm{i}\omega_n, \vec{p}) \text{ with } \omega_n = \frac{2\pi\mathrm{i}}{\beta}n = 2\pi\mathrm{i}nT, \ n \in \mathbb{Z}$$
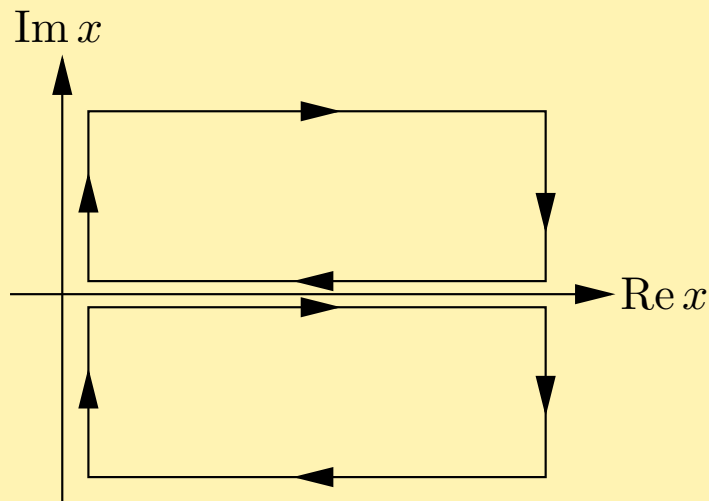
Matsubara Sums

Summing over Matsubara frequencies

- $F(z)$: analytic in an open strip around imaginary axis

$$\frac{1}{\beta} \sum_{n \in \mathbb{Z}} F(i\omega_n) = \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dx [F(x)+F(-x)] \left[\frac{1}{2} + \frac{1}{\exp(\beta x) - 1}\right]$$

- F(z): also analytic away from the real axis

$$\frac{1}{\beta} \sum_{n \in \mathbb{Z}} F(i\omega_n) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \frac{1}{2} [F(x - i\epsilon\sigma(x)) - F(x + i\epsilon\sigma(x))]+$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx n(x)[F(x - i\epsilon\sigma(x)) - F(x + i\epsilon\sigma(x))]$$

The Entropy

Thermodynamical properties

- Expectation values from thermal quantum field theory:

$$Z(\beta, V) = \operatorname{Tr} \exp(-\beta \mathbf{H})$$

$$\Rightarrow \varepsilon = \frac{1}{V} \langle \mathbf{H} \rangle = -\frac{1}{V} \partial_\beta \ln Z, \quad \frac{1}{V} \mathrm{d}(\beta \ln Z) = -\varepsilon \mathrm{d}\beta$$

- Define thermodynamical quantities:

$$P = \frac{\ln Z}{\beta V}, \ s = \beta(P + \varepsilon) \Rightarrow \mathrm{d}P = s \mathrm{d}T$$

- Solution of the real time Φ-derived self-consistent equations
$$\ln Z = \mathrm{i}\Gamma \Rightarrow P = \mathrm{i}\Gamma \Rightarrow s = \mathrm{i}\partial_T \Gamma$$

- Stationarity with respect to G_R: Need to derive only with respect to explicit temperature dependency

$$s = -2 \int_{p_0 > 0} \frac{\mathrm{d}^4 p}{(2\pi)^4} \partial_T n(p_0) \left\{ \operatorname{Im} \ln[-G_R^{-1}(p)] + \operatorname{Im}(\Sigma_R G_R) \right\} +$$
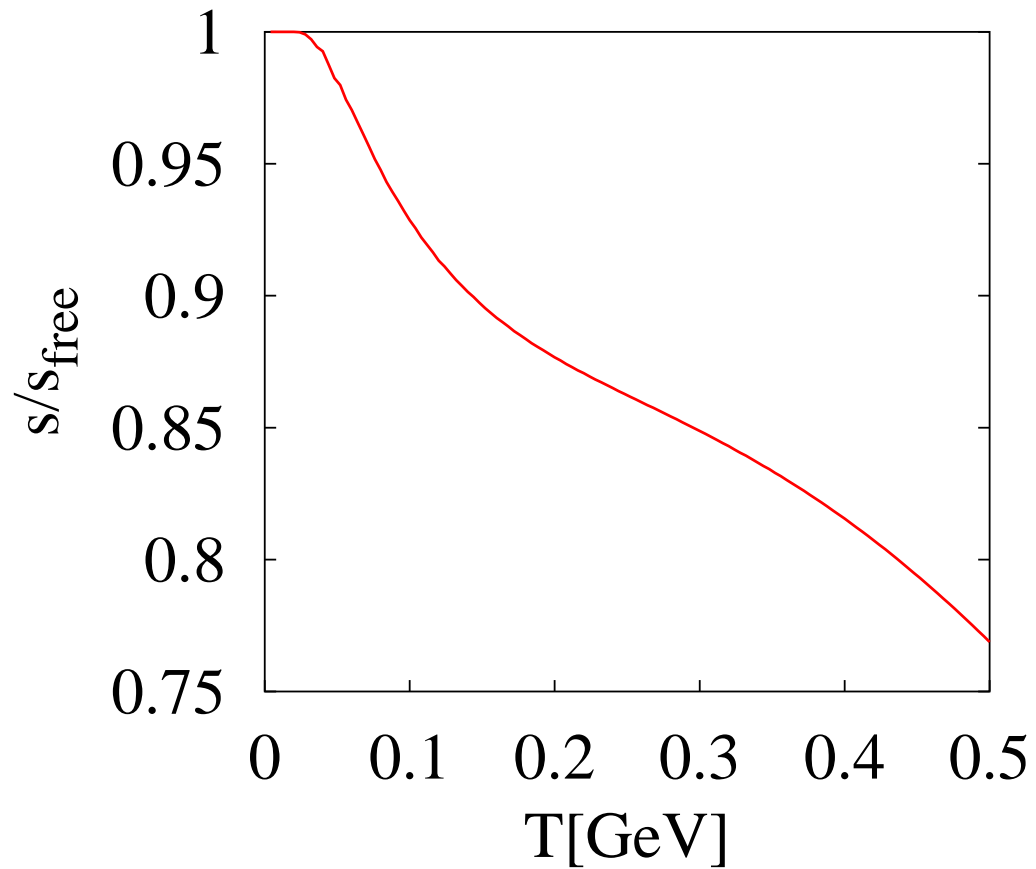
$$+ \mathrm{i} \left\{ \frac{\delta \Phi[\varphi, G]}{\delta n} \right\} \Bigg|_{G_R, \varphi \text{ fixed}} \partial_T n$$

- Especially for 2-point Φ-functionals

$$s = -2 \int_{p_0 > 0} \frac{\mathrm{d}^4 p}{(2\pi)^4} \partial_T n(p_0) \left\{ \operatorname{Im} \ln[-G_R^{-1}(p)] + (\operatorname{Im} \Sigma_R)(\operatorname{Re} G_R) \right\}$$
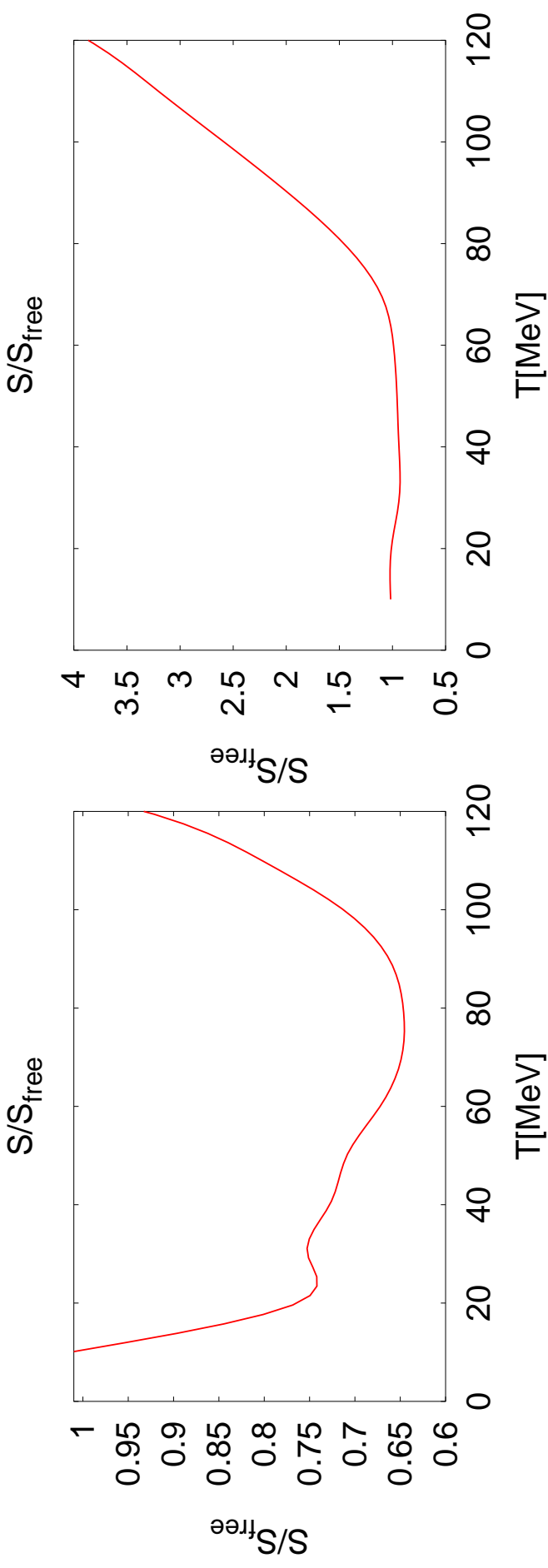
Result for Tadpole Resummation

The entropy density of the selfconsistent tadpole approximation per entropy of the free gas (for $\lambda = 50$).

Result for the sunset approximation

Entropy per entropy of the free gas (perturbative left, self-consistent right), the tadpole contribution was taken from the tadpole approximation. Numerics for full scheme is work in progress.

Conclusion and outlook

Conclusions

- Self-consistent Φ-derivable models can be renormalized and solved numerically

- Applications for the consistent treatment of particles and resonances with finite mass widths possible

- Applicable as well for dynamics as for thermodynamical quantities

- Consistent schemes for transport equations for such particles and resonances

Outlook

- Problem with Goldstone's theorem and phase transitions (Linear Σ-model)

- Development of numerics for more complicated cases beyond two-point level (ladder summations)

- Big fundamental problem:
- Most important physical theories involve gauge fields
☞ Standardmodel
☞ (Phasetransitions in QCD and electro-weak Theory)
☞ Effective meson theories (vector dominance and dileptons)
- Does there exist a gauge-invariant scheme?