The Contour Time Path

Ian Vernon*

September 25, 1997

Abstract

In this report various aspects of Finite Temperature and Density Quantum Field Theory are discussed. The main point being the calculation of Green's functions in this context, in which pertubation theory is used.

The motivation for calculating Green's functions comes from their wide application in modern Physics. The techniques that are presented here are used in such areas as condensed matter, super conductors and super fluidity, nuclear and elementary particle physics.

1 The Initial Problem

The initial situation is a non-relativistic many body problem in the Heisenberg picture. For this the Hamitonian Operator in the second quantisation is used:

$$\underset{\sim}{H}(t) = \int d^3\vec{x}\underset{\sim}{\psi}^\dagger(t,\vec{x})\left[-\frac{\Delta}{2m}\right]\underset{\sim}{\psi}(t,\vec{x}) + \frac{1}{2}\int d^3\vec{x}d^3\vec{y}\underset{\sim}{\psi}^\dagger(t,\vec{x})\underset{\sim}{\psi}^\dagger(t,\vec{y})V(\vec{x}-\vec{y})\underset{\sim}{\psi}(t,\vec{y})\underset{\sim}{\psi}(t,\vec{x}) \quad (1)$$

where the field operators which equal creation and annihilation operators obey the following equal time commutator or anticommutator relation:

$$[\underset{\sim}{\psi}(t,\vec{x}),\underset{\sim}{\psi}^\dagger(t,\vec{y})]_\mp = \delta^{(3)}(\vec{x}-\vec{y}). \quad (2)$$

The problem is to find the expectation value of an operator $\underset{\sim}{Q}(t)$. This is defined using the density operator $\underset{\sim}{\rho}$ (which is a statistical operator), as follows:

$$\left\langle \underset{\sim}{Q}(t)\right\rangle = \frac{\mathrm{Tr}(\underset{\sim}{\rho}\underset{\sim}{Q}(t))}{\mathrm{Tr}\,\underset{\sim}{\rho}} \quad (3)$$

This is the main aim: to calculate $\left\langle \underset{\sim}{Q}(t)\right\rangle$ using $\underset{\sim}{\rho}(t_0)$ and the dynamics of the system which are obtained from $\underset{\sim}{H}$.

2 $T = 0$ Field Theory

Before discusing the $T > 0$ case, the strategy for calculating physically important information about $T = 0$ systems is summarised:

*I.R.Vernon@sms.ed.ac.uk

1. Move to Interaction Picture (from the Heisenberg Picture) through a unitary transformation.
2. Obtain Dyson Series for time evolution (asymptotic states).
3. Obtain a Power Series after switching the pertubation on.
4. Use Wick's Theorem for vacuum expectation values and the Linked-Cluster-Theorem.
5. Results in FEYNMANN DIAGRAMS.

From these, diagramatic calculations can be made of:

- Transition Amplitudes
- Green's Functions
- Self Energy Functions

and from this, physically important information such as cross sections and transition probabilities can be abstracted. All this can be found in textbooks about quantum field theory, for instance, [Wei95].

For the $T \neq 0$ case the above strategy cannot be exploited since in a medium there are no asymptotic states (due to scattering). Also for a non-equilibrium system of $T > 0$ the vacuum expectation values do not correctly desribe the physical situation. Therefore a different approach is required i.e. $\underset{\sim}{\rho}(t_0)$ and the dynamics of the system are needed.

3 Schwinger-Keldysch Contour Ordering

Consider the time evolution in the interaction picture:

$$\left\langle \underset{\sim}{Q}(t) \right\rangle = \left\langle \underset{\sim}{C}^{\dagger}(t,t_0)\underset{\sim}{Q}_I(t)\underset{\sim}{C}(t,t_0) \right\rangle \tag{4}$$

where $\underset{\sim}{Q}_I(t)$ = operator in the interaction picture, and

$$\underset{\sim}{C}(t,t_0) = T^c \exp[-i\int_{t_0}^{t} \underset{\sim}{H}_I^{(1)}(t')dt'] \tag{5}$$

which is the unitary transformation operator between the interaction and Heisenberg pictures. Here T^c = the time ordering operator.

Substituting $\underset{\sim}{C}(t,t_0)$ into (4), the expectation value can then be expressed as a contour integral:

$$\left\langle \underset{\sim}{Q}(t) \right\rangle = \left\langle T\{\exp[-i\oint^{\mathcal{C}} \underset{\sim}{H}_I^{(1)}(t')dt']\underset{\sim}{Q}_I(t)\} \right\rangle \tag{6}$$

where $\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^+ +$ the contour time ordering and $t_{max} >$ the biggest time in the operators to be averaged. This technique is due to Schwinger [Sch61] and Keldysh [Kel64]. Reviews are found in [Dan84], [LP81] and [LW87].

Then using the Linked-Cluster Theorem in order to cancel closed vacuum diagrams, the following is obtained:

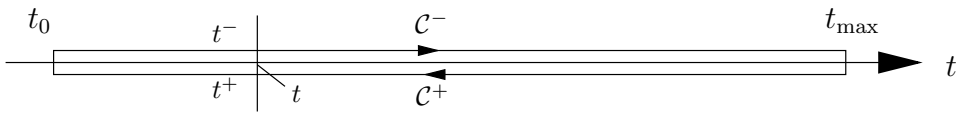

Figure 1: The Schwinger-Keldysh closed time contour.

$$\left\langle \underset{\sim}{Q}(t) \right\rangle = \frac{\left\langle T\{\exp[-i \oint^{\mathcal{C}} \underset{\sim}{H}_I^{(1)}(t')dt'] \underset{\sim}{Q}_I(t)\} \right\rangle}{\left\langle T\{\exp[-i \oint^{\mathcal{C}} \underset{\sim}{H}_I^{(1)}(t')dt']\} \right\rangle} \tag{7}$$

The above equations result in a system of closed diagrams without any external lines. There is a close analogy with Feynmann rules but in general in taking the expectation value of the Wick decomposed contour ordered products much more averages survive than in the vacuum case since normal ordered products don't need to vanish for a general statistical operator. One can show that the usual terms survive if and only if the initial time statistical operator is of the form $\exp(-\underset{\sim}{A})$ with $\underset{\sim}{A}$ a one-particle-operator [Dan84].

Further on we concentrate on this case. Then the only difference to the vacuum Feynmanrules is that one has to take contour- instead of real-time-integrals and the free Green's functions with respect to the statistical operator.

The contour integration can be substituted by summing over the branches of the contour and using real-time-integrals leading to a matrix formalism, which we shall discuss for the case of two-point-functions in the next sections.

4 Green's Functions in Real Time Formalism

For this technique, four different Green's functions (which can all be calculated with a little mathematics) are defined:

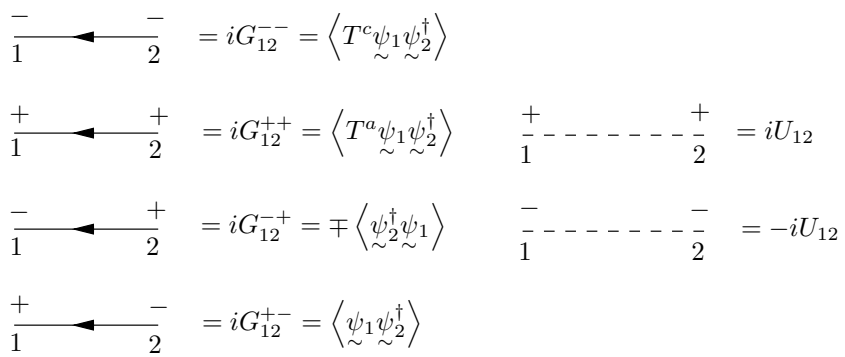

Figure 2: The four Green's functions in the Schwinger-Keldysh-formalism. The upper (lower) sign is for Fermions (Bosons). The dashed line is the interaction.

where $U_{12} = \delta(t_1 - t_2)V(\vec{x}_1 - \vec{x}_2)$ (a Newtonian instantaneous interaction).

The following is an example which demonstrates the use of the above definitions in the diagram technique where $\underset{\sim}{H}$ is split into $\underset{\sim}{H}_0$ and $\underset{\sim}{V}$ the interaction term. For a non-equilibrium system the Green's function G^{--} is given by:

$$iG_{12}^{--} = \left\langle \underset{\sim}{C}^{\dagger} T^c [\underset{\sim}{\psi}_1 \underset{\sim}{\psi}_2^{\dagger} \underset{\sim}{C}] \right\rangle \tag{8}$$

where

$$\underset{\sim}{C} = \underset{\sim}{C}(\infty, -\infty) = T^c \exp\left(-i \int_{-\infty}^{\infty} \underset{\sim}{H}_I^{(1)}(t) dt \right). \tag{9}$$

Expanding $\underset{\sim}{C}$ and $\underset{\sim}{C}^{\dagger}$ in a power series with respect to $\underset{\sim}{H}_I^{(1)}$ and substituting into (8) gives a series for iG_{12}^{--}. For the simple case of a system of particles (fermions say) in an external field $U(t, \vec{x}) = U(X)$, the 1st order terms are repsented in terms of the following Schwinger-Keldysh diagrams:

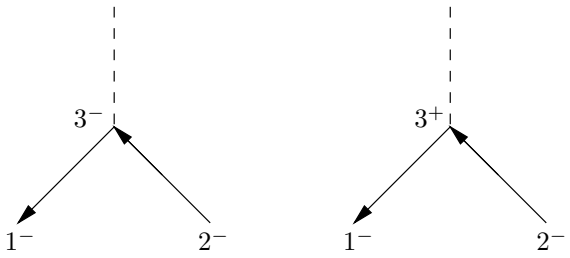

Figure 3: 1st order Schwinger-Kedysh diagrams for scattering by a external field.

From this we read off the analytic form due to the Schwinger-Keldysh diagrammatic rules:

$$iG_{12}^{(1)--} = \int \left[iG_{13}^{(0)--} iG_{32}^{(0)--} (-iU_3) + iG_{13}^{(0)-+} iG_{32}^{(0)+-} iU_3 \right] d^4 X_3 \tag{10}$$

since the four free Green's functions $G^{(0)}$ are known, G^{--} as well as the other Green's functions can be evaluated to any order using this technique.

5 Self Energy

Finally a brief discusion of Self Energy, beginning with the definition:

The sum over all 1 particle irreducible diagrams for amputated Green's functions = the Self Energy $= -i\Sigma$. In the normal diagram technique, the Dyson equation gives:

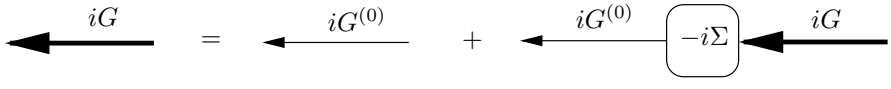

Figure 4: Definition of the Self Energy as the 1P-I amputated Green's functions.

Therefore the exact Green's functions can be expressed as a series:

In our case however, it is more complicated since there are four self energy terms: Σ^{--}, Σ^{-+}, Σ^{+-} and Σ^{++}, which means that we have to sum over the signs included in the definition of the Self Energy and Green's functions.

Thus the analytic form can be expressed in terms of matrix equations. For that we define

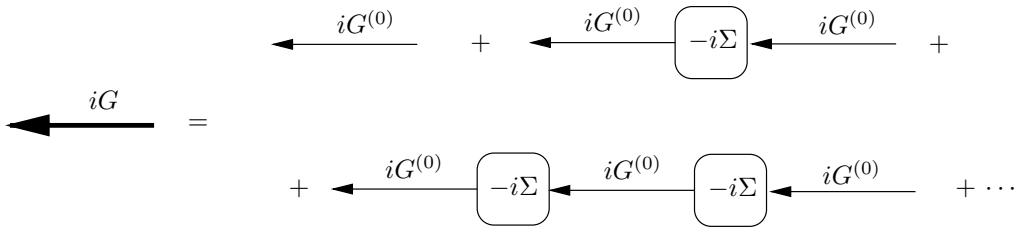

Figure 5: Series expansion of the Green's function in terms of self energy insertions.

$$G_{12} = \begin{pmatrix} G_{12}^{--} & G_{12}^{-+} \\ G_{12}^{+-} & G_{12}^{++} \end{pmatrix}, \quad \Sigma_{12} = \begin{pmatrix} \Sigma_{12}^{--} & \Sigma_{12}^{-+} \\ \Sigma_{12}^{+-} & \Sigma_{12}^{++} \end{pmatrix} \tag{11}$$

and can write the connection between Self Energy and the Green's function given by fig. (4) in the form

$$G_{12} = G_{12}^{(0)} + \int d^4X_3 d^4X_4 G_{14}^{(0)} \Sigma_{43} G_{32}. \tag{12}$$

Finally the advantages of this matrix method compared to the contour ordered formulism are:

1. Integration in Real time
2. Normal Diagrammatic rules for matrix Green's functions
3. Concrete functions as components of the matrices.

6 Summary

The problems of determining useful physical information in the Finite Temperature case were discussed and two formulations were presented to overcome these problems. Diagram techniques for calculating Green's functions were demonstrated and Self Energy was given as an example for the possibility to sum certain subsets of diagrams up to an infinite order and use them as building blocks for calculating physically relevant information. This can also be used as the starting point for selfconsistent calculations beyond perturbation theory. A well known example for such calculations is the Hartree-Fock method.

References

[Dan84] P. Danielewicz. Quantum Theory of Nonequilibrium Processes, I. Annals of Physics **152** (1984) 239

[Kel64] L. Keldysh. Diagram Technique for Nonequilibrium Processes. ZhETF **47** (1964) 1515. [Sov. Phys JETP **20** 1965 1018]

[LP81] E. Lifshitz, L. Pitaevsii. Physical Kinetics. Pergamon Press, (1981)

[LW87] N. P. Landsmann, C. G. van Weert. Real- and Imaginary-time Field Theory at Finite Temperature and Density. Physics Reports **145** (1987) 141

[Sch61] J. Schwinger. J. Math. Phys **2** (1961) 407

[Wei95] S. Weinberg. The Quantum Theory of Fields, vol. 1. Cambridge University Press, (1995)