Finite temperature field theory applied to the ϕ^4 model.

Josep F. Oliver

March 23, 2000

Abstract

The application of finite temperature field theory to the ϕ^4 model yields to a variation of the mass of the described bosons depending on the temperature. This new mass must be understood as an effective mass rather than a fundamental one. Calculations for the mass variations are shown for both types of potential: the symmetric potential and the potential that induces spontaneous symmetry breaking in the system.

1 Introduction

As is well known the main tool in quantum field theory is the propagator. It is the amplitude of emission of a single a particle in a given space-time point x and its annihilation at another point y ,i.e., it describes the motion of a particle in terms of fields. One of the most important results in this area is the fact that propagators can be derived from the vacuum expectation value of a pair of field operators.

$$iG(x-y) = \langle 0|T_c\phi(x)\phi(y)|0\rangle \tag{1}$$

The whole formalism of QFT has proven to be impressively useful for calculating cross sections and life times.

But many times the systems involve so many particles that it is impossible to handle them explicitly. We are no longer interested in scatterings that involve few particles but in the common and simultaneous interactions of many particles. In other words we are looking for the quantum equivalent of Classical Statistical Physics. We will see that the statistical operator is a very good formalism for handling this situation and finally we will show that under the conditions of a thermal bath particles evolve as if they had a different masses, in fact effective masses.

2 Fundamentals

For the general reader the fundamentals are outlined below

Statistical operator (Density matrix)

It is possible that in a given situation we do not know all the information of a system, i.e., we do not know its state. But we can assign it a probability distribution for finding it in the state $|\psi_k\rangle$. Let us call $\mathcal{P}_k$ to this probability. Then we can build the statistical operator as:

$$\hat{\rho} = \sum_k \mathcal{P}_k|\psi_k\rangle\langle\psi_k| \tag{2}$$

It can be proved that this operator satisfies:

$$Tr(\hat{\rho}) = 1 \qquad \langle O\rangle = Tr(\hat{\rho}\hat{O}) \tag{3}$$

We will take a result from information theory without proof. The amount of lost information in our system will be $I = -Tr(\hat{\rho}\ln\hat{\rho})$.

Suppose now we perform an energy measure over our system. By the postulates of Quantum Mechanics we know that after this measure the system will be in a Hamiltonian eigenstate. If the energy spectrum is non degenerate this measure is enough for knowing the exact state, but if it is not the case, all we can say is that the system ket belongs to the subspace generated by all the eigenstates associated with the energy we have measured. Our aim is to find how to describe the system we have obtained. One way for doing this is assume that it is going to be described by a statistical operator , in other words we look on an ensemble of systems with the measured value for the mean energy value. The required operator must be looked up from all the hermitian operators acting on the Hilbert space of the system that fulfilling the conditions (3) make maximum our lost information. This last imposition is just an assumption but doing it is possible to re-derive the entire statistical thermodynamics, coming out that our lost information measure is nothing than the entropy of the system. That is why it is going to be considered as a valid assumption. At this point we will assume our system is in equilibrium, that means $\hat{\rho}$ does not depend on time.

We cannot go on the details but the result of our query is :

$$\hat{\rho} = \frac{\exp(-\frac{\hat{H}}{kT})}{\sum_k \exp(-\frac{E_k}{kT})} \tag{4}$$

Notice the resemblance between the denominator and the partition function.

We have done a energy measure that is the reason why finally we have finished working with the canonical ensemble. The other ensembles can be developed in a similar way.

Basics required in field theory

In Field Theory the main aim is looking for a relativistic quantum formulation. For many reasons the standard substitutions:

$$E \longrightarrow i\hbar\frac{\partial}{\partial t} \qquad \vec{P} \longrightarrow -i\hbar\vec{\nabla} \qquad \vec{x} \longrightarrow \vec{x} \tag{5}$$

do not work with the basic relativistic formula $E^2 = p^2c^2 + m^2c^4$.

The new conserved quantities are not definite positive and the standard probability density $|\psi|^2$ is no longer conserved.

Second Quantization

This is one possible proceeding to achieve our goal. Not only (5) is able to quantize a classical system with a finite number of degrees of freedom but the next recipe can also do it:

$$\{f, g\} \longrightarrow \frac{1}{i\hbar}[\hat{f}, \hat{g}] \tag{6}$$

Where the Poisson brackets are used. The important feature is that it is perfectly applicable to an infinite degrees of freedom system. When this is done for classical fields all the observables show a quantized spectrum and it is interpreted as the single contributions of each particle. The eigenstates being $\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\}$.

Once the future predictions of this theory are compared with experiments the agreement is pretty good only with bosons. For fermions the commutator must be changed in (6) by the anticommutator.

Propagators

For interpreting them let it be a free field in its ground state,i.e.,vacuum $|0\rangle$. In this state $a$ll the expectation values are zero. There is a probability of a particle being created at one point of space in a definite instant x and being annihilated in y. The amplitude for this event is just (1). This is the fundamental vacuum fluctuation. And this amplitude is just the Green's function of the classical equations of motion of the classical free field,i.e., what classically describes the propagation of a signal by the field under consideration. The T_c is the time ordered operator, it causes the field operator with lower time coordinate acting first over the vacuum. A particle cannot be annihilated without previously being created!. This being is called propagator.

When interactive fields are taken into account the complete solution cannot be obtained and perturbation theory is used. For instance, let us consider the next Lagrangian that describes an self-interactive field, i.e., its particles interact among each other:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{7}$$

The unperturbated Hamiltonian in the theory is defined as the free term when no interaction takes place, at this order of approximation the propagator is:

$$\langle 0|T_c\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{4\pi}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)} \tag{8}$$

And the second term of the perturbation series, $\langle 0|T_c\phi(x)\phi(y)\frac{\lambda}{4!}\int d^4z\phi^4(z)|0\rangle$, describes the emission and re-absortion of a virtual particle at one instant of time during the travel of the created particle. Physically means that along the path the real particle is exciting the vacuum. This effect causes that the predicted mass of the particle goes off to infinity and renormalization is required.

ϕ^4 theory

Symmetric case

The Lagrangian for this system is no other than (7). The first term must be interpreted as a kinetic term and the two seconds as a potential term. Then this Lagrangian describes the interactions between particles of mass m.

Spontaneous Symmetry Breaking

If in the former Lagrangian is changed $m^2 \rightarrow -\mu^2$ then the potential changes as shown in the figure and it makes sense to expand the field ϕ around one of the minimums $\phi = \varphi + \psi$, where φ must be understood as a constant field called *mean field* and ψ the variation around the minimum. Now the symmetry

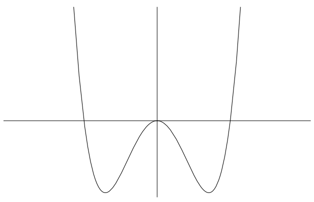

Figure 1: Sombrero shaped potential

of the Lagrangian is hidden in the solution and the new Lagrangian is describing the interaction between particles of mass $\sqrt{2}\mu$ but interacting in a rather more complicated way.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \mu^2\psi^2 - \sqrt{\frac{\lambda}{6}}\mu\psi^3 - \frac{\lambda}{4!}\psi^4 \quad (9)$$

Where it has been used the value of the minimum for φ. The physical idea is that the simplicity of the interaction is lost around the minimum. Pay attention that the symmetry $\phi \rightarrow -\phi$ now is lost.

Phase transitions

A first order phase transition occurs in a system when one of the variables that is describing it changes discontinuously. This phenomenon occurs always when the ground state is degenerate. In this situation if some external agent acts over the system then it could modify the potential in a way that one of the ground states is now preferred due to it has now the lower energy. If this agent acts in the opposite direction it can select between the different ground states. In most circumstances it is possible to change one of the parameter of the system in a way that the different ground states came closer, even coalesce in one single state. This is a second order phase transition. The point where this occurs is called *critical point*.

The most paradigmatic example is the phase transitions between liquid and gas. The parameter that causes the transitions being the temperature. But it is also well known that over the critic temperature it has no sense to distinguish between two phases.

In our system the transitions phases occur when our sombrero shaped potential evolves to a potential like:

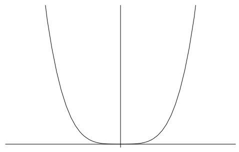

Figure 2: Potential for the symmetric case

3 Finite temperature field theory

Now we are no longer interested in vacuum fluctuations. A system of many particles is now under consideration. If the energy of the system is known then (4) will describe it perfectly. If it is also known the value of the number of particles grand canonical ensemble must be used. The important point is that fluctuations of this special state are searched. The behavior of few particles can be understood with vacuum excitations now it is expected to understand the behavior of many particles with this formalism.

The Finite Temperature Green's Functions

In such mixed state the expectation values are given by (3). Using the axioms of Quantum Mechanics and working in the interaction picture it can be shown that:

$$\langle O(t)\rangle = Tr\left(\hat{\rho}(t_0)\hat{C}^\dagger(t,t_0)\hat{O}(t)\hat{C}(t,t_0)\right) \quad (10)$$

Where $\hat{C}(t,t_0)$ is just the time evolution operator of the states in the interaction picture. It can be written as a Dyson series :

$$\hat{C}(t,t_0) = T_c \exp\left(-i\int_{t0}^{t} d\tau \hat{H}_I(\tau)\right) \quad (11)$$

$$\hat{C}^\dagger(t,t_0) = T_a \exp\left(i\int_{t0}^{t} d\tau \hat{H}_I(\tau)\right) \quad (12)$$

Where T_a is just the time anti-ordered operator. After doing few non trivial manipulations:

$$\langle O(t)\rangle = \sum_k \exp\left(-\frac{1}{kT}E_k^{(0)}\right) \times \\ \times Tr\left(T_{\mathcal{C}} \exp\left[-i\int_{\mathcal{C}} d\tau \hat{H}_I(\tau)\right]\hat{O}(t)\right) \tag{13}$$

Being $\mathcal{C}$ the path in the complex plane and $T_{\mathcal{C}}$ the operator that establish the correct order of the operators depending in which branch the integration is being done. As we are looking for fluctuations of a mixed state, it is sensible to permit this fluctuations be mixed states as well. That is why now our Green's Functions have to be matrices and is easy to prove that:

$$i\tilde{G}(x,y) = \langle T_{\mathcal{C}}\phi(x)\phi(y)\rangle \tag{14}$$

fulfills the equation:

$$\hat{\mathcal{M}}\left(i\tilde{G}(x,y)\right) = \delta^{(4)}(x-y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

showing that they are our searched object since $\hat{\mathcal{M}}$ stands for the motion equations of the classical field[1].

4 ϕ^4 revisited

A massive uncharged spin-less point particle is described by a scalar field. Relativity forces its Lagrangian to be:

$$\mathcal{L} = \partial^\mu\phi\partial_\mu\phi - m^2\phi^2 \tag{15}$$

So particles like free pions can be described with this Lagrangian. If it is added the term $-\frac{\lambda}{4!}\phi^4$ the interaction for this system is set.

As many other problems the complete solution of this system is impossible with elementary manipulations and perturbation theory is used. What is searched is (14) for this case.

Free case

Now it is considered to have many of such particles in a state of definite energy described by (4). When only the free part is considered the Green's functions happen to be in the momentum representation:

$$i\tilde{G}_{11} = \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi\delta(p^2 - m^2)n(p_0) \tag{16}$$

$$i\tilde{G}_{22} = \frac{i}{p^2 - m^2 - i\epsilon} + 2\pi\delta(p^2 - m^2)n(p_0) \tag{17}$$

$$i\tilde{G}_{12} = 2\pi\delta(p^2 - m^2)[\Theta(-p_0) + n(p_0)] \tag{18}$$

$$i\tilde{G}_{21} = 2\pi\delta(p^2 - m^2)[\Theta(p_0) + n(p_0)] \tag{19}$$

Where Θ is the Heaviside function and $n(p)$ is the occupation number for a Bose-Einstein statistics:

$$n(p) = \frac{1}{\exp\left(-\frac{p^2}{2mkT}\right) - 1} \tag{20}$$

The value of m depends on the potential taken and stands for the mass of the particles[2].

The physical meaning is easy to understand. The first terms in $\tilde{G}_{11}$ and $\tilde{G}_{22}$ are the vacuum contributions since they are the same than (8) but in momentum space. So they describe a virtual particle of mass m being created as a vacuum excitation. The second terms are the new contributions due to the thermal bath and describe the creation of only on-shell[3] particles. This particles are created in the thermal bath and have a mass m.

Interacting case

After mathematical operations the next term in the perturbation series for the matrix propagator is:

$$i\tilde{G}_{11} = \frac{i}{p^2 - M^2 + i\epsilon} + 2\pi\delta(p^2 - M^2)n(p_0) \tag{21}$$

$$i\tilde{G}_{22} = \frac{i}{p^2 - M^2 - i\epsilon} + 2\pi\delta(p^2 - M^2)n(p_0) \tag{22}$$

$$i\tilde{G}_{12} = 2\pi\delta(p^2 - M^2)[\Theta(-p_0) + n(p_0)] \tag{23}$$

$$i\tilde{G}_{21} = 2\pi\delta(p^2 - M^2)[\Theta(p_0) + n(p_0)] \tag{24}$$

where $M = m^2 + \frac{\lambda}{2}\varphi^2 + \Sigma_{11}$, Σ_{11} is the contribution to the mass of the particles due to the interactions with the thermal bath. It is the same phenomenon than in standard QFT but now fortunately the mass does not diverge, so no longer regularizations are required. φ is the mean field. Now it must be still considered as

[1] The -1 stands for causality reasons because in one branch must be used the time ordered operator and in the other the anti-time ordered

[2] For instance if the sombrero shaped potential is used then $m = \sqrt{2}\mu$.

[3] That means $p^2 = m^2$. Relation that virtual particles do not fulfill only real ones.

a constant but maybe temperature dependent. The mean point is that the particles still behave as free ones but with an effective mass what can be understood by comparison with the free case.

Now it comes out that the equations that governs φ value and the value of Σ_{11} are coupled:

$$\varphi\left(m^2+\Sigma_{11}+\frac{\lambda}{6}\varphi^2\right)=0 \qquad (25)$$

$$\Sigma_{11}=\frac{\lambda}{4\pi^2}\int_M^\infty d\omega\frac{\sqrt{\omega^2-M^2}}{\exp\left(\frac{\omega}{kT}\right)-1} \qquad (26)$$

Effective potential

The physical interpretation of (25) permits the introduction of an effective potential. If φ is fixed then M can be computed solving the implicit equation for it:

$$M^2=m^2+\frac{\lambda}{2}\varphi^2+\frac{\lambda}{4\pi^2}\int_M^\infty d\omega\frac{\sqrt{\omega^2-M^2}}{\exp\left(\frac{\omega}{kT}\right)-1} \qquad (27)$$

Once M is known Σ_{11} can be calculated, so Σ_{11} depends finally on φ once the temperature is fixed. Then (25) can be re-interpreted as $\frac{dV_{eff}}{d\varphi}=0$. Which is consistent with the free case since in that case $\Sigma_{11}=0$.

Symmetric case

One immediate solution is $\varphi=0$ and M being defined by the implicit (27). That case is the symmetric one. The effective mass, M, depends on temperature. That means that since the mean field is constant and zero the minimum of the effective potential does not change its position, but as M changes then the shape of the potential varies.

Spontaneous symmetry breaking

The other possible solution consists in fixing in (25) the bracket to zero and solve the coupled solution. This is more interesting since the mean field now must change for fulfilling (25) and it cannot be constant for all temperatures because Σ_{11} is temperature dependent. The effective potential then will vary in a more drastic way since its minimums will move. If both minimums coalesce in a single one then the system will suffer a phase transition.

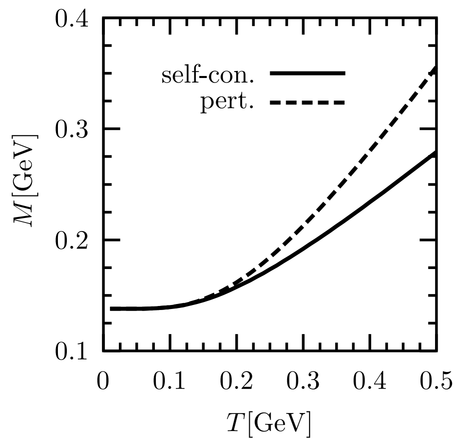

Figure 3: Results for the symmetric case.

5 Numerical computations

Symmetric case

With a fairly simple iterative algorithm it is very easy to compute $M=f(T)$. The results are shown in figure 3.

Solid line corresponds to complete numerical solution and the dashed one corresponds to perturbative solution when only one iteration is done.

Here M increases with T showing that the higher the temperature the higher interaction with thermal bath.

Spontaneous Symmetry Breaking

Now $M=f(T)$ and $\varphi=f(T)$ are shown in figures 4 and 5.

The mean field evolves until it reach the zero value and then a phase transition is shown by the system.

Further calculations

It would be very interesting to compute the evolution of V_{eff} as a function of temperature. It would show explicitly if our previous interpretations in terms of an evolving effective potential are true.

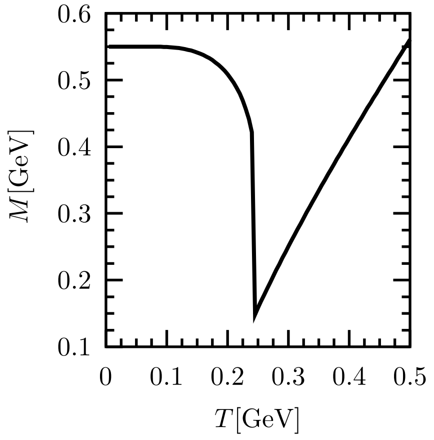

Figure 4: Mass Temperature dependence.

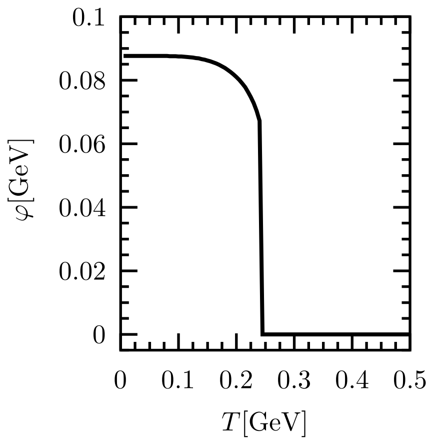

Figure 5: The self-consistent solution of the equations for M (left) and φ (right). The parameters were those of the linear σ-model ($\mu = 389$MeV and $\lambda = 118.26$)

6 Summary

After a brief introduction to field theory, ϕ^4 theory has been developed with the issues of Finite Temperature Field Theory. The has shown very interesting features like the possibility of treating the interactive case as a free one but with a temperature dependent potential and an effective mass. It has many resemblances with solid state physics.

One very important characteristic is the phase transition shown in the Spontaneous Symmetric Breaking case.

These ideas have a wide range of application from beam collisions until the previous instants of Big Bang passing by baryon asymmetry in the universe . It is possible that phenomena as phase transitions could be related with the mass of the actual particles or with the process that gives them mass.

References

[1] M. E. Peskin, D. V. Schroeder. *An Introduction to Quantum Field Theory.* Addison-Wesley Publishing Company.(1995).

A very good book for starting with Field Theory. It also contains a fairly good explanation of the linear σ model. A generalized model for the ϕ^4 theory.

[2] S. Weinberg. *The Quantum Theory of Fields* Cambridge University Press. (1996). It is a superb book for the readers already initiated in *QFT*.

[3] Pierre Ramond. *Field Theory: A Modern Primer.2nd Edition* Frontiers on Physics. Here it is shown in a formal way field theory. Quantization of physical systems is done with path integral not with Second Quantization.