Introduction to Relativistic Quantum Field Theory

Hendrik van Hees[1]
Goethe-Universität Frankfurt
Institut für Theoretische Physik
D-60438 Frankfurt am Main
Germany

October 21, 2023

[1]e-mail: hees@itp.uni-frankfurt.de

Contents

Preface

In the first chapter, we start with some facts known, from ordinary nonrelativistic quantum mechanics. We emphasise the picture of the evolution of quantum systems in space and time. The aim was to introduce the functional methods of path integrals on hand of the familiar framework of nonrelativistic quantum theory.

In this introductory chapter it was my goal to keep the story as simple as possible. Thus, all problems concerning operator ordering or interaction with electromagnetic fields were omitted. All these topics will be treated in terms of quantum field theory, beginning with the third chapter.

The second chapter is not yet written completely. It will be short and is intended to contain the vacuum many-body theory for nonrelativistic particles given as a quantum many-particle theory. It is shown that the same theory can be obtained by using the field-quantisation method (which was often called "the second quantisation", but on my opinion this is a very misleading term). I intend to work out the most simple applications to the hydrogen atom including bound states and exact scattering theory.

In the third chapter, we start with the classical principles of special relativity which are Lorentz covariance and the action principle in the covariant Lagrangian formulation, but we shall introduce only scalar fields to keep the stuff quite easy, since there is only one field degree of freedom. The classical part of the chapter ends with a discussion of Noether's theorem which is on the heart of our approach to observables which are defined from conserved currents caused by symmetries of space and time as well as by intrinsic symmetries of the fields.

After that introduction to classical relativistic field theory, we quantise the free fields, ending with a sketch about the nowadays well established facts of relativistic quantum theory: It is necessarily a many-body theory, because there is no possibility for a Schrödinger-like one-particle theory. The physical reason is simply the possibility of creation and annihilation of particle-antiparticle pairs (pair creation). It will come out that for a local quantum field theory the Hamiltonian of the free particles is bounded from below for the quantised field theory only if we quantise it with bosonic commutation relations. This is a special case of the famous spin-statistics theorem.

Then we show, how to treat ϕ^4 theory as the most simple example of an interacting field theory with help of perturbation theory, prove Wick's theorem and the LSZ-reduction formula. The goal of this chapter is a derivation of the perturbative Feynman-diagram rules. The chapter ends with the sad result that diagrams, which contain loops, correspond to integrals which do not exist since they are divergent. This difficulty is solved by renormalisation theory which will be treated later in these notes.

The fourth chapter starts with a systematic treatment of relativistic invariant theory using appendix B which contains the complete mathematical treatment of the representation theory of the Poincaré group, as far as it is necessary for physics. The most important result is the general proof of the spin-statistics theorem and the PCT theorem.

The rest of the chapter contains the foundations of path integrals for quantum field theories. Hereby, we shall find the methods helpful, which we have learnt in Chapter 1. This contains also the path integral formalism for fermions which needs a short introduction to the mathematics of Grassmann numbers.

After setting up these facts, we shall rederive the perturbation theory, which we have found with help of Wick's theorem in chapter 3 from the operator formalism. We shall use from the very beginning the diagrams as a very intuitive technique for book-keeping of the rather technically involved functional derivatives of the generating functional for Green's functions. On the other hand we shall also illustrate the „digram-less" derivation of the $\hbar$-expansion which corresponds to the number of loops in the diagrams.

We shall also give a complete proof of the theorems about generating functionals for subclasses of diagrams, namely the connected Green's functions and the proper vertex functions.

The chapter ends with the derivation of the Feynman rules for a simple toy theory involving a Dirac spin 1/2 Fermi field with the now completely developed functional (path integral) technique. As will come out quite straight forwardly, the only difference compared to the pure boson case are some sign rules for fermion lines and diagrams containing a closed fermion loop, coming from the fact that we have anticommuting Grassmann numbers for the fermions rather than commuting c-numbers for the bosons.

Chapter 5 is a detailed treatment of modern renormalisation theory. Here, we emphasise the calculation techniques, needed to calculate Feynman diagrams which have to be regularised in some way. I have chosen dimensional regularisation from the very beginning, because it leads to the most convenient treatment which is espacially true for the physically most important gauge field theories, about which we will learn in the later chapters of these notes. We will also prove Weinberg's theorem and the finiteness of the renormalised diagrams within the BPHZ formalism.

The sixth chapter is devoted to QED, including the most simple physical applications at tree-level. From the very beginning we shall take the gauge theoretical point of view. Gauge theories have proved to be the most important class of field theories, including the Standard Model of elementary particles. Thus, we use from the very beginning the modern techniques to quantise the theory with help of formal path integral manipulations, known as *Faddeev-Popov quantisation* in a certain class of covariant gauges. We shall also derive the very important *Ward-Takahashi identities*. As an especially useful gauge fixing we shall also formulate the background field gauge which is a manifestly gauge invariant procedure. We shall give the proof of renormalisability of QED in the background field gauge.

Chapter 7 contains a complete treatment of nonabelian gauge theories, including the notion of BRST invariance and renormalisability of these type of theories.

The appendix contains some mathematical material needed in the main parts.

Appendix A introduces some very basic facts about functionals and variational calculus.

Appendix B has grown a little lengthy, but on the other hand I think it is useful to write down all the stuff about the representation theory of the Poincaré groups. In a way it may be seen as a simplification of Wigner's famous paper from 1939.

Appendix C I hope the reader of my notes will have as much fun as I had when I wrote them!

Last but not least I come to the acknowledgements. First to mention are Robert Roth and Christoph Appel who gave me their various book style hackings for making it as nice looking as it is.

Also Thomas Neff has contributed by his patient help with all "mysteries" of the computer systems, I used at GSI, while I prepared this script.

Christoph Appel was always discussing with me about the hot topics of QFT, like, e.g., obtaining the correct symmetry factors of diagrams and the proper use of Feynman rules for various types of QFTs. He was also carefully reading the script and has corrected many spelling errors.

Literature

Finally I have to stress the fact that the lack of citations in these notes mean not that I claim that the contents are original ideas of mine. It was just my laziness in finding out all the references I used through my own tour through the literature and learning of quantum field theory.

I just cite some of the textbooks I found most illuminating during the preparation of these notes: For the fundamentals there exist a lot of textbooks of very different quality. For me the most important were [Wei95, Wei96, Sch14, BL86, Ram89, PS95, Kak93, Kug97]. Concerning gauge theories some of the clearest sources of textbook or review character are [Tay76, AL73, FLS72, Kug97, LZJ72a, LZJ72b, LZJ72c, Mut87]. One of the most difficult topics in quantum field theory is the question of renormalisation. Except the already mentioned textbooks, here I found the original papers very important, some of them are [BP57, Wei60, Zim68, Zim69, Zim70]. A very nice and concise monograph of this topic is [Col86]. Whenever I was aware of a URL with the full text of the paper, I cited it too, so that one can access these papers as easily as possible.

Chapter 1

Path Integrals

In this chapter we shall summarise some well known facts about nonrelativistic quantum mechanics in terms of path integrals, which where invented by Feynman in 1948 as an alternative formulation of quantum mechanics. It is thought to be an introduction to the tools of functional methods used in quantum field theory.

1.1 Quantum Mechanics

In this course we assume that the reader is familiar with quantum mechanics in terms of Dirac's bra- and ket formalism. We repeat the basic facts by giving some postulates about the structure of quantum mechanics which are valid in the nonrelativistic case as well as in the relativistic. In these notes, we emphasise that quantum theory is the description of physical systems in space and time. As we know, this picture is in some sense valid for a wider range of phenomena than the classical picture of particles and fields.

Although it is an interesting topic, we do not care about some problems with philosophy of quantum mechanics. In my opinion, the physicists have a well understood way in applying the formalism to phenomena in nature, and the problem of measurement is not of practical physical importance. That sight seems to be settled by all experimental tests of quantum theory, undertaken so far: They all show that quantum theory is correct in predicting and explaining the outcome of experiments with systems and there is no (practical) problem in interpreting the results from calculating "physical properties of systems" with help of the formalism given by the mathematical tool "quantum theory". So let us begin with the summary of the mathematical structure of quantum mechanics, as it is formulated in Dirac's famous book.

- Each quantum system is described completely by a ray in a Hilbert space $\mathscr{H}$. A ray is defined as the following equivalence class of vectors:

$$[|\psi\rangle] = \{c\,|\psi\rangle \,|\, |\psi\rangle \in \mathscr{H}, c \in \mathbb{C} \setminus \{0\}\}. \tag{1.1}$$

- The observables of the system are represented by selfadjoint operators $\mathbf{O}$ which build together with the unity operator an algebra of operators, acting on the Hilbert-space vectors. For instance, in the case of a quantised classical point particle this *algebra of observables* is built by the operators of the Cartesian components of position and (canonical) momentum

operators, which fullfil the *Heisenberg algebra*:

$$[\mathbf{x}_i, \mathbf{x}_k] = [\mathbf{p}_i, \mathbf{p}_k] = 0,\ [\mathbf{x}_i, \mathbf{p}_k] = \mathrm{i}\delta_{ik}\mathbf{1}. \tag{1.2}$$

Here and further on (except in cases when it is stated explicitly) we set Planck's constant $\hbar = 1$. In the next chapter, when we consider the relativistic theory, we shall also set the velocity of light $c = 1$ too. In this so called natural system of units, observables with the dimension of an action are dimensionless. Spatial distances and time intervals have the same unit, which is reciprocal to that of energy and momentum, and convenient unities in particle physics are MeV or GeV for energies, momenta and masses and fm for space or time intervals. The conversion within these units is given by the value $\hbar c \simeq 0.197$ GeV fm.

A possible result of a precise measurement of the observable O is necessarily an eigenvalue of the corresponding operator $\mathbf{O}$. Because $\mathbf{O}$ is selfadjoint, its eigenvalues are real, and the eigenvectors can be chosen such that they build a complete normalised set of mutually orthogonal vectors $|o, \alpha\rangle$. Here α is a set of discrete or continuous parameters, labeling the different eigenvectors of $\mathbf{O}$ with the same eigenvalue o. The orthonormalization condition reads

$$\left\langle o, \alpha \,\middle|\, o', \alpha' \right\rangle = \delta(o - o')\delta(\alpha - \alpha'), \tag{1.3}$$

where δ denotes either a Kronecker-δ symbol for discrete eigenvalues and parameters or a Dirac-δ distribution for continuous quantities. The completeness relation reads

$$\int \mathrm{d}o \int \mathrm{d}\alpha \, |o, \alpha\rangle \langle o, \alpha| = \mathbb{1}. \tag{1.4}$$

Here, the integrals may also denote sums for the case of decrete eigenvalues and/or parameters, α.

- If the system is prepared in a state, represented by a vector $|\psi\rangle$, fulfilling the the normalization condition $\langle \psi \,|\, \psi \rangle = 1$, then the probability to find the value o when measuring the observable O, is given by

$$P_\psi(o) = \int \mathrm{d}\alpha |\, \langle o, \alpha \,|\, \psi \rangle \,|^2. \tag{1.5}$$

This postulate is known as **Born's rule**.

The most famous consequence of this description of physical systems is Heisenberg's uncertainty relation, which follows from the positive definiteness of the scalar product in Hilbert space:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\mathbf{A}, \mathbf{B}] \rangle \right|. \tag{1.6}$$

Two observables can take simultaneously sharply determined values if and only if the corresponding operators commute. In this case both operators have a complete orthonormal basis of common eigenvectors.

A set of pairwise commutating observables is said to be complete if the simultaneous determination of the values of all these observables fixes the state of the system completely, i.e., if the simultaneous eigenspaces of these operators are one-dimensional (nondegenerate).

- Time is represented by a real parameter. There is a hermitian operator $\mathbf{H}$, associated with the system such that, if $\mathbf{O}$ is an observable, then

$$\mathring{\mathbf{O}} = \frac{1}{\mathrm{i}} [\mathbf{O}, \mathbf{H}] + \partial_t \mathbf{O} \tag{1.7}$$

is the operator representing the time derivative of this observable.

The partial time derivative is only for the explicit time dependence. The fundamental operators like space and momentum operators, which form a complete generating system of the algebra of observables, are not explicitly time dependent (by definition!). It should be emphasised that $\mathring{\mathbf{O}}$ is usually not the mathematical total derivative with respect to time. We shall see that the mathematical dependence on time is arbitrary in a wide sense, because, if we have a description of quantum mechanics, then we are free to redefine the operators and state kets by a time dependent unitary transformation without changing any physical prediction (probabilities, mean values of observables etc.).

- Due to our first assumption, the state of the quantum system is completely known, if we know a state ket $|\psi\rangle$ lying in the ray $[|\psi\rangle]$, in which the system is prepared in, at an arbitrary initial time. This preparation of a system is possible by the simultaneous determination of the values of a complete set of compatible observables.

  It is more convenient to have a description of the state in terms of Hilbert-space quantities than in terms of the projective space (built by the above defined rays). It is easy to see that the state is uniquely given by the projection operator

  $$\mathbf{P}_{|\psi\rangle} = \frac{|\psi\rangle \langle\psi|}{\||\psi\rangle\|^2}, \tag{1.8}$$

  with $|\psi\rangle$ an arbitrary ket contained in the ray (i.e., the state the system is in).

- In general, especially if we like to describe macroscopic systems within quantum mechanics, we do not know the state of the system completely. In this case, we can describe the system by a selfadjoint statistical operator $\boldsymbol{\rho}$ which is positive semidefinite (that means that for all kets $|\psi\rangle \in \mathscr{H}$ we have $\langle\psi|\boldsymbol{\rho}|\psi\rangle \geq 0$) and fulfills the normalisation condition $\operatorname{Tr}\boldsymbol{\rho} = 1$.

  The trace of an operator is defined with help of a complete set of orthonormal vectors $|n\rangle$ as $\operatorname{Tr}\boldsymbol{\rho} = \sum_n \langle n|\boldsymbol{\rho}|n\rangle$. The mean value of any operator $\mathbf{O}$ is given by $\langle\mathbf{O}\rangle = \operatorname{Tr}(\mathbf{O}\boldsymbol{\rho})$.

  The last assumption of quantum theory is that the statistical operator is associated with the system at all times. This implies the von Neumann equation,

  $$\mathring{\boldsymbol{\rho}} = \frac{1}{\mathrm{i}}[\boldsymbol{\rho}, \mathbf{H}] + \partial_t \boldsymbol{\rho} = 0. \tag{1.9}$$

  This equation is also valid for the special case that the system is prepared in a pure state, i.e., for $\boldsymbol{\rho} = \mathbf{P}_{|\psi\rangle}$.

1.2 Choice of the Picture

Now, having briefly summarised how quantum mechanics works, we like to give the time evolution a mathematical content, i.e., we settle the time dependence of the operators and states describing the system. As mentioned above, it is in a wide range arbitrary, how this time dependence is chosen. The only observable facts about the system are expectation values of its observables, so they should have a unique time evolution. To keep the story short, we formulate the result as a theorem and prove afterwards that it gives really the right answer. Each special choice of the mathematical time dependence of observables and state kets, that is consistent with the above given postulates of quantum mechanics, is called a *picture* of the time evolution. Now, we can state

Theorem 1. *The picture of quantum mechanics is uniquely determined by the choice of an arbitrary selfadjoint operator* $\mathbf{X}$ *which can be a local function of time. Local means in this context that it depends only on one time, so to say the time point "now" and not (as could be consistent with the causality property of physical laws) on the whole past of the system.*

This operator is the generator of the time evolution of the fundamental operators of the system. This means that it determines the unitary time evolution operator $\mathbf{A}(t,t_0)$ *of the observables by the initial-value problem*

$$\mathrm{i}\partial_t \mathbf{A}(t,t_0) = -\mathbf{X}(t)\mathbf{A}(t,t_0), \quad \mathbf{A}(t_0,t_0) = \mathbf{1} \tag{1.10}$$

such that for all observables, which do not depend explicitly on time,

$$\mathbf{O}(t) = \mathbf{A}(t,t_0)\mathbf{O}(t_0)\mathbf{A}^\dagger(t,t_0). \tag{1.11}$$

Then, the generator of the time evolution of the states is necessarily given by the selfadjoint operator $\mathbf{Y} = \mathbf{H} - \mathbf{X}$*, where* $\mathbf{H}$ *is the Hamiltonian of the system. This means that the unitary time evolution operator of the states is given by*

$$\mathrm{i}\partial_t \mathbf{C}(t,t_0) = +\mathbf{Y}(t)\mathbf{C}(t,t_0). \tag{1.12}$$

Proof. The proof of the theorem is not too difficult. At first one sees easily that all the laws given by the axioms like commutation rules (which are determined by the physical meaning of the observables due to symmetry requirements as will be shown later on) or the connection between states and probabilities is not changed by applying different unitary transformations to states and observables.

So there are only two statements to show: First we have to assure that the equation of motion for the time evolution operators is consistent with the time evolution of the entities themselves and second we have to show that this mathematics is consistent with the axioms concerning "physical time evolution" above, especially that the time evolution of expectation values of observables is unique and independent of the choice of the picture.

For the first task, let us look at the time evolution of the operators. Because the properties of the algebra, given by sums or series of products of the fundamental operators, especially their commutation rules, should not change with time, the time evolution has to be a linear transformation of operators, i.e., $\mathbf{O} \to \mathbf{A}\mathbf{O}\mathbf{A}^{-1}$, with an invertible linear operator $\mathbf{A}$ on Hilbert space. Because the observables are represented by selfadjoint operators, this property has to be preserved during the time evolution, leading to the constraint that $\mathbf{A}$ has to be unitary, i.e., $\mathbf{A}^{-1} = \mathbf{A}^\dagger$.

Now, for $t > t_0$, the operator $\mathbf{A}$ should be a function of t and t_0 only. Let us suppose the operators evolved with time from a given initial setting at t_0 to time $t_1 > t_0$ by the evolution operator $\mathbf{A}(t_0,t_1)$. Now, we can take the status of these operators at time t_1 as a new initial condition for their further time development to a time t_2. This is given by the operator $\mathbf{A}(t_1,t_2)$. On the other hand, the evolution of the operators from t_0 to t_2 should be given directly by the transformation with the operator $\mathbf{A}(t_0,t_2)$. One can easily see that this long argument can be simply written mathematically as the consistency condition:

$$\forall t_0 < t_1 < t_2 \in \mathbb{R}: \ \mathbf{A}(t_2,t_1)\mathbf{A}(t_1,t_0) = \mathbf{A}(t_2,t_0), \tag{1.13}$$

i.e., in short words: The time evolution from t_0 to t_1 and then from t_1 to t_2 is the same as the evolution directly from t_0 to t_2.

Now from unitarity of $\mathbf{A}(t,t_0)$ one concludes:

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{1} = \text{const.} \Rightarrow (\mathrm{i}\partial_t \mathbf{A})\mathbf{A}^\dagger = \mathbf{A}\partial_t(\mathrm{i}\mathbf{A})^\dagger, \tag{1.14}$$

so that the operator $\mathbf{X} = -\mathrm{i}(\partial_t \mathbf{A})\mathbf{A}^\dagger$ is indeed selfadjoint: $\mathbf{X}^\dagger = \mathbf{X}$. Now using eq. (1.13) one can immediately show that

$$[\mathrm{i}\partial_t \mathbf{A}(t,t_0)]\mathbf{A}^\dagger(t,t_0) = [\mathrm{i}\partial_t \mathbf{A}(t,t_1)]\mathbf{A}^\dagger(t,t_1) := -\mathbf{X}(t) \tag{1.15}$$

which in turn shows that $\mathbf{X}(t)$ does not depend on the initial time t_0, i.e., it is really local in time as stated in the theorem. Thus, the first task is done since the proof for the time evolution operator of the states is exactly the same: The assumption of a generator $\mathbf{X}(t)$ resp. $\mathbf{Y}(t)$ which is local in time is consistent with the initial value problems defining the time evolution operators by their generator.

Now, the second task, namely to show that this description of time evolution is consistent with the above mentioned axioms, is done without much sophistication. From $\mathbf{O}(t) = \mathbf{A}(t,t_0)\mathbf{O}(t_0)\mathbf{A}^\dagger(t,t_0)$ together with the definition (1.10) one obtains for an operator which may depend on time:

$$\frac{\mathrm{d}\mathbf{O}(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}}[\mathbf{O}(t), \mathbf{X}(t)] + \partial_t \mathbf{O}(t). \tag{1.16}$$

This equation can be written with help of the "physical time derivative" (1.7) in the following form:

$$\frac{\mathrm{d}\mathbf{O}(t)}{\mathrm{d}t} = \mathring{\mathbf{O}} - \frac{1}{\mathrm{i}}[\mathbf{O}, \mathbf{H} - \mathbf{X}]. \tag{1.17}$$

One sees that the eqs. (1.16) and (1.17) together with given initial values for an operator $\mathbf{O}$ at time t_0 are uniquely solved by applying a unitary time evolution operator which fulfils Eq. (1.10).

Now, the statistical operator $\boldsymbol{\rho}$ fulfils these equations of motion as any operator. But by the axiom (1.9), we conclude from (1.17)

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = -\frac{1}{\mathrm{i}}[\boldsymbol{\rho}(t), \mathbf{Y}(t)], \tag{1.18}$$

and this equation is solved uniquely by a unitary time evolution with the operator $\mathbf{C}$ fulfilling (1.12).

Q.E.D.

It should be emphasised that this evolution takes only into account the time dependence of the operators which comes from their dependence on the fundamental operators of the algebra of observables. It does not consider an explicit time dependence. The statistical operator is always time dependent. The only very important exception is the case of thermodynamical equilibrium where the statistical operator is a function of the constants of motion.

Now, we have to look at the special case that we have full quantum theoretical information about the system. Then we know that this system is in a pure state, given by $\rho = \mathbf{P}_{|\psi\rangle} = |\psi\rangle\langle\psi|$ (where $|\psi\rangle$ is normalised). It is clear that for this special statistical operator the general eq. (1.18) and from that (1.12) is still valid. It follows immediately, that up to an unimportant phase factor the state ket evolves with time by the unitary transformation

$$|\psi, t\rangle = \mathbf{C}(t,t_0)\,|\psi, t_0\rangle\,. \tag{1.19}$$

From this, one sees that the normalisation of $|\psi, t\rangle$ is 1, if the ket was renormalised at the initial time t_0. The same holds true for a general statistical operator, i.e., $\operatorname{Tr}\boldsymbol{\rho}(t) = \operatorname{Tr}\boldsymbol{\rho}(t_0)$ (exercise: show this by calculating the trace with help of a complete set of orthonormal vectors).

1.3 Formal Solution of the Equations of Motion

Now we want to integrate the equations of motion for the time-evolution operators formally. Let us do this for the case of $\mathbf{A}$, introduced in (1.11). Its equation of motion, which we like to solve now, is given by (1.10).

The main problem comes from the fact that the selfadjoint operator $\mathbf{X}(t)$, generating the time evolution, depends in general on time t, and operators at different times need not commute. Because of this fact we can not solve the equation of motion like the analogous differential equation with functions, having values in $\mathbb{C}$.

At first, we find by integration of (1.10) with help of the initial condition $\mathbf{A}(t_0, t_0) = \mathbf{1}$ an integral equation which is equivalent to the initial-value problem (1.10):

$$\mathbf{A}(t, t_0) = \mathbf{1} + \mathrm{i} \int_{t_0}^{t} \mathrm{d}\tau \mathbf{X}(\tau)\mathbf{A}(\tau, t_0). \tag{1.20}$$

The form of this equation leads us to solve it by defining the following iteration scheme.

$$\mathbf{A}_n(t, t_0) = \mathbf{1} + \mathrm{i} \int_{t_0}^{t} \mathbf{X}(\tau)\mathbf{A}_{n-1}(\tau, t_0)\mathrm{d}\tau, \ \mathbf{A}_0(t, t_0) = \mathbf{1}. \tag{1.21}$$

The solution of the equation should be given by taking the limit $n \to \infty$. We will not think about the convergence because this is a rather difficult task and, as far as I know, yet unsolved problem.

One can prove by induction that the formal solution is given by the series

$$\begin{aligned} \mathbf{A}(t, t_0) &= \sum_{k=0}^{\infty} \mathbf{A}^{(k)}(t, t_0) \text{ with} \\ \mathbf{A}^{(k)}(t, t_0) &= \int_{t_0}^{t} \mathrm{d}\tau_1 \int_{t_0}^{\tau_1} \mathrm{d}\tau_2 \ldots \int_{t_0}^{\tau_{k-1}} \mathrm{d}\tau_k \mathbf{X}(\tau_1)\mathbf{X}(\tau_2)\ldots\mathbf{X}(\tau_k). \end{aligned} \tag{1.22}$$

To bring this series in a less complicated form, let us first look at $\mathbf{A}^{(2)}(t, t_0)$:

$$\int_{t_0}^{t} \mathrm{d}\tau_1 \int_{t_0}^{\tau_1} \mathrm{d}\tau_2 \mathbf{X}(\tau_1)\mathbf{X}(\tau_2). \tag{1.23}$$

The range of the integration variables is the triangle in the $\tau_1\tau_2$-plane shown at figure 1.1:

Using Fubini's theorem we can interchange the integrations

$$\mathbf{A}^{(2)} = \int_{t_0}^{t} \mathrm{d}\tau_1 \int_{\tau_1}^{t} \mathrm{d}\tau_2 \mathbf{X}(\tau_2)\mathbf{X}(\tau_1). \tag{1.24}$$

A glance at the operator ordering in (1.23) and (1.24) shows that it is such that the operator at the later time is always on the left. For this, one introduces the *causal time ordering operator* T_c, invented by Dyson. With help of T_c, one can add both equations, leading to the result

$$2\mathbf{A}^{(2)}(t, t_0) = T_c \int_{t_0}^{t} \mathrm{d}\tau_1 \int_{t_0}^{t} \mathrm{d}\tau_2 \mathbf{X}(\tau_1)\mathbf{X}(\tau_2). \tag{1.25}$$

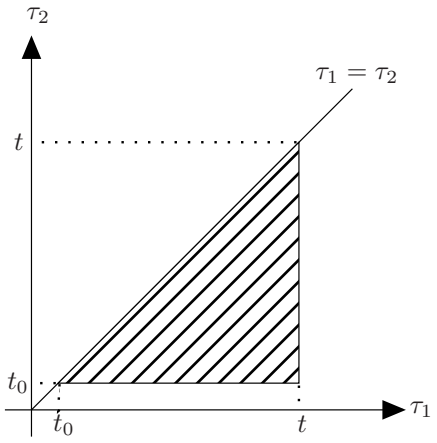


Figure 1.1: *Range of integration variables in (1.23)*

We state that this observation holds for the general case of an arbitrary summand in the series (1.22), i.e.,

$$\mathbf{A}^{(k)}(t,t_0) = \frac{1}{k!} T_c \int_{t_0}^{t} \mathrm{d}\tau_1 \cdots \int_{t_0}^{t} \mathrm{d}\tau_n \mathbf{X}(\tau_1) \cdots \mathbf{X}(\tau_n). \tag{1.26}$$

To prove this assumption, we use an induction argument. Assume the assumption is true for $k = n-1$ and look at the nth summand of the series. Because the assumption is true for $k = n-1$, we can apply it to the $n-1$ inner integrals:

$$\mathbf{A}^{(n)}(t,t_0) = \frac{1}{(n-1)!} T_c \int_{t_0}^{t} \mathrm{d}\tau_1 \int_{t_0}^{\tau_1} \mathrm{d}\tau_2 \cdots \int_{t_0}^{\tau_1} \mathrm{d}\tau_n \mathbf{X}(\tau_1) \cdots \mathbf{X}(\tau_n). \tag{1.27}$$

Now we can do the same calculation as we did for $A^{(2)}$ with the outer integral and one of the inner ones. Adding all the possibilities of pairing and dividing by n one gets immediately

$$\mathbf{A}^{(n)}(t,t_0) = \frac{1}{n!} T_c \int_{t_0}^{t} \mathrm{d}\tau_1 \cdots \int_{t_0}^{t} \mathrm{d}\tau_n \mathbf{X}(\tau_1) \cdots \mathbf{X}(\tau_n), \tag{1.28}$$

and that is (1.26) for $k = n$. So, our assumption is proven by induction.

With this little combinatorics we can write the series formally as

$$\mathbf{A}(t,t_0) = T_c \exp\left[\mathrm{i} \int_{t_0}^{t} \mathrm{d}\tau \mathbf{X}(\tau)\right]. \tag{1.29}$$

This is the required formal solution of the equation of motion. For the operator $\mathbf{C}(t,t_0)$ one finds the solution by the same manipulations to be:

$$\mathbf{C}(t,t_0) = T_c \exp\left[-\mathrm{i} \int_{t_0}^{t} \mathrm{d}\tau \mathbf{Y}(\tau)\right]. \tag{1.30}$$

1.4 Example: The Free Particle

The most simple example is the free particle. For calculating the time development of quantum mechanical quantities, we chose the Heisenberg picture defined in terms of the above introduced

time evolution operators $\mathbf{X} = \mathbf{H}$ and $\mathbf{Y} = 0$. We take as an example a free point particle moving in one-dimensional space. The fundamental algebra is given by the space and the momentum operator which fulfil the Heisenberg algebra

$$\frac{1}{\mathrm{i}}\,[\mathbf{x}, \mathbf{p}] = \mathbf{1}, \tag{1.31}$$

which follows from the rules of canonical quantisation from the Poisson bracket relation in Hamiltonian mechanics or from the fact that the momentum is defined as the generator of translations in space.

As said above, in the Heisenberg picture only the operators, representing observables, depend on time, and the states are time independent. To solve the problem of time evolution we can solve the operator equations of motion for the fundamental operators rather than solving the equation for the time evolution operator. The Hamiltonian for the free particle is given by

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m}, \tag{1.32}$$

where m is the mass of the particle. The operator equations of motion can be obtained from the general rule (1.16) with $\mathbf{X} = \mathbf{H}$:

$$\frac{d\mathbf{p}}{dt} = \frac{1}{\mathrm{i}}\,[\mathbf{p}, \mathbf{H}] = 0, \;\; \frac{d\mathbf{x}}{dt} = \frac{1}{\mathrm{i}}\,[\mathbf{x}, \mathbf{H}] = \frac{\mathbf{p}}{m}. \tag{1.33}$$

This looks like the equation for the classical case, but it is an operator equation. But in our case that doesn't affect the solution which is given in the same way as the classical one by

$$\mathbf{p}(t) = \mathbf{p}(0) = \text{const}, \;\; \mathbf{x}(t) = \mathbf{x}(0) + \frac{\mathbf{p}}{m}t. \tag{1.34}$$

Here, without loss of generality, we have set t_0=0.

Now let us look on the time evolution of the wave function given as the matrix elements of the state ket and a complete set of orthonormal eigenvectors of observables. We emphasise that the time evolution of such a wave function is up to a phase independent of the choice of the picture. So we may use any picture we like to get the answer. Here, we use the Heisenberg picture, where the state ket is time independent. The whole time dependence comes from the eigenvectors of the observables. As a first example we take the momentum eigenvectors and calculate the wave function in the momentum representation. From (1.33) we get up to a phase:

$$|p, t\rangle = \exp(\mathrm{i}\mathbf{H}t)\,|p, 0\rangle = \exp\left(\mathrm{i}\frac{p^2}{2m}t\right)|p, 0\rangle\,, \tag{1.35}$$

and the time evolution of the wave function is simply

$$\psi(p, t) = \langle p, t \,|\, \psi \rangle = \exp\left(-\mathrm{i}\frac{p^2}{2m}t\right)\psi(p, 0). \tag{1.36}$$

This can be described by the operation of an integral operator in the form

$$\psi(p, t) = \int \mathrm{d}p' \underbrace{\langle p, t \,|\, p', 0 \rangle}_{U(t,p;0,p')} \langle p', 0 \,|\, \psi \rangle = \int \mathrm{d}p' U(t, p; 0, p')\psi(p', 0). \tag{1.37}$$

From (1.34) one finds

$$U(t,p,0,p') = \exp\left(-\mathrm{i}\frac{p^2}{2m}t\right)\delta(p-p'). \tag{1.38}$$

It should be kept in mind from this example that the time evolution kernels or *propagators* which define the time development of wave functions are in general distributions rather than functions.

The next task, we like to solve, is the propagator in the configuration-space representation of the wave function. We will give two approaches: First we start anew and calculate the space eigenvectors from the solution of the operator equations of motion (1.34). We have by definition:

$$\mathbf{x}(t)\,|x,t\rangle = \left(\mathbf{x}(0) + \frac{\mathbf{p}(0)}{m}t\right)|x,t\rangle = x\,|x,t\rangle\,. \tag{1.39}$$

Multiplying this with $\langle x',0|$ we find by using the representation of the momentum operator in space representation $\mathbf{p} = 1/\mathrm{i}\partial_x$:

$$(x'-x)\,\langle x',0\,|\,x,t\,\rangle = \frac{\mathrm{i}t}{m}\partial_{x'}\,\langle x',0\,|\,x,t\,\rangle \tag{1.40}$$

which is solved in a straight forward way:

$$U(t,x;0,x')^* = \langle x',0\,|\,x,t\,\rangle = N\exp\left[-\mathrm{i}\frac{m}{2t}(x'-x)^2\right]. \tag{1.41}$$

Now we have to find the complex normalisation factor N. It is given by the initial condition

$$U(0,x;0,x') = \delta(x-x') \tag{1.42}$$

which also determines its phase and the fact that U is the matrix element of a unitary operator:

$$\int \mathrm{d}x' U(t,x_1;0,x')U^*(t,x_2;0,x') = \delta(x_1-x_2). \tag{1.43}$$

Using (1.41), this integral gives

$$\int \mathrm{d}x' U(t,x_1;0,x')U^*(t,x_2;0,x') = |N|^2\frac{2\pi t}{m}\delta(x_1-x_2), \quad \to N = \sqrt{\frac{m}{2\pi t}}\exp(\mathrm{i}\alpha). \tag{1.44}$$

To determine the phase, we use (1.42). It is most simple to fold U with an arbitrary L^2 function, for which we choose the Gaussian $\exp(-ax^2)$:

$$\int \mathrm{d}x' U(t,x;0,x')\exp(-ax'^2) = \sqrt{\frac{1}{2at/m-\mathrm{i}}}\exp\left(-\frac{amx^2}{m+2\mathrm{i}at}+\mathrm{i}\alpha\right). \tag{1.45}$$

The square root is understood as its principle value, i.e., for $t>0$ it has a positive imaginary part. Now taking the limit $t\to 0_+$ it becomes $\exp(+\mathrm{i}\pi/4)$ and thus we must have $\alpha = -\pi/4$, yielding the final result or the propagator:

$$U(t,x;0,x') = \sqrt{\frac{m}{2\pi\mathrm{i}t}}\exp\left[\frac{\mathrm{i}m}{2t}(x-x')^2\right]. \tag{1.46}$$

An alternative possibility to get this result is to use the momentum space result and transform it to space representation. We leave this nice calculation as an exercise for the reader. For help we give the hint that again one has to regularise the distribution to give the resulting Fourier integral a proper meaning.

1.5 The Feynman-Kac Formula

Now we are at the right stage for deriving the path integral formalism of quantum mechanics. In these lectures we shall often switch between operator formalism and path integral formalism. We shall see that both approaches to quantum theory have their own advantages and disadvantages. The operator formalism is quite nice to see the unitarity of the time evolution. On the other hand the canonical quantisation procedure needs the Hamiltonian formulation of classical mechanics to define Poisson brackets which can be mapped to commutators in the quantum case. This is very inconvenient for the relativistic case because we have to treat the time variable in a different way than the space variables. So the canonical formalism hides relativistic invariance leading to non covariant rules at intermediate steps. Relativistic invariance will be evident at the very end of the calculation.

Additional to this facts which are rather formal we shall like to discuss gauge theories like electrodynamics or the standard model. The quantisation of theories of that kind is not so simple to formulate in the operator formalism but the path integral is rather nice to handle. It is also convenient to use functional methods to derive formal properties of quantum field theories as well as such practical important topics like Feynman graphs for calculating scattering amplitudes perturbatively.

In this section we shall take a closer look on path integrals applied to nonrelativistic quantum mechanics.

For sake of simplicity we look again on a particle in one configuration space dimension moving in a given potential V. Again we want to calculate the time evolution kernel $U(t', x'; t, x)$ which was given in the previous chapter in terms of the Heisenberg picture space coordinate eigenstates:

$$\left\langle x', t' \,\middle|\, x, t \right\rangle = \left\langle x', 0 \left| \exp[-\mathrm{i}\mathbf{H}(t'-t)] \right| x, 0 \right\rangle \tag{1.47}$$

where we have used the solution of the equation of motion for Hamiltonian which is explicitly time independent, i.e. in the Heisenberg picture it is a function of the fundamental operators, here taken as $\mathbf{x}$ and $\mathbf{p}$ alone. We consider at the moment the most simple case which in fact covers a wide range of application in the case of nonrelativistic quantum mechanics:

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}). \tag{1.48}$$

We will take into account more general cases later. The idea behind our derivation of the path integral formula is quite simple. Because we know the time evolution explicitly for very small time intervals, namely it is given by the Hamiltonian, it seems to be sensible to divide the time interval (t,t') in N equal pieces (in the following called time slices) of length $\Delta t = (t'-t)/N$. Since the Hamiltonian is not explicitly time dependent which means in the Heisenberg picture that it is a constant of motion (see eq. (1.16) and keep in mind that in the Heisenberg picture we have by definition $\mathbf{X} = \mathbf{H}$) we can write the time evolution operator in the "time sliced" form

$$\exp[-\mathrm{i}\mathbf{H}(t'-t)] = \underbrace{\exp(-\mathrm{i}\mathbf{H}\Delta t)\exp(-\mathrm{i}\mathbf{H}\Delta t)\ldots\exp(-\mathrm{i}\mathbf{H}\Delta t)}_{N \text{ times}}. \tag{1.49}$$

Now there is the problem that $\mathbf{x}$ and $\mathbf{p}$ are not commuting. But one can show easily, that there holds the following formula

$$\exp[\lambda(\mathbf{A}+\mathbf{B})] = \exp\lambda\mathbf{A}\exp\lambda\mathbf{B} + O(\lambda^2) \tag{1.50}$$

by expanding both sides of the equation in orders of λ.

From this we can hope that with $N \to \infty$ the error made by splitting the time evolution operator in the form

$$\exp(-\mathrm{i}\Delta t\mathbf{H}) = \exp\left(-\mathrm{i}\Delta t\frac{\mathbf{p}^2}{2m}\right)\exp[-\mathrm{i}\Delta t V(\mathbf{x})] + O(\Delta t^2) \tag{1.51}$$

and neglecting the terms of order Δt^2 becomes negligible. Now splitting the time evolution operator in this way we may put a unity operator in between which is written as the spectral representation $\int \mathrm{d}x\,|x\rangle\langle x|$ or $\int \mathrm{d}p\,|p\rangle\langle p|$ in the following way:

$$\begin{aligned} U(t',x';t,x) &= \int \mathrm{d}p_1\ldots\mathrm{d}p_N\mathrm{d}x_2\ldots\mathrm{d}x_N \times \\ &\times \left\langle x'\left|\exp\left(-\mathrm{i}\Delta t\frac{\mathbf{p}^2}{2m}\right)\right|p_N\right\rangle \langle p_N\,|\exp[-\mathrm{i}\Delta t V(\mathbf{x})]|\,x_N\rangle \times \\ &\times \left\langle x_N\left|\exp\left(-\mathrm{i}\Delta t\frac{\mathbf{p}^2}{2m}\right)\right|p_{N-1}\right\rangle \langle p_{N-1}\,|\exp[-\mathrm{i}\Delta t V(\mathbf{x})]|\,x_{N-1}\rangle \times \\ &\times \cdots \times \\ &\times \left\langle x_2\left|\exp\left(-\mathrm{i}\Delta t\frac{\mathbf{p}^2}{2m}\right)\right|p_1\right\rangle \langle p_1\,|\exp(-\mathrm{i}\Delta t V)|\,x\rangle\,, \end{aligned} \tag{1.52}$$

where, in the latter sum, we have set $x_1 = x'$ and $x_{N+1} = x$. Now the two different sorts of matrix elements arising in this expression are trivially calculated to be

$$\left\langle x_{k+1}\left|\exp\left(-\mathrm{i}\Delta t\frac{\mathbf{p}^2}{2m}\right)\right|p_k\right\rangle = \exp\left(-\mathrm{i}\Delta t\frac{p_k^2}{2m}\right)\frac{\exp(\mathrm{i}p_k x_{k+1})}{\sqrt{2\pi}} \tag{1.53}$$

$$\langle p_k\,|\exp(-\mathrm{i}\Delta t V)|\,x_k\rangle = \exp[-\mathrm{i}\Delta t V(x_k)]\frac{\exp(-\mathrm{i}x_k p_k)}{\sqrt{2\pi}}, \tag{1.54}$$

where we have used the correctly "normalised" momentum eigenstate in the configuration-qspace representation:

$$\langle x\,|\,p\rangle = \frac{\exp(\mathrm{i}xp)}{\sqrt{2\pi}}. \tag{1.55}$$

Putting all this together we obtain the time sliced form of the path integral formula which is named after its inventors Feynman-Kac formula:

$$\begin{aligned} U(t',x';t,x) &= \lim_{N\to\infty}\int \mathrm{d}p_1\ldots\mathrm{d}p_N\mathrm{d}x_2\ldots\mathrm{d}x_{N-1} \times \\ &\times \left(\frac{1}{2\pi}\right)^N \exp\left[-\mathrm{i}\Delta t\sum_{k=1}^{N}\left(\frac{p_k^2}{2m}+V(x_k)\right)+\mathrm{i}\sum_{k=1}^{N}p_k(x_{k+1}-x_k)\right], \end{aligned} \tag{1.56}$$

Now we interpret this result in another way than we have obtained it. The pairs (x_k, p_k) together can be seen as a discrete approximation of a path in phase space parametrised by the time. The mentioned points are defined to be $(x(t_k), p(t_k))$ on the path. Then the sum in the argument of the exponential function is an approximation for the following integral along the given path:

$$\int_t^{t'} \mathrm{d}t\left[-H(x,p)+p\frac{dx}{dt}\right]. \tag{1.57}$$

Now we should remember that we have fixed the endpoints of the path in configuration space to be $x(t) = x$ and $x(t') = x'$. So we have the following interpretation of the Feynman-Kac

formula after taking the limit $N \to \infty$: The time evolution kernel $U(x', t'; x, t)$ is the sum of the functional $\exp(\mathrm{i}S[x, p])$ over all paths beginning at time t at the point x ending at the point x' at time t'. For the momenta there is no boundary condition at all. This is quite o.k., because we have no restriction on the momenta. Because of the uncertainty relation it does not make any sense to have such conditions on both x and p at the same time! The action S is here seen as a functional depending on the paths in phase space which fulfil this boundary conditions:

$$S[x, p] = \int_t^{t'} \mathrm{d}t \left[p\frac{dx}{dt} - H(x, p) \right]. \tag{1.58}$$

We conclude that the formula (1.56) may be taken as the definition of the continuum limit of the path integral, written symbolically as

$$U(t', x'; t, x) = \int_{(t,x)}^{(t',x')} \mathrm{D}p\mathrm{D}x \exp\left\{\mathrm{i}S[x, p]\right\}. \tag{1.59}$$

The physical interpretation is now quite clear: The probability that the particle known to be at time t exactly at the point x is at time t' exactly at the point x' is given with help of the time evolution kernel in space representation as $|U(t', x'; t, x)|^2$ and the amplitude is given as the coherent sum over all paths with the correct boundary conditions. All paths in phase space contribute to this sum. Because the boundary space points x and x' are exactly fixed at the given times t and t' respectively it is quantum mechanically impossible to know anything about the momenta at this times. Because of that typical quantum mechanical feature there are no boundary conditions for the momenta in the path integral!

Now let us come back to the discretised version (1.56) of the path integral. Since the Hamiltonian is quadratic in p the same holds for the p-dependence of the exponential in this formula. So the p-integrals can be calculated exactly. As seen above we have to regularise it by giving the time interval Δt a negative imaginary part which is to be tent to zero after the calculation. For one of the momentum integrals this now familiar procedure gives the result

$$I_k = \int \mathrm{d}p_k \exp\left[-\mathrm{i}\Delta t \frac{p_k^2}{2m} + \mathrm{i}p_k(x_k - x_{k-1})\right] = \sqrt{\frac{2\pi m}{\mathrm{i}\Delta t}} \exp\left[\frac{\mathrm{i}m(x_k - x_{x-1})^2}{2\Delta t}\right]. \tag{1.60}$$

Inserting this result in eq. (1.56) we find the configuration space version of the path integral formula:

$$U(t', x'; t, x) = \lim_{N\to\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \sqrt{\frac{m}{2\pi\mathrm{i}\Delta t}}^N \exp\left\{\mathrm{i}\sum_{k=1}^{N} \left[\frac{m(x_k - x_{k-1})^2}{2\Delta t} - V(x_i)\Delta t\right]\right\}. \tag{1.61}$$

As above we can see that this is the discretised version of the path integral

$$U(t', x'; t, x) = \int_{t,x}^{t',x'} \mathrm{D}'x \exp\{\mathrm{i}S[x]\}, \tag{1.62}$$

where we now obtained $S[x] = \int_t^{t'} \mathrm{d}tL$, i.e. the action as a functional of the path in configuration space. The prime on the path integral measure is to remember that there are the square root factors in (1.61).

With that manipulation we have obtained an important feature of the path integral: It is a description which works with the Lagrangian version of classical physics rather than with the

Hamiltonian form. This is especially convenient for relativistic physics, because then the Hamiltonian formalism is not manifestly covariant.

It was Feynman who invented the path integrals in 1942 in his Ph.D. thesis. Later on he could use it as a tool for finding the famous Feynman graphs for perturbative QED which we shall derive later in our lectures. That Feynman graphs give a very suggestive picture of the scattering processes of the particles due to electromagnetic interaction among them. In the early days of quantum field theory Schwinger and Feynman wondered why they obtained the same results in QED. Schwinger was using his very complicated formal field operator techniques and Feynman his more or less handwaving graphical arguments derived from his intuitive space-time picture. Later on Dyson derived the Feynman rules formally from the canonical quantisation of classical electrodynamics and that was the standard way getting the rules for calculating scattering cross sections etc. With the advent of non-Abelian gauge theories in the late fifties and their great breakthrough in the early seventies (electro weak theory, renormalisability of gauge theories) this has changed completely: Nowadays the path integral formalism is the standard way to obtain the content of the theory for all physicists who are interested in theoretical many body quantum physics.

After this little historical sideway let us come back to the path integrals themselves. Now it is time to get some feeling for it by applying it to the most simple nontrivial example which can be calculated in a closed form: The harmonic oscillator.

1.6 The Path Integral for the Harmonic Oscillator

The harmonic oscillator is defined by the Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2. \tag{1.63}$$

The corresponding Hamiltonian is quadratic not only in p but also in x. This is the reason, why we can calculate the path integral exactly in this case. We will use the discretised version (1.61) of the configuration space path integral.

The biggest problem is the handling of the boundary conditions of the path. Fortunately this problem can be solved by parameterising the path relative to the classical one defined as that path which extremises the action $S[x]$:

$$\left.\frac{\delta S[x]}{\delta x}\right|_{x=x_{\mathrm{cl}}} = 0 \text{ with } x(t) = x,\ x(t') = x'. \tag{1.64}$$

Since the action is quadratic it can be expanded around the classical path and the series will end with the summand of second order in $y = x - x_{\mathrm{cl}}$:

$$S[y + x_{cl}] = S[x_{cl}] + \frac{1}{2}\left\langle \left.\frac{\delta^2 S}{\delta x_1 \delta x_2}\right|_{x=x_{\mathrm{cl}}} y_1 y_2 \right\rangle_{12}, \tag{1.65}$$

where the bracket is a shorthand notation with the following meaning

$$\langle f_{12\ldots n}\rangle_{12\ldots n} = \int_t^{t'} \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n f(t_1, t_2, \ldots, t_n). \tag{1.66}$$

The term linear in y does not contribute because of (1.64).

Since we have to sum over all paths x with the boundary conditions fulfilled by the classical path this can be expressed as sum over all paths y with the easier to handle boundary conditions $y(t) = y(t') = 0$. Formally this is done by substitution $y = x - x_{\mathrm{cl}}$ into the path integral. Thinking in terms of the discretised version of the path integral one immediately sees that the path integral measure is invariant under time dependent translations of x, i.e. $\mathrm{D}'x = \mathrm{D}'y$. So we get the important result

$$U(t', x'; t, x) = \exp\{\mathrm{i}S[x_{\mathrm{cl}}]\} \int_{(t,0)}^{(t',0)} \mathrm{D}'y \exp\left[\frac{\mathrm{i}}{2}\left\langle \frac{\delta S[x_{\mathrm{cl}}]}{\delta x_1 \delta x_2} y_1 y_2 \right\rangle\right] \tag{1.67}$$

As the first step we calculate the action along the classical path. We have to calculate the functional derivative of S with fixed boundary conditions.

$$\delta S = \int_t^{t'} \mathrm{d}t \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right]. \tag{1.68}$$

By integration by parts with taking into account the boundary conditions $\delta x(t) = \delta x(t') = 0$ we obtain

$$\delta S = \int_t^{t'} \mathrm{d}t \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] \delta x. \tag{1.69}$$

So the equations of motion defining the classical path are given by the Euler Lagrange equations with the Lagrangian L:

$$0 = \left.\frac{\delta S}{\delta x}\right|_{x=x_{\mathrm{cl}}} = \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{x=x_{\mathrm{cl}}}. \tag{1.70}$$

It is clear that we get the equation of the classical motion of the harmonic oscillator. The equation with the correct boundary conditions defined in (1.64) is simple to solve:

$$x_{\mathrm{cl}}(\tau) = x\cos[\omega(\tau - t)] + \frac{x' - x\cos[\omega(t' - t)]}{\sin[\omega(t' - t)]}\sin[\omega(\tau - t)]. \tag{1.71}$$

From this result the action along the classical path is given by

$$S[x_{\mathrm{cl}}] = \frac{m\omega\{(x^2 + x'^2)\cos[\omega(t' - t)] - 2xx'\}}{2\sin[\omega(t' - t)]}. \tag{1.72}$$

To finish the calculation now we are left with the path integral in (1.67) with the homogeneous boundary conditions. This has to be calculated in the discretised version. We call the path integral the amplitude A:

$$A = \lim_{N\to\infty}\left(\frac{mN}{2\pi\mathrm{i}(t' - t)}\right)^{N/2} \int \mathrm{d}y_1 \ldots \mathrm{d}y_{N-1} \exp\left\{\mathrm{i}\sum_{k=1}^{N}\left[\frac{m(y_k - y_{k-1})^2}{2\Delta t} - \frac{m^2}{2}\omega^2 y_k^2 \Delta t\right]\right\} \tag{1.73}$$

Since $y_0 = y_N = 0$ the argument of the exponential function can be written as

$$\sum_{k=1}^{N}\left[\frac{m(y_k - y_{k-1})^2}{2\Delta t} - \frac{m^2}{2}\omega^2 y_k^2 \Delta t\right] = \frac{m}{2\Delta t}\vec{y}^{\,t} M_N \vec{y}, \tag{1.74}$$

where $\vec{y}$ is the column vector $(y_1, y_2, \ldots, y_{N-1})$ and

$$M_N = \begin{pmatrix} C & -1 & 0 & 0 & 0 & \cdots \\ -1 & C & -1 & 0 & 0 & \cdots \\ 0 & -1 & C & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ with } C = 2 - \omega^2 \Delta t^2 \tag{1.75}$$

Now we calculate the k-dimensional Gaussian integral with a symmetric positive definite matrix M. Since such matrices can be diagonalised by an orthogonal transformation we get

$$\int \mathrm{d}^k y \exp(-y^t M y) = \prod_{j=1}^{k} \int \mathrm{d}x_j \exp(-\lambda_j x_j^2), \tag{1.76}$$

where we have substituted $z = Ox$. The λ_j are the eigenvalues of the matrix M. So the problem reduces to the product of single Gaussian integrals:

$$\int \mathrm{d}^k y \exp(-y^t M y) = \sqrt{\frac{\pi^k}{\prod_{j=1}^{k} \lambda_j}} = \sqrt{\frac{\pi^k}{\det M}}. \tag{1.77}$$

So after giving Δt a negative imaginary value and analytic continuation back to the real value $(t' - t)/N$ (determinants are analytic functions of the matrix elements) our problem to calculate (1.73) is reduced to calculate the determinant of M_N. That will be done as an exercise because the calculation is lengthy and tedious. The result after taking the continuum limit $N \to \infty$ gives the result for the amplitude:

$$A = \sqrt{\frac{m\omega}{2\pi \mathrm{i} \sin[\omega(t' - t)]}}. \tag{1.78}$$

Thus the final result is

$$U[x', t'; x, t] = \sqrt{\frac{m\omega}{2\pi \mathrm{i} \sin[\omega(t' - t)]}} \exp\left\{ \frac{\mathrm{i}m\omega\{(x^2 + x'^2)\cos[\omega(t' - t)] - 2xx'\}}{2\sin[\omega(t' - t)]} \right\}, \tag{1.79}$$

where we have put together (1.67), (1.68) and (1.78).

Exercise

Calculate the determinant of M_N, put it into (1.73-1.74) to prove (1.78)! Hint: It is useful to set $C = 2\cos\phi$ in (1.75).

1.7 Some Rules for Path Integrals

Now that we have calculated a closed solvable example, we can derive some general properties of path integrals. The first one we get by writing down the composition rule (1.13), which is valid for our time evolution kernel too, in terms of path integrals. For $t_1 < t_2 < t_3$ we have

$$\int_{(t_1,x_1)}^{(t_3,x_3)} \mathrm{D}'x \exp\{\mathrm{i}S[x]\} = \int \mathrm{d}x_2 \int_{(t_1,x_1)}^{(t_2,x_2)} \mathrm{D}'x \exp\{\mathrm{i}S[x]\} \int_{(t_2,x_2)}^{(t_3,x_3)} \mathrm{D}'x \exp\{\mathrm{i}S[x]\}. \tag{1.80}$$

This formula can be proved with help of the discretised version of the path integral formula. That easy task is left as an exercise for the reader.

We can give a more intuitive explanation of this composition rule. The left hand side of the equation gives the sum of the functional $\exp\{\mathrm{i}S[x]\}$ over all paths of the particle with the boundary conditions $x(t_1) = x_1$ to $x(t_3) = x_3$. On the right hand side the both path integrals are the sum of the same functional over all paths with the same boundary conditions but with the constraint $x(t_2) = x_2$ for a time in the interval (t_1, t_3), i.e. over all paths with the particle at a given place x_2 at time t_2. Now this restriction is compensated by the single integral over all intermediate positions x_2. So the left and the right hand side of the equation are indeed the same!

Now there is no problem to obtain a little bit more complicated quantities in terms of path integrals. As an example which will become crucial in the quantum field case we shall calculate the expectation value of a product of Heisenberg configuration space operators at different times. If this times are ordered in the sense of the causal time ordering operator introduced after eq. (1.34), that means that the time values of the operators increase from the right to the left, there is no problem. Using the unit operator given in terms of x-eigenstates at the given times and applying the composition rule of path integrals we obtain the following rule

$$\left\langle x', t' \left| T_c \mathbf{x}(t_1)\mathbf{x}(t_2)\dots\mathbf{x}(t_k) \right| x, t \right\rangle = \int_{(t,x)}^{(t',x')} \mathrm{D}'x x(t_1)x(t_2)\dots x(t_k) \exp\{\mathrm{i}S[x]\}. \tag{1.81}$$

1.8 The Schrödinger Wave Equation

In this section we want to derive Schrödinger's wave equation for the kernel of time evolution. The main idea is to show, how to find such equations from the path integral. We start again with the Feynman-Kac formula for the time evolution kernel for $t' > t$:

$$U(t', x'; t, x) = \int_{(t,x)}^{(t',x')} \mathrm{D}'x \exp\{\mathrm{i}S[x]\}. \tag{1.82}$$

Now we want to calculate the time derivative of this expression with respect to t'. So let ϵ be a little increment of t'. Now by the composition rule one has

$$U(t' + \epsilon, x'; t, x) = \int \mathrm{d}\xi U(t' + \epsilon, x'; t', \xi) U(t', \xi; x, t). \tag{1.83}$$

Now we try to find the expansion with respect to ϵ. Since it can be thought to be very small compared to the typical time scale of the problem we can approximate the first kernel on the right hand side by a path integral with just one time slice namely the interval $(t', t' + \epsilon)$. So we get

$$U(t', x'; t, x) = \int \mathrm{d}\xi \sqrt{\frac{m}{2\pi\mathrm{i}\epsilon}} \exp\left[\mathrm{i}\frac{m(x' - \xi)^2}{2\epsilon}\right] [1 - \mathrm{i}\epsilon V(x') + O(\epsilon^2)] U(t', \xi; t, x). \tag{1.84}$$

Since we like to calculate this expression for $\epsilon \to 0$ the exponential is oscillating rapidly except for values $\xi \approx x'$. In a mathematically more correct treatment we could have continued the time to be imaginary instead of regularising the path integral by only a little negative imaginary part. If we had done so the exponential is not oscillating but damped. From the imaginary time formalism we obtain the correct time evolution kernel by analytic continuation. So we see

that the main contribution of the ξ-integral comes from $\xi \approx x'$. By changing the variable of integration to $\delta = x' - \xi$ and expanding $U(t', \xi; t, x)$ around δ we obtain

$$\begin{aligned} U(t'+\epsilon, x'; t, x) &= \int \mathrm{d}\delta \sqrt{\frac{m}{2\pi \mathrm{i}\epsilon}} \exp\left(\frac{\mathrm{i}m\delta^2}{2\epsilon}\right) [1 - \epsilon V(x') + O(\epsilon^2)] \times \\ &\times \left[\sum_{k=0}^{\infty} \left(\frac{\partial}{\partial x'}\right)^k U(t', x'; t, x) \frac{(-\delta)^k}{k!}\right]. \end{aligned} \tag{1.85}$$

Now interchanging integration and summation of the series we need integrals of a Gaussian multiplied by any potence of δ. This integrals can be obtained by defining a generating function:

$$f(\Lambda) = \int \sqrt{\frac{m}{2\pi \mathrm{i}\epsilon}} \exp\left(\frac{\mathrm{i}m\delta^2}{2\epsilon} + \Lambda\delta\right). \tag{1.86}$$

After the usual regularisation by an negative imaginary part for ϵ we can calculate this integral and from this we find

$$I_k = \int \mathrm{d}\delta \sqrt{\frac{m}{2\pi \mathrm{i}\epsilon}} \delta^k \exp\left(\frac{\mathrm{i}m\delta^2}{2\epsilon}\right) = \left.\frac{d^k f(\Lambda)}{d\Lambda^k}\right|_{\Lambda=0}. \tag{1.87}$$

For $\epsilon \to 0$ we need only the following identities

$$I_0 = 1,\ I_1 = 0,\ I_2 = \frac{\mathrm{i}\epsilon}{m},\ I_{2n} = O(\epsilon^n),\ I_{2n+1} = 0 \text{ for } n \in \mathbb{N}. \tag{1.88}$$

Inserting this into (1.85) we get

$$U(t'+\epsilon, x'; t, x) = [1 - \mathrm{i}\epsilon V(x') + O(\epsilon^2)] \left[1 + \frac{\mathrm{i}\epsilon}{2m}\partial_{x'}^2\right] U(t', x'; t, x). \tag{1.89}$$

Subtracting $U(t', x'; t, x)$ and letting $\epsilon \to 0$ we obtain the important result

$$\mathrm{i}\partial_{t'} U(t', x'; t, x) = \left[-\frac{1}{2m}\partial_{x'}^2 + V(x')\right] U(t', x'; t, x) \text{ for } t' > t. \tag{1.90}$$

This is the Schrödinger equation, since the differential operator on the right hand side is the Hamiltonian expressed in terms of configuration space representation. The initial condition for the kernel, necessary to make the solution of the Schrödinger equation unique, is of course

$$U(t+0, x'; t, x) = \delta(x' - x). \tag{1.91}$$

This is the first time we see that the time evolution kernel is nothing but the Green's function of the Schrödinger equation. We will get a deeper insight in this relation by looking on an important physical application of all the developed formalism, namely the perturbative treatment of potential scattering with help of path integrals.

1.9 Potential Scattering

Potential scattering is an example where the path integral formalism is very intuitive. The setup of a scattering experiment can described in the following idealised way: The physicist prepares

a beam of particles with a definite momentum $\vec{p}_i$ far away from the target where it will be scattered. After the scattering process the physicist will use detectors, again far away from the scatterer, to find the rate of scattered particles as a function of their momentum $\vec{p}_f$.

Now let us describe this from a little bit more mathematical point of view: Since we want to look on the most simple situation now, we give the scatterer in terms of a external potential. So we have only one particle scattered by this external potential. This is approximately realized by the scattering of a particle on another particle with a much greater mass (e.g. scattering of an electron at a heavy nucleus). This potential should have only a finite range, i.e., $V(\vec{x})$ should be very small for $|\vec{x}| > R$, where R is a typical range scale of the potential. So the particles prepared before the scattering and registered after the scattering, far away from $\vec{x} = 0$, that means now $|\vec{x}| > R$, can be treated as quasi free particles. This is what is widely known as an *asymptotic free state*.

Now let us analyse this situation with help of quantum mechanics. What we like to calculate is the transition rate of particles which are prepared at an initial time $t_0 \to -\infty$ as asymptotic free particles of definite momentum $\vec{p}_i$ to a given asymptotic free state of definite momentum $\vec{p}_f$ at a time t_f. In the Heisenberg picture this is simply written as

$$S_{fi} = \lim_{t_i \to -\infty, t_f \to \infty} \langle t_f, \vec{p}_f \,|\, \vec{p}_i, t_i \rangle \,. \tag{1.92}$$

This defines the *scattering matrix*, shortly called the *S-Matrix*, as the transition rate from a given initial asymptotic free state (in our case of definite momentum) to a given final asymptotic free state (in our case also of definite momentum).

It can be seen as the main application of vacuum quantum field theory to calculate the S-Matrix for this and much more complicated processes. We will give a description of scattering from the quantum field theory point of view later. We will see that in the case of nonrelativistic potential scattering the both approaches are equivalent. But for example the scattering of two identical particles is much easier to get from the field theoretic point of view than from our space-time point of view because the space of two identical particles becomes much more complicated than one should expect from classical physics. We come back to this in the next section when we show a very convincing argument made by Cecile Morette-de Witt about the fact that there can only be bosons and fermions in a configuration space with three or more dimensions. But let us come now back to our potential scattering problem of one particle!

The whole sections before we always calculated the transition amplitudes, i.e. the time evolution kernel, dependent on space and time variables. Now of course we can look at the scattering process in configuration space too, and we can introduce the time evolution kernel simply by setting some identity operators between the momentum eigenvectors in (1.92):

$$S_{fi} = \lim_{t_i \to -\infty, t_f \to \infty} \int \mathrm{d}^3\vec{x}_1 \mathrm{d}^3\vec{x}_2 \underbrace{\langle t_f, \vec{p}_f \,|\, t_f, \vec{x}_1 \rangle}_{\varphi^*_{\vec{p}_f}(t_f, \vec{x}_1)} \underbrace{\langle t_f, \vec{x}_1 \,|\, t_i, \vec{x}_2 \rangle}_{U(t_1, \vec{x}_1; t_x, \vec{x}_2)} \underbrace{\langle t_i, \vec{x}_2 \,|\, t_i, \vec{p}_i \rangle}_{\varphi_{\vec{p}_f}(t_i, \vec{x}_2)} . \tag{1.93}$$

So we can write the S-Matrix in the form

$$S_{fi} = \lim \int \mathrm{d}^3\vec{x}_1 \mathrm{d}^3\vec{x}_2 \varphi^*_{\vec{p}_f}(t_f, \vec{x}_1) U(t_f, \vec{x}_1; t_i, \vec{x}_2) \varphi_{\vec{p}_i}(t_i, \vec{x}_2), \tag{1.94}$$

where lim is meant to remember to take the limit $t_i \to -\infty, t_f \to \infty$ *at the very end of the calculation.* Herein the wave functions $\varphi_{\vec{p}}$ have to be taken as the solution of the time dependent

(!) Schrödinger equation for a free particle with definite momentum $\vec{p}$:

$$\varphi_{\vec{p}}(t,\vec{x}) = \frac{1}{(2\pi)^{(3/2)}} \exp\left(-\mathrm{i}\frac{\vec{p}^2}{2m}t + \mathrm{i}\vec{p}\vec{x}\right). \tag{1.95}$$

So again the main task is solved if we find a solution for the time evolution kernel or an approximation appropriate for our situation. In the most cases one has to do it in the former sense because it is impossible to find an analytical solution for the kernel. The best method one can use is to solve the Schrödinger equation for the kernel. But here we like to give a derivation of the approximation series known as the *Born series* with help of the path integral. This has the great advantage to give a very intuitive picture of scattering in terms of processes in space-time.

For this purpose we assume now that the potential is small in the sense of a small coupling constant as is the case, for instance, for electromagnetic forces were the coupling constant is of order of Sommerfeld's fine structure constant $\alpha = e^2/(\hbar c) \approx 1/137$. In this case it is sensible to expand the time evolution kernel in powers of the potential.

Our starting point is again the Feynman-Kac formula for the time evolution kernel

$$U(t_f,\vec{x}_1;t_i,\vec{x}_2) = \int_{(t_i,\vec{x}_1)}^{(t_f,\vec{x}_2)} \mathrm{D}'^3\vec{x}\exp\{\mathrm{i}S[\vec{x}]\}, \tag{1.96}$$

which is a straight forward generalisation from the one dimensional case, treated so far, to the case of three dimensional configuration space.

Now we have

$$S[x] = \int_{t_i}^{t_f} [L_0(\vec{x},\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}) - V(x)]\mathrm{d}t \text{ with } L_0(\vec{x},\frac{d\vec{x}}{dt}) = S_0[x] + S_I[x] = \frac{m}{2}\left(\frac{dx}{dt}\right)^2. \tag{1.97}$$

Here $S_0[x]$ is the action functional of free particles and $S_I[x]$ the interaction part of the full action of the problem. Now we expand the integrand of the path integral in powers of S_I:

$$\begin{aligned} U(t_f,\vec{x}_f;t_i,\vec{x}_i) &= \int_{(t_i,\vec{x}_i)}^{(t_f,\vec{x}_f)} \mathrm{D}'^3\vec{x}\exp\{\mathrm{i}S_0[x]\}\times \\ &\times \left\{1 - \mathrm{i}\int_{t_i}^{t_f}\mathrm{d}\tau V[x(\tau)] + \frac{(-\mathrm{i})^2}{2!}\int_{t_i}^{t_f}\mathrm{d}\tau_1\int_{t_i}^{t_f}\mathrm{d}\tau_2 V[x(\tau_1)]V[x(\tau_2)] + \ldots\right\}. \end{aligned} \tag{1.98}$$

From this we see that the 0th approximation is given by the noninteracting time evolution kernel. Now using the same arguments as used above when calculating the path integral for the time ordered product of space operators we find as the first order correction

$$U^{(1)}(t_f,\vec{x}_f;t_i,\vec{x}_i) = -\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}\tau_1\int \mathrm{d}^3\vec{y}U_0(t_f,\vec{x}_f;\tau_1,\vec{y}_1)V(\vec{y}_1)U_0(\tau_1,\vec{y}_1;t_i,\vec{x}_i). \tag{1.99}$$

Herein U_0 is the time evolution kernel for free particles. Since under the time integral in (1.99) there is always $t_i \le \tau_1 \le t_f$, we have no problem with the definition of the used time evolution kernels.

Before proceeding to the second approximation let us interpret the result. For this purpose we think about the integrand under the integral in (1.99) at a fixed time τ_1 . Then we can get the intuitive idea behind the formula as follows (we have to read it from the right to the left!): first the particle propagates as a free particle from the initial position $\vec{x}_i$ at time t_i to the point $\vec{y}_1$,

then it is scattered by the potential at this place, and it moves again as a free particle to the end point $\vec{x}_f$ at the final time t_f. The integrals over the time interval and the intermediate point $\vec{y}_1$ can be interpreted again as the coherent sum over all possibilities for the particle to scatter once at the potential: This may take place at all times in the interval (t_i, t_f) and at all places $\vec{y}$. This is again consistent with our summation over paths picture of quantum mechanics introduced by the path integral. Now we see the meaning of the kth order of the expansion is clear: This term is in the series to take into account the possibility that the particle is scattered k times by the potential and that this can happen at each point in space and at each time in the interval (t_i, t_f).

Because of the time ordering problem when using the time evolution kernel we take a closer look on the second order of our series expansion. Here we have to distinguish the two cases $\tau_1 < \tau_2$ and $\tau_1 > \tau_2$. This can be written with help of Heaviside's unit step function defined as

$$\Theta(\tau) = \begin{cases} 0 \text{ for } \tau < 0 \\ 1 \text{ for } \tau \geq 0. \end{cases} \tag{1.100}$$

With this function we can write the second order of the series as

$$\begin{aligned} U^{(2)}(t_f, \vec{x}_1; t_i, \vec{x}_2) &= \frac{(-\mathrm{i})^2}{2!} \int_{t_i}^{t_f} \mathrm{d}\tau_1 \int_{t_i}^{t_f} \mathrm{d}\tau_2 \times \\ &\times [\Theta(\tau_1 - \tau_2) U_0(t_f, \vec{x}_2; \tau_1, \vec{y}_1) V(\vec{y}_1) U_0(\tau_1, \vec{y}_1; \tau_2, \vec{y}_2) V(\vec{y}_2) U_0(\tau_2, \vec{y}_2; t_i, \vec{x}_2) + \\ &+ \Theta(\tau_2 - \tau_1) U_0(t_f, \vec{x}_2; \tau_2, \vec{y}_2) V(\vec{y}_2) U_0(\tau_2, \vec{y}_2; \tau_1, \vec{y}_1) V(\vec{y}_1) U_0(\tau_1, \vec{y}_1; t_i, \vec{x}_2)]. \end{aligned} \tag{1.101}$$

For convenience now we introduce the retarded time evolution kernel by:

$$U_0^{(R)}(\tau_1, \vec{y}_1; \tau_2, \vec{y}_2) = \begin{cases} U_0(\tau_1, \vec{y}_1; \tau_2, \vec{y}_2) \text{ for } \tau_1 > \tau_2 \\ 0 \text{ for } \tau_1 \leq \tau_2 \end{cases} = \Theta(\tau_1 - \tau_2) U_0(\tau_1, \vec{y}_1; \tau_2, \vec{y}_2) \tag{1.102}$$

and with this we find by interchanging the integration variables in the second integral on the right hand side of (1.101) that both integrals are the same and can be written as one integral with the retarded kernel which contains the time Θ-functions. This line of arguing can be done using induction with the general kth summand of the series:

$$\begin{aligned} U^{(k)}(t_f, \vec{x}_f; t_i, \vec{x}_i) &= (-\mathrm{i})^k \int \mathrm{d}^4 y_1 \int \mathrm{d}^4 y_2 \ldots \int \mathrm{d}^4 y_k \times \\ &\times U_0^{(R)}(t_f, \vec{x}_f; y_1) V(\vec{y}_1) U_0^{(R)}(\tau_1, \vec{y}_1; y_2) V(\vec{y}_2) \ldots V(\vec{y}_k) U_0^{(R)}(y_1; t_i, \vec{x}_i). \end{aligned} \tag{1.103}$$

Here we have introduced four-dimensional space-time variables $(\tau_k, \vec{y}_k)$ which we have abbreviated with y_k. The integrals have to be understood with the boundaries (t_i, t_f) for the time arguments of the y_k and over all space for the $\vec{y}_k$-components. This is the form of the Born series for the time evolution kernel known from wave mechanics.

Now it is time to introduce the Feynman rules for this problem. They can be read off the eqs. (1.98) and (1.103). They reflect the "history" of the particle passing the potential. For each retarded time evolution kernel $U_0^{(R)}(x_1; x_2)$ we draw a solid line with an arrow pointing from $x_2 = (t_2, \vec{x}_2)$ to $x_1 = (t_1, \vec{x}_1)$, representing the direction of propagation. Each potential $-\mathrm{i}V$ is drawn as a sticker with a dashed line. This sticker is connected to two directed solid lines, i.e., an ingoing and an outgoing time evolution kernel. The potential is to take at the point where the three lines come together. Finally one has to integrate over all inner points, also called *vertices*. The propagator kernels describe the propagation of the particle between scattering

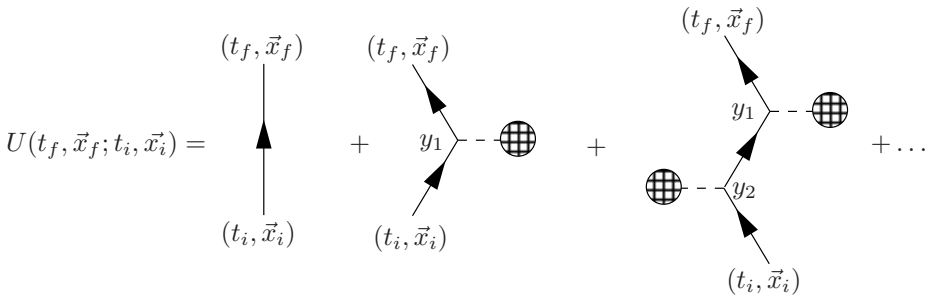


Figure 1.2: *Diagrammatic representation of the Born series*

events, and is therefore called *propagator*. Thus with this rules we can write the Born series (1.98) diagrammatically in the form shown in fig. 1.2.

From this diagrams one reads off immediately that $U^{(R)}(x_1; x_2)$ fulfils the following integral equation

$$U^{(R)}(x_1, x_2) = U_0^{(R)}(x1, x2) - \mathrm{i} \int \mathrm{d}^4 y U_0^{(R)}(x_1, y) V(y) U^{(R)}(y, x_2). \tag{1.104}$$

From (1.90) and the analogous equation for the free propagator $U_0^{(R)}(x_1; x_2)$, valid for $t_1 > t_2$, one reads off as matching condition

$$\left(\mathrm{i}\partial_{t_1} + \frac{\Delta_1}{2m}\right) U_0^{(R)}(x1, x2) = \mathrm{i}\delta^{(4)}(x_1 - x_2), \tag{1.105}$$

which leads us to the important conclusion that the free retarded propagator is the Green's function for the free Schrödinger equation with the appropriate bounding condition, namely

$$U_0^{(R)}(x_1; x_2) \propto \Theta(t_1 - t_2), \tag{1.106}$$

which expresses the retarded nature of the propagator and reflects the causal structure of the Born series: At the time t there is no influence from the potential which comes from times later than t!

Using (1.106) together with (1.104) we find that $U^{(R)}$, the exact propagator of the particle under influence of the potential, is the retarded Green's function of the Schrödinger equation with the potential V:

$$\left[\mathrm{i}\partial_{t_1} + \frac{\Delta_1}{2m} - V(\vec{x}_1)\right] U^{(R)}(x1, x2) = \mathrm{i}\delta^{(4)}(x_1 - x_2). \tag{1.107}$$

This shows the *particle wave dualism* of quantum mechanics: The problem of scattering of a particle at a potential V is equivalent to a scattering problem of Schrödinger's waves. So starting from the particle picture we ended here with the field picture of the behaviour of particles in quantum theory. Here in the nonrelativistic case both pictures are equivalent. We will see in the next chapter that this is not true for the relativistic case. The physical reason is simply explained: In the nonrelativistic case there is a conservation law for the number of particles in a closed system, but in the quantum case there is always the possibility to create pairs of particles and antiparticles, for instance, when a particle is scattered at an external potential. The classical analogue is known from electrodynamics, namely the effect of *Bremsstrahlung*, i.e., the emission of light, when charged particles are accelerated. In the particle picture this is the creation of *photons*. In the relativistic case it is thus physically not adequate to work with a Hilbert space of a

fixed number of particles. Fortunately there is a formalism which exactly describes this situation, namely quantum field theory. In the former days this was known as the "second quantisation", but this name is not consistent with our modern understanding of quantum field theory which is nothing else than the quantum theory of particles with the possibility to describe the creation and annihilation of particles consistent with the conservation laws.

The space-time picture is nice to get an idea of the meaning of the scattering process in the quantum sense, but it is complicated to calculate with because of the involved time dependence of the free propagator, shown by (1.46). It is much more simple to work in the momentum representation. This we obtain by inserting the Fourier representation of the free retarded propagator. This is a nice example for the application of (1.105). So we make the ansatz:

$$U_0(x_1;x_2) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \exp[\mathrm{i}\vec{p}(\vec{x}_1-\vec{x}_2)]\tilde{U}_0(t_1;t_2;\vec{p}). \tag{1.108}$$

With this (1.105) reads:

$$\left(\mathrm{i}\partial_{t_1} - \frac{\vec{p}^2}{2m}\right) U_0^{(R)}(t_1;t_2;\vec{p}) = \mathrm{i}\delta(t_1-t_2). \tag{1.109}$$

The δ-function comes from the Θ function shown in the boundary condition (1.106). With this we obtain the unique solution of the boundary value problem to be

$$U_0^{(R)}(t_1;t_2;\vec{p}) = \exp\left[-\mathrm{i}\frac{\vec{p}^2}{2m}(t_1-t_2)\right]\Theta(t_1-t_2). \tag{1.110}$$

The time dependence in this representation is much easier to handle with than with that of the configuration space representation.

Now we insert the obtained results into the first order Born approximation (1.99) to find the first order result for the S-Matrix (1.94). The somewhat lengthy but straightforward calculation is left as an exercise for the reader.

It should be emphasised that the result after doing all space and momentum integrations can be taken without ambiguities in the limit $t_i \to -\infty$ and $t_f \to \infty$ which shows that the concept of asymptotic states is well defined in the sense of a *weak* limit, i.e. it is a limit to be taken after calculating the matrix elements.

Here we give the well known result of the first order Born approximation

$$S_{fi}^{(1)} = \frac{-\mathrm{i}}{(2\pi)^2}\tilde{V}(\vec{p}_i-\vec{p}_f)\delta(E_f-E_i), \tag{1.111}$$

where E_f and E_i are the energy of the outgoing and the ingoing asymptotically free particle, respectively, i.e., $E_{f/i} = \vec{p}_{f/i}^{\,2}/(2m)$. The δ-function reflects the fact that the energy is conserved by the scattering process. The meaning of $\tilde{V}$ is simply to be the Fourier transform of the potential, namely

$$\tilde{V}(\vec{p}) = \int \mathrm{d}^3\vec{x}\exp[\mathrm{i}\vec{p}\vec{x}]V(\vec{x}). \tag{1.112}$$

Exercise

Calculate the first and second order result for the S-Matrix in terms of the momentum version of $U_0^{(R)}$ and V. Hint: Calculate at first $U^{(1)}(t_f,\vec{x}_f;t_i,\vec{x}_i)$ and then the integrals with the initial and final asymptotic states. You can use the result for calculation of the second order!

Now we want to calculate the cross section for scattering at the external potential. It is defined as the ratio of in a momentum $\vec{p}_f$ scattered particles per unit time and the incoming current (i.e. number of particles per unit time and unit area).

The main difficulty is to calculate the number of particles per unit scattered in a given momentum state, because for this we have to square $S_{fi} - \delta_{fi} \approx S_{fi}^{(1)}$. But this expression, given by (1.111), contains a δ distribution. So what is the meaning of the δ distribution squared? From a mathematical point of view there is no meaning. On the other hand this square gives the transition probability, not the probability per time unit. So we have to go back to a finite time interval, and take again the limit after all other calculations are done. The simple calculation yields:

$$S_{fi}^{(1\text{reg})} = \frac{V(\vec{p}_f - \vec{p}_i)\{\exp[-\mathrm{i}(E_f - E_i)t_f] - \exp[-\mathrm{i}(E_f - E_i)t_i]\}}{(2\pi)^3(E_f - E_i)}. \tag{1.113}$$

Taking the modulus of this expression squaring it and dividing by $t_f - t_i$ gives after some trigonometry

$$w_{fi}^{(1\text{reg})} = \frac{|V(\vec{p}_f - \vec{p}_i)|^2}{(2\pi)^5} \left(\frac{\sin[(E_f - E_i)(t_f - t_i)/2]}{(E_f - E_i)(t_f - t_i)/2} \right)^2 \frac{t_f - t_i}{2\pi}. \tag{1.114}$$

Now we are ready to take the limit $t_i \to -\infty$ and $t_f \to \infty$:

$$w_{fi}^{(1)} = \frac{|V(\vec{p}_f - \vec{p}_i)|^2}{(2\pi)^5} \delta(E_f - E_i). \tag{1.115}$$

We have again obtained an energy conserving δ distribution, but as before in the case of the S-Matrix there is no ambiguity to take the weak limit at the end of the calculation.

For the definition of the cross section we need the current of incoming particles. We know from elementary quantum mechanics that the density of particles $\rho(t, \vec{x}) = |\psi(t, \vec{x})|^2$, where ψ is the asymptotic free wave function of the incoming particle, has a current defined by

$$\vec{j}(t, \vec{x}) = \frac{1}{2im}(\psi^* \nabla \psi - cc.) = \frac{\vec{p}_i}{(2\pi)^3 m}. \tag{1.116}$$

The conservation of the particle number is represented by the continuity equation

$$\partial_t \rho(t, \vec{x}) + \mathrm{div} \vec{j}(t, \vec{x}) = 0, \tag{1.117}$$

which again shows in the integrated form, together with the validity of the Schrödinger equation, the unitarity of time evolution.

Taking the absolute value of the current and dividing the transition probability yields the cross section per unit momentum volume:

$$\frac{\mathrm{d}\sigma^{(1)}}{\mathrm{d}^3 \vec{p}_f} = \frac{m|V(\vec{p}_f - \vec{p}_i)|^2}{(2\pi)^2 |\vec{p}_i|} \delta(E_f - E_i). \tag{1.118}$$

Integrating this result over all momentum lengths yields the cross section per solid angle. Here we use $\delta(E_f - E_i) = m/p_i \delta(p_f - p_i)$. The result is

$$\frac{\mathrm{d}\sigma^{(1)}}{\mathrm{d}\Omega} = \frac{m^2 |V(\vec{p}_f - \vec{p}_i)|^2}{(2\pi)^2} \text{ with } |\vec{p}_f| = |\vec{p}_i|. \tag{1.119}$$

Exercise

Calculate the first order Born approximation for the cross section in the cases of a Yukawa and a Coulomb potential!

1.10 Generating functional for Vacuum Expectation Values

Now let us come back to the operator formalism of quantum mechanics for a short while. The time evolution kernel is defined in terms of eigenvectors of the configuration space variable as

$$U(t_1, x_1; t_2, x_2) = \langle t_1, x_1 \,|\, t_2, x_2 \rangle = \langle x_1 \,|\exp[-\mathrm{i}\mathbf{H}(t_1 - t_2)]|\, x_2 \rangle \tag{1.120}$$

where $|x_1\rangle$ and $|x_2\rangle$ are eigenvectors of $\mathbf{x}$ at a fixed time and $\mathbf{H}$ is the explicitely time-independent Hamiltonian of the system. Now let $|n\rangle$ denote the complete set of eigenvectors of the Hamiltonian, i.e.,

$$\mathbf{H}\,|n\rangle = E_n\,|n\rangle\,. \tag{1.121}$$

Since $\mathbf{H}$ is hermitian these kets can be taken to be orthogonal and normalised. That means we have

$$\mathbf{1} = \sum_n |n\rangle\,\langle n|\,,\ \langle n \,|\, m \rangle = \delta_{nm}, \tag{1.122}$$

and they are time-independent since $\mathbf{H}$ is time-independent in the here used Heisenberg picture. With help of (1.122) we can write the time evolution kernel in the form

$$U(t_1, x_1; t_2, x_2) = \sum_n \exp[-\mathrm{i}E_n(t_1 - t_2)]\varphi_n(x_1)\varphi_n^*(x_2) \text{ with } \varphi_n(x) = \langle x \,|\, n \rangle\,. \tag{1.123}$$

In quantum field theory we shall be interested in expectation values of time-ordered operator products with respect to the ground state of the system, i.e., the energy eigenstate for the lowest energy eigenvalue. The existence of a lower bound of the energy spectrum is necessary for physical reasons. In the case of quantum fields this ground state has the meaning that there are no particles around and therefore it is called *the vacuum*.

Now we like to show how to calculate the vacuum expectation values of time-ordered operator products with help of the time evolution kernel by application of path integral methods. Here we will apply the external current formalism the first time. It will lead us to the Feynman rules for perturbation theory in a quite straightforward manner. Now let us introduce the Lagrangian

$$L_J = L + xJ. \tag{1.124}$$

Herein J is an external force applied to the system. We restrict the influence of this external perturbation to a finite interval in time:

$$J(\tau) = 0 \text{ for } \tau \notin [t, t']. \tag{1.125}$$

From that we get for $T < t < t' < T'$ together with the composition rule for the time evolution kernel

$$U_J(T', x'; T, x) = \int \mathrm{d}y_1 \mathrm{d}y_2 U(T', x'; t', y_1) U_J(t', y_1; t, y_2) U(t, y_2; T, x) \tag{1.126}$$

where U_J and U are the kernels with and without the influence of the external force respectively. Now from (1.123) we have

$$U(t, y_2; T, x) = \sum_n \varphi_n(y_2)\varphi_n^*(x) \exp[-\mathrm{i}E_n(t-T)]. \tag{1.127}$$

Now if we have a Hamiltonian which is bounded from below, we can multiply the sum by $\exp(-\mathrm{i}E_0T)$ and analytically continue the kernel to the upper complex plane. So we obtain

$$\lim_{T\to i\infty} \exp(-\mathrm{i}E_0T)U(t, y_2; T, x) = \varphi_0(y_2)\varphi_0^*(x) \exp(-\mathrm{i}E_0t). \tag{1.128}$$

With the same calculation for the other non perturbed kernel in (1.126) but with $T' \to -\mathrm{i}\infty$ we get the desired result

$$Z[J] := \left\langle t', 0 \,\middle|\, t, 0 \right\rangle_J = \lim_{T'\to -\mathrm{i}\infty, T\to i\infty} \frac{U_J(T', x'; T, x)}{\exp[-\mathrm{i}E_0(T'-T)]\varphi_0(x')\varphi_0^*(x)}. \tag{1.129}$$

The whole discussion shows that it would have been enough to rotate the time axes clockwise with a little angle δ. So we can take the limit in (1.129) as $T \to \infty \exp(-\mathrm{i}\epsilon)$ and $T' \to -\infty \exp(-\mathrm{i}\epsilon)$. Another way, which we shall prefer in the context of path integrals, is to introduce a small positive definite imaginary part into the Lagrangian, for instance we take $\mathrm{i}m\epsilon x^2/2$.

Since for $t' \to \infty$ and $t \to -\infty$ a constant factor in $Z[J]$ is not of interest, we take as boundary conditions such paths for which

$$\lim_{\tau\to\pm\infty} x(\tau) = 0. \tag{1.130}$$

This assures that the action integral along the whole time axis exists. Then we conclude from (1.129) the path integral formula for the Z-functional:

$$Z[J] = N \int \mathrm{D}x \exp\left[\mathrm{i}\int_{-\infty}^{\infty} dt \left(L + Jx + \frac{\mathrm{i}m\epsilon}{2}x^2\right)\right]. \tag{1.131}$$

At the very end of the calculation the expression is understood to be taken for $\epsilon \to +0$. The quantum field theoretical analogue of this functional will play a crucial role in the following calculations.

1.11 Bosons and Fermions, and what else?

Section in preparation!

Chapter 2

Nonrelativistic Many-Particle Theory

In this chapter we sketch shortly the many particle theory for the nonrelativistic case, which is done to show that field quantisation is nothing else than many-particle theory in Fock space representation.

2.1 The Fock Space Representation of Quantum Mechanics

First we like to describe many-particle states with a fixed number of identical bosons or fermions. At first we have to think about the correct Hilbert space for this problem.

Each single particle state can be spanned by generalised momentum-spin eigenstates. Since the momentum and spin operators for different particles commute the basis kets for the many particles are given by the tensor products of the n single particle basis states. Thus the first guess for the Hilbert space for the many-particle states is

$$\mathscr{H}_n = \underbrace{\mathscr{H} \otimes \ldots \otimes \mathscr{H}}_{\text{n times}}. \tag{2.1}$$

Unfortunately this Hilbert space is too big if the n particles are indistinguishable. This assumption is understandable if one thinks about basic quantum mechanic considerations about the nature of particles. In classical mechanics we can label the particles by their initial positions and momenta and then we know the position of each individual particle over there whole trajectory. But in quantum mechanics there is no trajectory in the classical sense. Thus the only label of particles are its *intrinsic* quantum numbers, as are the spin sort (0, 1/2, ...), mass, charge numbers etc. Thus if we have indistinguishable particles there is no possibility to find an individual particle along the time evolution.

In other words the multi-particle product states have to be unchanged (up to a phase which we choose arbitrarily to be 1) under permutation of the single particle quantum numbers of a complete compatible set of single particle observables. Thus each state in the n-particle Hilbert space has to build a one-dimensional representation of the permutation group S_n. These are either the trivial representation (such particles are called *bosons*) or the alternating representation (such particles are called *fermions*). The latter representation is defined by that it just multiplies the state with the sign of the permutation applied to the one-particle states contained in the

n-particle product state. Thus we have

$$|q_1, q_2, \ldots, q_n\rangle_\pm = \begin{cases} \frac{1}{N} \sum_{P \in S_n} \left|q_{P(1)}\right\rangle \otimes \ldots \otimes \left|q_{P(n)}\right\rangle & \text{for bosons} \\ \frac{1}{N} \sum_{P \in S_n} \operatorname{sign}(P) \left|q_{P(1)}\right\rangle \otimes \ldots \otimes \left|q_{P(n)}\right\rangle & \text{for fermions.} \end{cases} \tag{2.2}$$

Herein q_k is the complete set of single-particle quantum numbers (for instance $(\vec{p}_k, \sigma_k)$ with $\vec{p}_k \in \mathbb{R}^3$, $\sigma_k = -s, -s+1, \ldots, s$. N is a (state dependent) normalisation constant which normalises the states to δ-functions as usual in quantum mechanics.

Thus the correct Hilbert space for n indistinguishable particles is given by the span of all this *symmetrised or anti-symmetrised* products of single particle states. In the following we denote the bosonic or fermionic n-particle Hilbert space by $\mathscr{H}_n^+$ or $\mathscr{H}_n^-$.

Now sometimes it is more convenient to describe a system of many particles without fixing the numbers of particles contained in the system. As we shall see in the next chapter this is necessary to describe relativistic particles consistently. The physical reason is well-known nowadays, namely the possibility to create and annihilate particles. In the nonrelativistic theory the particle numbers are fixed[1], as we shall see later in this chapter.

The appropriate Hilbert space is given as the orthogonal sum over all n-particle bosonic or fermionic Hilbert spaces:

$$\mathscr{H}^\pm = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2^\pm \oplus \cdots \tag{2.3}$$

This space is called the *Fock space* of many particle states with a non-fixed number of particles. Clearly there is also a Hilbert space which describes the possibility that there is no particle at all. This Hilbert space is by definition one-dimensional and this state is therefore called *the vacuum*.

Now we come to a nice feature of the Fock space, namely the possibility to define so called *creation operators*. Both are defined as linear operators by their operation on the above given symmetrised or anti-symmetrised product states. They map a n-particle state to a $n+1$-particle state. We define these operators by

$$\mathbf{a}^\dagger(q) \left|q_1, \ldots, q_n\right\rangle^\pm = \left|q, q_1, \ldots, q_n\right\rangle^\pm . \tag{2.4}$$

Thus $\mathbf{a}^\dagger(q)$[2] adds one particle in the state $|q\rangle$ to the n-particle state leading to a $n+1$-particle state. It is therefore called a *creation operator*.

Now we want to calculate the hermitian conjugate of the creation operator. This is most conveniently done by noting that the identity operator on the Hilbert space can be written as

$$\mathbf{1} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{q_1, \ldots q_n} \left|q_1, \ldots, q_n\right\rangle^\pm \left\langle q_1, \ldots, q_n\right|^\pm . \tag{2.5}$$

Multiplying this identity from the left with $\mathbf{a}^\dagger(q)$ we obtain by definition (2.4):

$$\mathbf{a}^\dagger(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{q_1, \ldots q_n} \left|q, q_1, \ldots, q_n\right\rangle^\pm \left\langle q_1, \ldots, q_n\right|^\pm . \tag{2.6}$$

From this we find the adjoint

$$\mathbf{a}(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{q_1, \ldots q_n} \left|q_1, \ldots, q_n\right\rangle^\pm \left\langle q, q_1, \ldots, q_n\right|^\pm \tag{2.7}$$

[1] This is only true for elementary particles, not for quasiparticles (like phonons in condensed matter physics).
[2] It is written as $\mathbf{a}^\dagger$, not as $\mathbf{a}$ simply by convention in literature.

and by using the fact that the states used are orthonormalised we find

$$\mathbf{a}(q)\,|q_1,\ldots,a_n\rangle^{\pm} = \sum_{j=1}^{n}(\pm 1)^{j-1}\delta(l_j,l)\left|l_1,\ldots,\hat{l}_j,\ldots,l_n\right\rangle. \tag{2.8}$$

With help of (2.8) and (2.4) we obtain the important commutator relations

$$[\mathbf{a}(l),\mathbf{a}^{\dagger}(k)]_{\mp} = \delta(l,k),\ [\mathbf{a}(l),\mathbf{a}(k)]_{\mp} = 0,\ [\mathbf{a}^{\dagger}(l),\mathbf{a}^{\dagger}(k)]_{\mp} = 0. \tag{2.9}$$

Now we can write any operator in terms of creation and annihilation operators which shows that these are a complete set of generating operators of the algebra of observables of our many body system. To show how the prove for this works, we calculate this representation for the two most simple cases, namely *one-particle* and *two-particle-operators*.

In the n-particle subspace of the Fock-space a one-particle operator is defined to be of the form

$$\mathbf{O}^{(n)} = \sum_{j=1}^{n}\mathbf{1}\otimes\cdots\otimes\underbrace{\mathbf{o}}_{\text{jth place}}\otimes\mathbf{1}\cdots\mathbf{1} := \sum_{j=1}^{n}\mathbf{o}_j. \tag{2.10}$$

It is important that this operator is symmetric under all interchanges of the positions of the one-particle operator $\mathbf{o}$ because otherwise one could use it to distinguish a part of the identical particles from another part in the sample. This would contradict our understanding of identical particles!

Further it is clear that a one-particle operator is defined on the Fock space by

$$\mathbf{O} = \sum_{n=1}^{n}\mathbf{O}^{(n)}. \tag{2.11}$$

On the unsymmetrised Fock space the operator can be written as

$$\mathbf{O}^{(n)} = \sum_{j=1}^{n}\sum_{k,l}|k_1\rangle\otimes\cdots\otimes|k_n\rangle\langle k_1|\otimes\cdots\otimes\langle k_n|\,\mathbf{o}_j\,|l_1\rangle\otimes\cdots\otimes|l_n\rangle\langle l_1|\otimes\cdots\otimes\langle l_n|. \tag{2.12}$$

Now we introduce the symmetrisers and anti-symmetrisers:

$$\mathbf{S}^{\pm} = \frac{1}{\sqrt{n!}}\sum_{\mathbf{P}\in S_n}P(\pm)\mathbf{P}. \tag{2.13}$$

Herein we have set $P(-) = \sigma(P)$ and $P(+) = 1$ for $P \in S_n$. The operator $\mathbf{P}$ interchanges the one-particle states of a direct product of n one-particle Hilbert space vectors given by the permutation $P \in S_n$ and is further defined on the n-particle space to be a linear operator. Thus the symmetriser or anti-symmetriser maps the unsymmetric product space to the boson or fermion Fock space. It is a simple exercise to show that $\mathbf{S}^{\pm}$ is a hermitian projection operator, i.e., $(\mathbf{S}^{\pm})^2 = \mathbf{S}^{\pm}$ and $\mathbf{S}^{\pm\dagger} = \mathbf{S}^{\pm}$. With help of this operator we can show for the operator to be defined on the n-particle subspace of the boson or fermion Fock space by

$$\mathbf{O}^{(n)}_{\pm} = \mathbf{S}^{\pm}\mathbf{O}^{(n)}\mathbf{S}^{\pm} = \sum_{j=1}^{n}\sum_{k,l}|k_1\ldots k_j\ldots k_n\rangle_{\pm}\,o_{k_jl_j}\,\langle l_1\ldots l_j\ldots l_n|. \tag{2.14}$$

Here we have used the one-particle matrix elements $o_{kl} = \langle k \,|\mathbf{o}|\, l\rangle$. With help of (2.4) and summing over n we find on the whole Fock space[3]:

$$\mathbf{O}_{\pm} = \sum_{kl} o_{kl} \mathbf{a}^{\dagger}(k)\mathbf{a}(l). \tag{2.15}$$

This has a simple interpretation: The one-particle operator destroys a particle in state l and creates a particle in state k. The "weight" with which this happens is given by the one-particle matrix element o_{kl}.

We give now a very simple *example* for this formalism. As the one-particle basis we chose the generalised position-spin basis $|\vec{x}\sigma\rangle$. As a typical one-particle operator we take the momentum of the particle. The one-particle matrix element is given by

$$\left\langle \vec{x}, \sigma \left|\tilde{\mathbf{p}}\right| \mathbf{x}', \sigma' \right\rangle = -\mathrm{i}\frac{\partial}{\partial \vec{x}} \delta_{\sigma\sigma'} \delta^{(3)}(\vec{x} - \vec{x'}), \tag{2.16}$$

which is already known from elementary quantum mechanics. For a reason we will give in the next section below the creation and annihilation operators for this basis will be denoted by $\boldsymbol{\psi}(\vec{x}, \sigma)$. This operators are called *field operators*. We shall see that these operators can be obtained directly by a canonical quantisation formalism for Schrödinger's equation[4].

Thus by inserting (2.16) into the general formula (2.15) we find

$$\tilde{\mathbf{p}}_{\pm} = \int \sum_{\sigma} d^3\vec{x} \boldsymbol{\psi}^{\dagger}(\vec{x}\sigma) \frac{1}{\mathrm{i}} \frac{\partial}{\partial \vec{x}} \boldsymbol{\psi}(\vec{x}, \sigma). \tag{2.17}$$

[3] making use of the representation of the identity operator (2.5)

[4] In the ancient days of quantum mechanics this formalism was called "second quantisation". Hereby the physicist had in mind that there was the first quantisation procedure which quantised Hamiltonian classical mechanics by making the canonical variables hermitian operators. As we try to show in this chapter this is a misleading idea since the field quantisation is nothing else than many particle quantum mechanics for indistinguishable particles

Chapter 3

Canonical Field Quantisation

In the 19th century the physicists changed their point of view about the fundamental forces (as far as they were known in these days) dramatically. Before, the paradigm of a fundamental force law was Newton's $1/r^2$ behaviour of gravitation, which was extremely successful in describing the motion of the planets around the sun. This law was the description of an instantaneous action at a distance. The force caused by the masses of the bodies attracting each other depends only on the distance of these bodies in space. Changing the relative position of the bodies affects the force without any delay in time.

On the other hand there was Maxwell's theory of electromagnetism, which introduced a new type of *dynamical* quantities into physics, namely *fields*. This description showed that the forces are not acting instantaneously when the sources of this forces change their position, but the forces need a certain time to propagate from one point in space to another. The main physical phenomenon in this context is the propagation of waves. In the case of electromagnetic waves that included the propagation of light, known to be a wave phenomenon a for a long time.

In this chapter we shall give an introduction to field quantisation and its meaning in terms of the operator formalism. This will be done using the most primitive examples, namely on free scalar particles and scalar real ϕ^4-theory. For the rest of our lectures we shall use the more convenient path integral formalism and functional methods. The main goal of this chapter is to get an understanding of the physical meaning of field quantisation as a many-particle theory.

In the beginning we shall recapitulate some facts about the special-relativistic space-time structure in terms of the Poincaré group. Then we shall investigate the unitary representations of this group, which are important in physics. Equipped with these fundamentals we can introduce the action functional for fields in the Lagrangian as well as in the Hamiltonian formulation and prove the classical part of Noether's theorem.

Having done the classical part we can quantise the free theory.

The chapter ends with a short introduction to perturbation theory from the operator point of view applied to ϕ^4 theory.

3.1 Space and Time in Special Relativity

In 1905 Einstein wrote his famous article about "electrodynamics of moving bodies". It was a revolution in our knowledge about space and time. His approach to special relativity was

to take the problems of electrodynamics and Galilean invariance seriously and to give up the Newtonian picture of space and time. He simply postulated that on one hand the velocity of light is independent of the velocity of the source the light comes out (which was an empirical fact at this time) and on the other hand there is no possibility to distinguish between two coordinate frames which are in relative motion with constant velocity. With these two postulates he could investigate what it means to measure time and space intervals and the transformation laws from one inertial frame to another equivalent one.

Nowadays we are in the convenient position that we have Minkowski's famous four-dimensional space-time geometry at hand. Together with the Ricci formalism for tensor algebra and calculus we can formulate relativity in a straightforward mathematical manner. Space and time build together a four-dimensional real vector space. The vectors are given by their components with respect to a coordinate system and are denoted by x^μ with the upper index running from 0 to 3.

On this space we define the Minkowski inner product as the symmetric indefinite bilinear form

$$x^t \hat{g} y = g_{\mu\nu} x^\mu y^\nu \quad \text{with} \quad \hat{g} = (g_{\mu\nu}) = \operatorname{diag}(1,-1,-1,-1). \tag{3.1}$$

Herein a summation over a pair of indices, one upper and one lower, is understood. The upper index t on a column vector means the row vector with the same components (transposition of a matrix).

The physical meaning of the components is that $x^0 = ct$, where c is the velocity of light. In the following we set $c = 1$ which is convenient in relativistic physics. The structure of space and time and so the kinematics of special relativity is now fixed by the statement that a linear transformation, i.e., the transformation from one inertial system to another, is defined as the set of linear matrices acting on the column vectors (x^μ) which leave the Minkowski inner product invariant. That means that $\hat{L}$ is such a transformation matrix if and only if

$$\forall x, y \in \mathbb{R}^4 : (\hat{L}x)^t \hat{g} \hat{L} y = x^t \hat{g} y. \tag{3.2}$$

Since this has to hold for all pairs of vectors x and y we can conclude that $\hat{L}$ is an "allowed" transformation matrix if and only if

$$\hat{L}^t \hat{g} \hat{L} = \hat{g}. \tag{3.3}$$

All linear transformations fulfilling this restriction are called *Lorentz transformations* and build a group of transformations (prove this as an exercise!). This group is called $\mathrm{O}(1,3)$, i.e., the orthogonal group with respect to an inner product with one positive and one negative eigenvalue of its defining symmetric matrix.

Since all points in our four-dimensional space-time are indistinguishable the physical laws are also invariant under a translation in time and space. Together with the Lorentz group these translations build the inhomogeneous Lorentz group or the *Poincaré group*. The main task addressed with our investigation of these groups is to find the mathematical structures which are invariant under the action of this group. Fortunately there are some further restrictions on the physical laws (like causality) which restrict the possible physically important representations of this symmetry. Because of this it is important to know the structure of the Poincaré group quite well.

But at first let us give a physical meaning to the Lorentz group. Since this group leaves the Minkowski inner product invariant there are three distinct classes of vectors, namely:

- (x^μ) time-like $\Leftrightarrow x_\mu x^\mu > 0$,

- (x^μ) light-like $\Leftrightarrow x_\mu x^\mu = 0$,
- (x^μ) space-like $\Leftrightarrow x_\mu x^\mu < 0$.

These three classes of vectors are invariant under Lorentz transformations. They can be visualised by using a $1+2$-dimensional space-time (cf. Fig. 3.1)

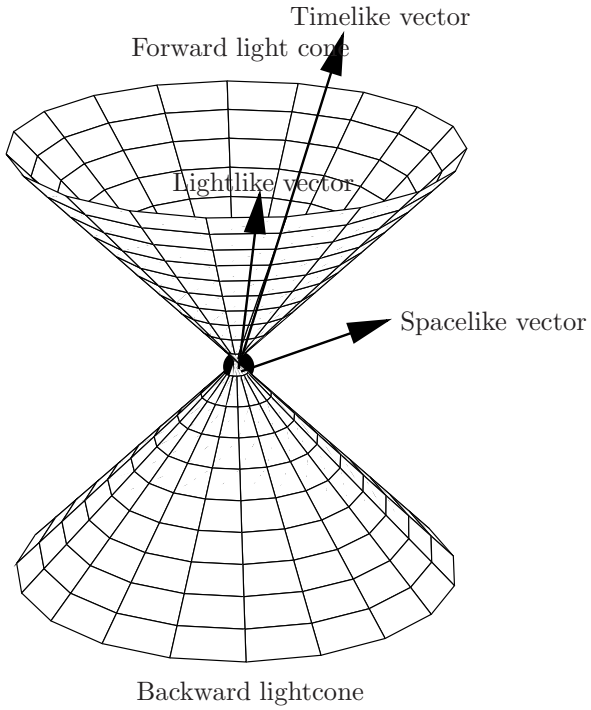


Figure 3.1: $1+2$-*dimensional space-time*

Now we think about the physical meaning of this space-time. It is clear that the geometry of this space-time is different from that of the more familiar Euclidean spaces used in geometry. The reason is, of course, the indefiniteness of the scalar product. From elementary classical electrodynamics we know that the free plain waves are a solution of the Maxwell equations, i.e., the equations of motion for free electromagnetic waves (e.g. light). The propagation of the waves is described by its phase factor which is of the form (here setting c the velocity of light):

$$\exp(-\mathrm{i}\omega t + \mathrm{i}\vec{k}\vec{x}) = \exp(-\mathrm{i}kx) \quad \text{with} \quad x^0 = ct. \tag{3.4}$$

The physical meaning is evidently a plane wave propagating in space along the direction of $\vec{k}$. The Maxwell equations are solved by waves with such phase factors if $k^2 = 0$ which leads to the dispersion relation $\omega = |\vec{k}|/c$. In our covariant language that means that a free electromagnetic wave has a light-like four-vector as the wave vector.

Now in relativistic physics there is no causal signal or event which can travel faster than the light. In more geometrical language this means that two events can only be causally related if their relative four-vector is time-like. If x and y are such events this means $(x-y)^2 > 0$. Now we say that x has happened later than y if $x^0 - y^0 > 0$, i.e. if $x - y$ is located within the forward light-cone. We can say the forward (backward) light-cone defines the future relative to

the origin. All the space-like points are such points which can happen simultaneously from the point of view of an appropriate coordinate frame.

We shall come back now to the Lorentz transformation itself. If we have a transformation that leaves the zero-components of four-vectors invariant, this transformation is of the form $\hat{D} = \mathrm{diag}(1, \hat{d})$ where $\hat{d}$ is a 3×3-matrix. This matrix fulfils the condition $|\hat{d}\vec{x}| = |\vec{x}|$ for all three vectors $\vec{x}$. One can easily show that from this we can conclude that for all three vectors $\vec{x}$ and $\vec{y}$ this transformation has to have the property to leave the three dimensional Euclidean scalar product invariant: $(\hat{d}\vec{x})(\hat{d}\vec{y}) = \vec{x}\vec{y}$. So for the matrix $\hat{d}$ we have $\hat{d}^t\hat{d} = \hat{d}\hat{d}^t = 1$. Such matrices are called *orthogonal matrices*. These matrices describe *rotations* or the composition of a *rotation and a reflection*. All these matrices together build a linear group, called $\mathrm{O}(3)$, i.e. the group of orthogonal transformations of the Euclidean space $\mathbb{R}^3$. The pure rotations are given by orthogonal matrices with $\det \hat{d} = +1$ and form a subgroup of $\mathrm{O}(3)$, namely the so called special orthogonal group $\mathrm{SO}(3)$.

From this reasoning it follows that the whole Lorentz group has $\mathrm{O}(3)$ and $\mathrm{SO}(3)$ as subgroups.

Now we come to another subgroup. As explained above the four-dimensional space-time contains a "causal structure": Namely events which are causally connected are in the light-cone, i.e. its relative vector is time-like. The direction of time is given by the sign of the zero-component of the relative vector, namely if it is in the forward or backward light-cone. Now a physical change of the frame, i.e. the change of the coordinate system of $\mathbb{R}^4$ by a Lorentz transformation, should not change the time direction of causally connected events. Now a Lorentz transformation maps a time-like vector to a time-like vector, because it does not change the Minkowski product. So the time-like basis vector of the original frame e_0 is mapped by the Lorentz transformation to the time-like vector e_0' of the new frame. In order to be in the forward light-cone, defined by the direction of e_0, the applied Lorentz transformation $\hat{L}$ should have a positive matrix element ${L^0}_0$. Now from (3.3) we have $|{L^0}_0| \geq 1$.

So we have ${L^0}_0 \geq 1$ for such Lorentz transformations which do not change the direction of time. They are called *orthochronous Lorentz transformations*. It is clear that these transformations build a subgroup of the whole Lorentz group called the *orthochronous Lorentz group* $\mathrm{O}(1,3)^\uparrow$. This can be proved by the following intuitive argument: Since an orthochronous Lorentz transformation maps all time-like vectors in the forward (backward) light-cone to time-like vectors which remain in the same light-cone, this is also the case for compositions of such transformations and their inverses.

From (3.3) we can also conclude that $\det \hat{L} = \pm 1$ for all Lorentz transformations $\hat{L}$. Now it is clear that the matrices with determinant 1 form a subgroup of all linear coordinate transformations in $\mathbb{R}^4$ called the special linear group $\mathrm{SL}(4, \mathbb{R})$. Since the intersection of two subgroups of a bigger group builds again a subgroup of this group the Lorentz transformations with determinant 1 build a subgroup of the whole Lorentz group, namely the special Lorentz group $\mathrm{SO}(1,3)$ also called *proper Lorentz group*. With the same argument we see that also the intersection $\mathrm{SO}(1,3)^\uparrow = \mathrm{O}(1,3)^\uparrow \cap \mathrm{SO}(1,3)$ is a subgroup of the whole Lorentz group. It is called the proper orthochronous Lorentz group, which is the part of the Lorentz group, which is connected to the identity of the group (this is shown in appendix B).

Now let us investigate what is the meaning of a transformation acting on both the time and the space indices. For this purpose it is enough to find the general form of a transformation keeping x^2 and x^3 constant. This is enough since all three space-like basis vectors are equivalent in this context.

So we write

$$(L^{\mu}{}_{\nu}) = \mathrm{diag}(\hat{l}, 1, 1) \quad \text{with} \quad \hat{l} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.5}$$

Now since $\hat{L}$ is assumed to be in $\mathrm{SO}(1,3)\uparrow$ we have the following restrictions on the submatrix $\hat{l}$:

$$a \geq 1, \quad \det \hat{L} = ad - bc = 1, \quad a^2 - c^2 = 1, \quad b^2 - d^2 = -1, \quad ab - cd = 0. \tag{3.6}$$

The three last conditions are derived from (3.3). The first of these together with the first inequality we use to parameterise

$$a = \cosh\lambda, \quad c = -\sinh\lambda. \tag{3.7}$$

From the last condition we conclude

$$b/d = c/a = -\tanh\lambda \tag{3.8}$$

and since $\det \hat{l} = 1$ we have

$$d\cosh\lambda + b\sinh\lambda = 1 \Rightarrow d(\cosh\lambda - \sinh\lambda\tanh\lambda) = 1 \Rightarrow d = \cosh\lambda, \quad b = -\sinh\lambda. \tag{3.9}$$

So we find finally

$$\hat{l} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{3.10}$$

This applied to the four-vector x gives

$$x' = \begin{pmatrix} x^0\cosh\lambda - x^1\sinh\lambda \\ -x^0\sinh\lambda + x^1\cosh\lambda \\ x^2 \\ x^3 \end{pmatrix} \tag{3.11}$$

which shows that we describe the change of the frame of reference from one coordinate system to another equivalent one which moves with constant velocity along the x^1 axes relative to each other. The origin of the primed system is given by $\vec{x}' = 0$ and is thus moving with respect to the other system with velocity:

$$v = c\frac{x^1}{x^0} = c\tanh\lambda = c\beta. \tag{3.12}$$

Here again we set c for the velocity of light. This is an important physical conclusion of this calculation: The relative velocity of two inertial frames can not be greater than the velocity of light. Since we can express the hyperbolic functions with help of tanh, we have

$$\cosh\lambda = \frac{1}{\sqrt{1-\beta^2}}, \quad \sinh\lambda = \frac{\beta}{\sqrt{1-\beta^2}}. \tag{3.13}$$

Inserting this into (3.12) we find the more familiar form of this physically most important sort of proper Lorentz transformations, namely the so called *Lorentz boosts*. In our case we have found the boost in 1-direction:

$$x'^0 = ct' = \frac{x^0 - \beta x^1}{\sqrt{1-\beta^2}}, \quad x'^1 = \frac{x^1 - vt}{\sqrt{1-\beta^2}}. \tag{3.14}$$

In appendix B we prove the theorem that each proper orthochronous Lorentz transformation can be obtained by composition of a boost in 1-direction and rotations in the original frame and the boosted frame. This should not be too surprising from intuition. It shows the physical meaning of proper orthochronous Lorentz transformations: This matrix group gives the correct relativistic transformations from one inertial frame to another.

It is easy to show that all boosts in 1-direction form a subgroup (exercise: show this by multiplying two matrices of the form (3.10) and application of the addition theorem for the hyperbolic functions). But this is *not* the case for Lorentz boosts in different directions. Two boosts in different directions applied after each other always give a proper orthochronous Lorentz transformation but containing rotations in addition to a boost.

The most famous physical implication of this is the magnetic moment of the electron: In the early days of quantum mechanics, when the physicists calculated the spin-orbit coupling in atomic physics, they miscalculated the electromagnetic moment by a factor 1/2! The reason was that they had to transform from the rest frame of the electron, where it was easy to calculate the magnetic field induced by the motion of the nucleus, back to the rest frame of the nucleus. Thomas has shown that one has to use the Lorentz transformation rather than the Galilean transformation, even for the relatively slow motion of electrons around a nucleus in an atom. This rather complicated calculation yields the correct result 2 for the gyro magnetic factor of the electron (which is valid up to small radiative corrections, which we shall explain in the context of quantum electrodynamics (QED) later). We shall obtain this result immediately from the calculation of a relativistic electron in a homogeneous external magnetic field in lowest order perturbation theory, a calculation much simpler than Thomas's original one. That was the first success in Dirac's relativistic hole theory, the most important precursor of our today's view of relativistic quantum electrodynamics!

3.2 Tensors and Scalar Fields

In physics we have to look for structures which are consistent with the space-time structure chosen to describe nature. In our case the space-time is not too complicated, since the Minkowski space is a flat space with an indefinite metric as we have described above.

In classical physics the question, which structures are the right ones is relatively straightforward. The mathematical objects used to describe nature should have a definite behaviour under Lorentz transformations (at least proper orthochronous Lorentz transformations). Since the Lorentz transformations are linear transformations the most simple covariant quantities are scalars, vectors, and tensors of higher rank. A tensor is defined by its components with respect to a given coordinate system and their transformation rules under Lorentz transformations. So far we know how to transform a vector, namely like the space-time vectors, $x^\mu \to L^\mu{}_\nu x^\nu$, where we have used the *contravariant* vector components (with upper indices). The *covariant* components have lower indices and are defined with help of the Minkowski metric to be $x_\mu = g_{\mu\nu} x^\nu$. It is clear that with the same matrix one can raise the indices again, now written as $x^\mu = g^{\mu\nu} x_\nu$ with $g_{\mu\nu} = (g^{-1})^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

A tensor of rank k is a quantity with k indices (upper or lower) whose components transform under Lorentz transformations like a product of vector components. It is clear that $g^{\mu\nu}$ is a tensor of rank 2 which is invariant under Lorentz transformations due to their defining property (3.3): $g'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma g^{\rho\sigma} = g^{\mu\nu}$. This property ensures that summation over an upper and an

lower index of a tensor yields again a tensor. This operation is called the *contraction* of these two indices.

The covariant vector components transform as

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda^\nu{}_\rho x^\rho = g_{\mu\nu} g^{\rho\sigma} \Lambda^\nu{}_\rho x_\sigma. \tag{3.15}$$

From $\hat{g}\hat{\Lambda}^t\hat{g} = \hat{\Lambda}^{-1}$, which follows immediately from (3.3) one finds

$$x'_\mu = x_\nu (\Lambda^{-1})^\nu{}_\mu. \tag{3.16}$$

By definition it is clear that the product of two tensors with ranks k and l, defined by the products of their components, results in a tensor of rank $k+l$. The most simple example is the scalar product itself. Two vectors x^μ and y^μ build together the tensor $T^{\mu\nu} = x^\mu y^\nu$ of second rank. Now the contraction of the tensor $T^\mu{}_\mu = T^{\mu\nu} g_{\mu\nu}$ gives the invariant Minkowski product of the two vectors.

The next simple quantities are tensor fields, especially scalar or vector fields. They are functions of the space variables obeying certain transformation properties. For instance, in the case of a tensor field of second rank the transformation law under a Lorentz transformation is given by

$$T'^{\mu\nu}(x') = L^\mu{}_\rho L^\nu{}_\sigma T^{\rho\sigma}(x) = L^\mu{}_\rho L^\nu{}_\sigma T^{\rho\sigma}(L^{-1}x') \text{ with } x'^\mu = L^\mu{}_\rho x^\rho. \tag{3.17}$$

The analogous transformation property holds for tensors of any rank. The most simple case is that of a tensor field of rank zero, i.e., a scalar field:

$$\phi'(x') = \phi(x) = \phi(L^{-1}x') \text{ with } x'^\mu = L^\mu{}_\rho x^\rho. \tag{3.18}$$

Under translations all fields behave like scalar fields.

An infinitesimal translation in space and time is thus given by

$$\phi'(x) - \phi(x) = \phi(x - \delta a) - \phi(x) = -\delta a^\mu \frac{\partial}{\partial x^\mu} \phi(x) := \delta a^\mu \mathrm{i}\mathbf{p}_\mu \phi(x). \tag{3.19}$$

$$\phi'(x') = \phi(x) = \phi(x' - \delta a) = \phi(x') - \delta a^\mu \frac{\partial}{\partial x'^\mu} \phi(x') := \delta a^\mu \mathrm{i}\mathbf{p}_\mu \phi(x'). \tag{3.20}$$

Hence the generators of these transformations, which are the operators of energy and momentum, are given by

$$\mathbf{p}_\mu = \mathrm{i}\partial_\mu. \tag{3.21}$$

Now for a free particle energy and momentum are definite quantum numbers. So we look for eigenstates of energy and momentum:

$$\mathrm{i}\partial_\mu \phi_p(x) = p_\mu \phi_p(x) \;\Rightarrow\; \phi_p(x) = N_p \exp(-\mathrm{i} p_\mu x^\mu). \tag{3.22}$$

N is a normalisation constant which will be fixed in a convenient way when we develop the quantum field theory.

With help of the four-momenta we can build a scalar operator $m^2 = \mathbf{p}_\mu \mathbf{p}^\mu$. For a causal propagation of the waves described by the fields the momentum p has to be time-like. So we have $m^2 \geq 0$. Here we shall concentrate on the case $m^2 > 0$. Since m^2 is a scalar it commutes with generators of Lorentz transformations as well as with energy and momentum operators. Thus it is represented by a number in any irreducible representation of the Poincaré group. So it

is a definite quantum number of the system, called the mass. If we think about wave functions of quantum mechanics this is the mass of the particles described by the wave function. The particle at rest has an energy fulfilling the condition $E^2 = m^2$. So there are plane-wave solutions for the free particle with a negative energy. This is a big problem we have to solve since the Hamilton operator should be bounded from below.

Otherwise one could always find an energy state which is lower than any given other one by just adding more particles in a negative-energy state. Then the whole system could never be stable. We could produce as much energy as we like and could construct a perpetuum mobile of first kind which is claimed not to exist.

We shall see that there is an elegant solution in terms of how to interpret the free-particle states in terms of a quantum field theory. In the case of a scalar field the mass is the only intrinsic quantity describing the particles. There can only be intrinsic degrees of freedom like electric charge which will be defined later.

The fields with a definite mass are given by the wave equation for free particles, namely

$$(\mathbf{p}_\mu \mathbf{p}^\mu - m^2)\phi = (-\Box - m^2)\phi = 0 \quad \text{with} \quad \Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta. \tag{3.23}$$

This is the *Klein-Gordon equation* for free particles.

So a general field with definite mass and three momentum consists of a part with a positive and one with a negative energy eigenvalue

$$\phi(x) = N^+ \exp(-\mathrm{i}\omega_{\vec{p}} t + \mathrm{i}\vec{p}\vec{x}) + N^- \exp(+\mathrm{i}\omega_{\vec{p}} t + \mathrm{i}\vec{p}\vec{x}) \quad \text{with} \quad \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}. \tag{3.24}$$

From a physical point of view we cannot see the meaning of the negative energy states which is a problem since in the case of interacting fields we need the *complete* set of plane waves to describe the time evolution of the system. So starting with a state without negative energy parts time evolution of interacting particles most probably will mix in such states.

The (modern) interpretation of this ambiguity is to say that there does not exist a one-particle picture of relativistic quantum mechanics like in the case of the nonrelativistic theory in terms of the Schrödinger equation. Nowadays we know the physical reason since in accelerators there are *produced* a lot of particles. So we have to use a theory to describe these phenomena correctly which has not a fixed number of particles. As we have seen for the nonrelativistic case, quantum field theory fulfils exactly this idea.

Here is a short historical remark at place: In the old days of quantum mechanics relativity (at least special relativity) was a well established theory. So it is not surprising that the physicists started from the very beginning to seek for a relativistic generalisation of quantum mechanics as it was formulated by Schrödinger (which was in these days thought to be the most convenient formulation of quantum mechanics). Since the heuristic concept to find wave equations by assuming a dispersion relation due to the energy momentum relation of the underlying classical mechanics (that was known as the *correspondence principle*), it is easy to imagine that very quickly the physicists found the Klein-Gordon equation. Really it was found by Schrödinger just before he developed his nonrelativistic equation! But the problem were the negative energy eigenstates and the fact that nobody could find a positive definite scalar product with a current-conservation law which settles the time evolution to be unitary as is needed for conserving the probability (i.e. the normalisation of the wave packets). So the pioneers of quantum mechanics concluded that there is not such a simple relativistic one-particle quantum mechanics as is Schrödinger's

nonrelativistic one. On the other hand in these days the only known "elementary particles" were electrons and protons both with spin 1/2 (as was known since the famous work by Goudsmit and Uhlenbeck). So P.A.M. Dirac came to the conclusion that the correct wave equation has to take into account the spin of the "elementary particles". The same time he liked to solve the problem with the negative energy states. So since the Klein Gordon equation was of second order in both time and space derivatives and the successful Schrödinger equation had a time derivative only in first order, he had the ingenious idea to seek a first-order relativistic equation. Since the equation has to be covariant the space derivatives had to be of first order too. As you might guess what he found by that intuitive reasoning was the famous Dirac equation. We shall derive it in the next chapter by studying the systematic representation theory of the Poincaré group (first done by E. P. Wigner in 1939). The Dirac equation indeed did solve one of the two problems, namely that of a positive definite scalar product. But already there were the negative energy eigenstates and they could not be omitted by the same reason as for the scalar case. But now Dirac had another ingenious idea: Since electrons are fermions they can occupy each single-particle quantum state with one and only one particle. So he concluded that the world we live in is given by filling all the negative-energy states with electrons! Since there is an energy gap of $2m_e c^2 = 1022$ MeV no electrons of that Dirac sea can come to our positive-energy world except in cases of hard scattering processes where one can create a pair with an electron and an in these days unknown partner, the "anti-electron", which was realized to have the same mass and spin as the electron but an opposite sign of electric charge. This lead Dirac to the prediction (!) of the existence of antiparticles.

Nowadays we have another sight of these facts, which will be given later in detail: The formalism of quantum field theory gives us the possibility to reinterpret the negative energy states to be antiparticles with positive energy (Feynman-Stueckelberg interpretation). Together with fixing the ground state energy of the now bounded from below Hamiltonian to be zero there is no longer any need for a Dirac sea occupied completely by infinitely many electrons! The advantage of this approach comes from the fact that nowadays there are known hundred kinds of bosons: If the Hamiltonian for them was not bounded from below all the bosons must crash into this infinitely low energy state (at least at zero temperature) and as far as I know there would not be any chance of stable bosons. There is another more physical reasoning against the sea interpretation also for fermions: Since we know that our space-time is much more complicated than in special relativity namely due to general relativity (Wheeler gave it the nice name "geometrodynamics"). This theory of space and time at large scales (there are models of the whole universe like the cosmological standard model with the big bang as the beginning) tells us that the energy-momentum tensor of the matter and fields enters Einstein's equations as source terms for the field of gravity. But gravity is not a field as those we are taking into account in this course! It is nothing else than the curvature of the four-dimensional space-time. Since the Dirac sea should give an infinitely big energy density our universe would look completely different from our experience. From this yet not completely formulated quantum theory of the curved space-time itself it follows that we are on the save side with our Feynman-Stueckelberg interpretation which renormalises the vacuum of the free theory by introducing normal-ordering which is also necessary to give the field-operator products introduced in one of the next sections in the canonical quantisation approach a unique meaning.

Now let us come back to the theory and the most simple case of a scalar field. It is important to realize that the Klein Gordon equation can be formulated by an action functional in the same way as one can formulate Newton's point mechanics with help of an action functional. Here we think of the fields as infinitely many dynamical degrees of freedom, labelled by $\vec{x}$. Thus the

Lagrangian is defined as a space integral over a Lagrange density:

$$L(t) = \int \mathrm{d}^3\vec{x}\mathscr{L}(\phi, \partial_\mu\phi). \tag{3.25}$$

Exercise

To give a very intuitive idea what is meant by this, look at a spring with masses equidistantly connected to it. This masses can move as transversal waves. Find the equation of motion of such a system! After doing this think the masses to be smeared out over the whole spring. You will end with a string and the wave equation for its motion. This is indeed a very intuitive picture of what is meant by a system with infinitely many degrees of freedom!

Then the action is given by integrating (3.25) over a finite time interval (t_1, t_2).

$$S[\phi] = \int \mathrm{d}^4x\mathscr{L}(\phi, \partial_\mu\phi). \tag{3.26}$$

In order to be invariant under Lorentz transformations the Lagrange density, in the context of quantum field theory widely called *Lagrangian*, it is suffcient if it is a scalar field built by ϕ and its gradient.

Now the equation of motion for the field is given by the stationarity condition of the action functional. One immediately finds by applying the tools shown in appendix A that the functional derivative and with it the stationarity condition is given by

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} = 0. \tag{3.27}$$

Now we like to find relativistically invariant equations of motion for fields which are intended to describe particles in the same way as the Schrödinger field does for nonrelativistic particles. We shall find that this fails and that this problem can be solved by thinking about the field equations as operator equations of motion in the sense of a many-particle theory, described in terms of a quantum field theory we have found in the previous chapter from the particle point of view for the nonrelativistic case. There this was only a convenient description for a many-particle system which could as well be formulated in an equivalent way with help of Schrödinger wave functions for a fixed number of particles. As we shall see soon, this is not the case for the relativistic realm, since there is always the possibility to create and annihilate pairs of particles and antiparticles. Thus the number of a certain sort of particles is not conserved but only other quantities like the electric charge. So in the relativistic case the Fock space of a non-definite number of particles is not only a convenient but a physically necessary concept to describe relativistic particles!

Nevertheless at first we stay for a little moment at the classical point of view and find a covariant equation of motion for the scalar field. In the next chapter we shall do this in a more formal and more complete way. Here, we give a simple heuristic argument to find the free particles' equation of motion. This equation of motion should be derivable from a Lagrangian which is quadratic in the field and depends only on the field and its gradient. It should also be a scalar field. Now the only scalar one can build out of the gradient of the field is $\partial_\mu\phi\partial^\mu\phi$. So a good candidate for the Lagrangian of the free real scalar field is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2. \tag{3.28}$$

Using (3.27) we obtain the Klein-Gordon equation for the real scalar field:

$$\Box\phi + m^2\phi = 0 \quad \text{with} \quad \Box = \partial_\mu\partial\mu = \partial_t^2 - \Delta. \tag{3.29}$$

Now we can interpret this equation with help of the meaning of the gradient operator known from nonrelativistic quantum mechanics to be the momentum operator: $\mathbf{p}_\mu = \mathrm{i}\partial_\mu$. So the Klein-Gordon equation gives the relativistic relation between energy and momentum of a free particle, namely $E^2 = \vec{p}^2 + m^2$.

To find a consistent definition of physical quantities we now prove Noether's theorem.

3.3 Noether's Theorem (Classical Part)

As shown above, the classical field theory is defined by an action functional $S[\phi]$. The physical fields are given by the condition of stationarity of S

$$\frac{\delta S}{\delta\phi} = 0, \tag{3.30}$$

which is the equation of motion for the fields. The action functional is given as the four-dimensional integral of the Lagrange density, which is a function of the fields ϕ and their gradients $\partial_\mu\phi$:

$$S[\phi] = \int \mathrm{d}^4x \mathscr{L}(\phi, \partial_\mu\phi). \tag{3.31}$$

The only constraint on the field is that it must vanish at infinity of four-dimensional space such that S exists.

Calculating the functional derivative with help of the techniques shown in appendix A shows that the stationarity condition (3.30) gives the Euler-Lagrange equations for the fields:

$$\frac{\delta S[\phi]}{\delta\phi} = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)} = 0. \tag{3.32}$$

Now we look at a rather general class of symmetry transformations which are described by the operation of a Lie group on the fields and space-time variables. The only assumption we want to make is that the action of the group on the space-time variables is independent on the fields. Then the operation of an infinitesimal transformation can be described by

$$x'^\mu = x^\mu + \delta x^\mu, \quad \phi'(x') = \phi(x) + \delta\phi(x). \tag{3.33}$$

The field variation $\delta\phi(x)$ contains the operation of global and local internal symmetries of the fields as well as the action of the Lorentz group on the fields. Now we have to calculate the change of the action under the change of such an operation:

$$\delta S[\phi] = \int \mathrm{d}^4x' \mathscr{L}(\phi', \partial'_\mu\phi') - \int \mathrm{d}^4x \mathscr{L}(\phi, \partial_\mu\phi). \tag{3.34}$$

In the first integral we have to change the variables of integration to the original space-time variables. Up to first order in δx the Jacobian of the transformation is given by

$$\det\left(\frac{\partial x'^\mu}{\partial x_\nu}\right) = \det\left(\delta^\mu_\nu + \partial_\nu\delta x^\mu\right) = 1 + \partial_\mu\delta x^\mu. \tag{3.35}$$

This can be seen easily using the definition of a determinant as the sum over permutations of matrix elements. In first order in δx only the product of the diagonal elements is important. All other products appearing in the definition of the determinant contain at least two factors δx.

Now we have to take into account that variation and differentiation of the fields do not commute since we are varying the space-time variables as well as the fields:

$$\delta(\partial_\mu\phi) = \partial'_\mu\phi'(x') - \partial_\mu\phi = \partial_\nu(\phi+\delta\phi)\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)^{-1} - \partial_\mu\phi = \partial_\mu(\delta\phi) - (\partial_\mu\delta x^\nu)\partial_\nu\phi. \tag{3.36}$$

Taking (3.35) and (3.36) together with (3.34) we obtain after integrations by parts

$$\delta S[\phi] = \int \mathrm{d}^4x \left[\frac{\delta S[\phi]}{\delta\phi(x)}\delta\phi + \partial_\mu\left((\partial_\nu\phi)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} - \delta^\mu_\nu\mathscr{L}\right)\delta x^\nu\right] = 0. \tag{3.37}$$

The vanishing of the variation of the action functional for all fields (not only for solutions of the equations of motion!) is the definition for symmetry transformations. Now the identical vanishing of the integral for all fields can only be true if the integrand is a four-divergence.

A little calculation concerning the derivative in the second term gives

$$-\int \mathrm{d}^4x \frac{\delta S[\phi]}{\delta\phi(x)}(\partial_\nu\phi)\delta x^\nu. \tag{3.38}$$

Now the infinitesimal operation of the group can be written in the form

$$\delta\phi(x) = \tau_a(x,\phi)\delta\eta^a, \quad \delta x^\mu = -T^\mu_a(x)\delta\eta^a, \tag{3.39}$$

where τ and T are bases of the Lie algebra of the group in the representation on the fields and space-time variables respectively. The $\delta\eta$ are real parameters independent on x and ϕ. All together we draw the conclusion that the integrand of (3.37) has to be a four-divergence:

$$\left[\frac{\delta S[\phi]}{\delta\phi(x)}\tau_a + \frac{\delta S[\phi]}{\delta\phi(x)}(\partial_\nu\phi)T^\nu_a\right]\delta\eta^a = -\partial_\mu j^\mu_a\delta\eta^a. \tag{3.40}$$

For the solutions of the field equations, i.e., such fields which fulfil the stationarity condition (3.30) we have

$$\partial_\mu j^\mu_a = 0 \tag{3.41}$$

since the generators of the Lie group T^a and τ^a are linearly independent.

This is *Emmy Noether's Theorem*:

For each generator of a symmetry group of the action functional there exists a current j^μ_a with vanishing four-divergence. These currents are called the *Noether currents* of the symmetry.

Now we have to find the explicit expressions for the currents. That means we have to express the vanishing of the four-divergence and the constraint on the group to be a symmetry of the action with help of the Lagrange density rather than with help of the action functional. Using (3.37 we find

$$-\delta\eta^a\partial_\mu j^\mu_a = \partial_\mu\left[\left(\partial_\nu\phi\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} - \delta^\mu_\nu\mathscr{L}\right)\delta x^\nu - \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] + \delta\mathscr{L} + \mathscr{L}\partial_\mu\delta x^\mu. \tag{3.42}$$

So we conclude that the group operation is a symmetry of the action, if there exists a field Ω such that

$$\exists\Omega^\mu_a(\phi,x): \quad \delta\mathscr{L} + \mathscr{L}\partial_\mu\delta x^\mu = -\partial_\mu\Omega^\mu_a\delta\eta^a, \tag{3.43}$$

and then the Noether currents are given by

$$\delta\eta^a j_a^\mu = -\left(\partial_\nu\phi\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} - \delta_\nu^\mu\mathscr{L}\right)\delta x^\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi + \Omega_a^\mu\delta\eta^a. \tag{3.44}$$

Now we show that Noether's theorem is the local formulation of a conservation law. This can be done by integrating the vanishing four-divergence over an arbitrary four-dimensional volume V which has a boundary ∂V built by three dimensional space-like hypersurface of the four-dimensional space-time. Hereby a hypersurface is called space-like if all its normal vectors are time-like. Integrating (3.41) over this volume and applying the four-dimensional version of Gauss's integral theorem we obtain

$$\int_{\partial V} j_a^\mu \mathrm{d}S_\mu = 0. \tag{3.45}$$

Now a set of space-like hypersurfaces parameterised by $f(x) = \tau = \text{const}$ (with f an arbitrary scalar field with time-like gradients) defines an invariant measure of time τ. Now we assume that the four-volume V is bounded by the two hypersurfaces $S_1 : f(x) = \tau_1$ and $S_2 : f(x) = \tau_2$:

$$Q_a(\tau_1) = \int_{S_1} j_a^\mu dS_\mu = \int_{S_2} j_a^\mu dS_\mu = Q_a(\tau_2). \tag{3.46}$$

This means that the integral over the space-like hypersurface is a quantity constant in time and that the Qs are independent of the special choice of the space-like hypersurface. For convenience we may use the hypersurface $x^0 = t$ in a given reference frame:

$$Q_a = \int \mathrm{d}^3\vec{x}\, j_a^0(x). \tag{3.47}$$

The Q_a are called the *Noether charges* of the symmetry group.

Now we look at space-time translations. The four Noether charges of this group are the total energy and momentum of the fields. An infinitesimal translation in space and time is defined by

$$\delta\phi(x) = 0, \quad \delta x = -\delta a = \text{const} \Rightarrow \tau_a(x,\phi) = 0, \quad T_a^\mu(x) = \delta_a^\mu, \quad \delta\eta^a = -\delta a^a. \tag{3.48}$$

It is easy to see that the symmetry condition is fulfilled with setting $\Omega^\mu \cong 0$. Then, with help of (3.40), we obtain the corresponding Noether currents $\Theta^\mu{}_a$:

$$\Theta^\mu{}_a = \left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\partial_a\phi - \delta_a^\mu\mathscr{L}. \tag{3.49}$$

This is the so called *canonical energy momentum tensor*, which has no direct physical meaning because it is not unique as we shall show in a moment. On the other hand, the Noether charges are unique physical quantities, namely total energy and momentum of the field system:

$$P_\nu(t) = \int \mathrm{d}^3\vec{x}\Theta^0{}_\nu. \tag{3.50}$$

The Noether currents can be changed with an arbitrary four-divergence without changing the Noether charges. With the j_a^μ defined above there are equivalent Noether currents given by

$$j'^\mu_a = j_a^\mu + \partial_\nu k_a^{\nu\mu} \tag{3.51}$$

Since the divergence of this currents should vanish, we just have to take the $k_a^{\mu\nu}$ antisymmetric with respect of μ and ν, which is a rather weak assumption.

As we shall see, in the case of electrodynamics the canonical energy-momentum tensor cannot be interpreted as density of energy and momentum in all cases of physical interest. For instance in electrodynamics this tensor is not gauge invariant, but we shall see that we can define a physical energy-momentum tensor (the so called Belinfante tensor) which is gauge invariant and gives the well-known expressions for energy and momentum density in form of the familiar Poynting vector.

The spatial components of the physical energy-momentum tensor have the physical meaning of a tension. This can be seen by derivation of the total momentum (3.49) with respect to time and using its conservation. In the case of electrodynamics the space components of the physical energy momentum tensor is Maxwell's tensor of tension.

Now we apply Noether's theorem to the case of Lorentz transformations. An infinitesimal Lorentz transformation acts on the fields and space-time coordinates as follows:

$$\delta\phi(x) = \frac{1}{2}\delta\omega_{\mu\nu}\hat{\sigma}^{\mu\nu}\phi(x), \quad \delta x_\mu = \delta\omega_{\mu\nu}x^\nu, \tag{3.52}$$

where $\hat{\sigma}^{\mu\nu} = -\hat{\sigma}^{\nu\mu}$ are the six generators of the representation of the $\mathrm{SL}(2,\mathbb{C})$ which is the covering group of the $\mathrm{SO}(1,3)$. Because we have $\partial_\mu \delta x^\mu = 0$ the Lorentz invariance (which means that the Lorentz transformations are a symmetry group of the action) is the constraint on the Lagrange density to be a scalar field (seen as a function of x). The six Noether currents are then given by

$$J^{\rho,\mu\nu} = x^\mu \Theta^{\rho\nu} - x^\nu \Theta^{\rho\mu} - \frac{\partial \mathscr{L}}{\partial(\partial_\rho \phi)}\hat{\sigma}^{\mu\nu}\phi. \tag{3.53}$$

Here it is important to anti-symmetrise the currents $J^{\rho,\mu\nu}$ with respect to μ and ν since $\delta\omega^{\mu\nu}$ is antisymmetric. The μ and ν indices label the six Noether currents. Thus the six Noether charges are given by

$$J^{\mu\nu} = \int_{\partial V} \mathrm{d}S_\rho J^{\rho,\mu\nu}. \tag{3.54}$$

Herein V has the same meaning as in (3.45). The three space components coming from the invariance under rotations build therefore the total angular momentum of the system. By looking at (3.53) and the meaning of the energy-momentum tensor one sees that the angular-momentum tensor contains an orbital and a spin part. However, in relativistic physics there is no specific meaning to distinguish orbital and spin angular momentum. Only the total angular momentum has a definite physical meaning.

The conservation of the three mixed temporal-spatial components of $J^{\mu\nu}$ originates from the invariance under boost transformations. So this is the relativistic analogue of the centre of mass motion in the nonrelativistic case.

We close this section with the construction of a symmetric energy-momentum tensor. This is important for general relativity since there the energy-momentum tensor is necessarily symmetric. We shall see in the next chapter that in the case of electrodynamics the tensor can be chosen to be gauge-invariant, which is important to show that energy and momentum densities are sensible physical quantities in this case.

We start with (3.53) and the fact that it is conserved,

$$0 = \partial_\rho J^{\rho,\mu\nu} = \Theta^{\mu\nu} - \Theta^{\nu\mu} - \partial_\rho \frac{\partial \mathscr{L}}{\partial(\partial_\rho \phi)}\hat{\sigma}^{\mu\nu}\phi, \tag{3.55}$$

which shows that the canonical energy-momentum tensor is in general not symmetric in the indices μ and ν.

Now we make the ansatz

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \omega^{\rho\mu\nu}, \tag{3.56}$$

where $\omega^{\rho\mu\nu}$ is an arbitrary tensor field which is antisymmetric in the indices ρ and μ.

We try to choose $\omega^{\rho\mu\nu}$ such that $T^{\mu\nu}$ is a symmetric tensor. Since it differs from the canonical energy-momentum tensor only by a total divergence it yields the same total energy and momentum for the field configuration. The antisymmetry of $\omega^{\rho\mu\nu}$ in ρ and μ makes the divergence $\partial_\mu T^{\mu\nu}$ vanish if $\partial_\mu \Theta^{\mu\nu} = 0$. This means $T^{\mu\nu}$ is as well a representant of the energy-momentum density as the canonical energy-momentum tensor $\Theta^{\mu\nu}$.

Inserting this ansatz into (3.55) shows that it is consistent with setting

$$\partial_\rho \left(\omega^{\rho\mu\nu} - \omega^{\rho\nu\mu}\right) = \partial_\rho \left[\frac{\partial \mathscr{L}}{\partial(\partial_\rho \phi)} \hat{\sigma}^{\mu\nu} \phi\right]. \tag{3.57}$$

The general solution of this equation is given by

$$\omega^{\rho\mu\nu} - \omega^{\rho\nu\mu} = \frac{\partial \mathscr{L}}{\partial(\partial_\rho \phi)} \hat{\sigma}^{\mu\nu} \phi + \partial_\sigma \eta^{\sigma\rho\mu\nu} := \eta^{\rho\mu\nu}, \tag{3.58}$$

where $\eta^{\sigma\rho\mu\nu}$ is an arbitrary tensor field which is antisymmetric in σ and ρ as well as in μ and ν. It is clear that then $\eta^{\rho\mu\nu}$ is antisymmetric in μ and ν.

Now using

$$\omega^{\rho\mu\nu} - \omega^{\rho\nu\mu} = \eta^{\rho\mu\nu}, \quad \omega^{\rho\mu\nu} + \omega^{\mu\rho\nu} = 0 \tag{3.59}$$

we find that with given $\eta^{\rho\mu\nu}$ (3.58) is solved uniquely by

$$\omega^{\rho\mu\nu} = \frac{1}{2}\left[\eta^{\rho\mu\nu} + \eta^{\mu\nu\rho} - \eta^{\nu\rho\mu}\right]. \tag{3.60}$$

It is easy to show by an algebraic calculation that indeed ω fulfills the conditions that we derived for it above. So we find the theorem, proven first by Belinfante in 1939 [Bel39], that we can always find a symmetric energy-momentum tensor.

We shall see in the next chapter that by a clever choice of $\eta^{\sigma\rho\mu\nu}$ which is the only freedom we have to make the energy-momentum tensor symmetric, makes the energy-momentum tensor of the electromagnetic field also gauge invariant.

3.4 Canonical Quantisation

Now we like to solve our problem with the particle interpretation and causality raised by the negative energy states. For this purpose let us consider a free complex scalar field with the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi. \tag{3.61}$$

Although there seems to be no solution in terms of a Schrödinger-like theory, i.e., to interpret the ϕ-field as a one-particle wave function, we try to build a many-particle theory by quantising the fields.

For this purpose we need another formulation of the classical field theory, namely the Hamiltonian one, known from point mechanics and the canonical quantisation as an approach to Dirac's operator formulation of quantum mechanics in terms of the algebra of observables.

This approach has the disadvantage to destroy manifest Lorentz invariance since we have to introduce canonical momentum densities by the definition

$$\Pi = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)}, \quad \Pi^* = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi^*)}. \tag{3.62}$$

Now let us look at the variation of the action,

$$S[\phi] = \int \mathrm{d}^4 x \mathscr{L} \;\Rightarrow\; \delta\phi \frac{\delta S}{\delta \phi} = \int \mathrm{d}^4 x \left[\frac{\partial \mathscr{L}}{\partial \phi} \delta\phi + \delta(\partial_t \phi)\Pi + (\nabla \delta\phi) \frac{\partial \mathscr{L}}{\partial(\nabla\phi)} \right], \tag{3.63}$$

where the nabla symbol ∇ acts only on the spatial components of x. This is what is meant by saying that the formalism is not manifestly covariant since we have now fixed the reference frame by splitting in space and time components.

But now we can proceed in the same way as we would do in the case of point mechanics: We introduce the Hamiltonian density

$$\mathscr{H} = \Pi \partial_t \phi - \mathscr{L}. \tag{3.64}$$

Varying it with respect to the fields we find

$$\delta\mathscr{H} = \delta\Pi \partial_t \phi + \Pi \delta \partial_t \phi + \text{cc.} - \delta\mathscr{L} = \delta\Pi \partial_t \phi - \frac{\partial \mathscr{L}}{\partial \phi}\delta\phi - \delta(\nabla\phi)\frac{\partial \mathscr{L}}{\partial(\nabla\phi)} + \text{cc.} \tag{3.65}$$

This shows that the natural variables for $\mathscr{H}$ are Π, ϕ, $\nabla\phi$ and their conjugate complex counterparts. The Hamiltonian is defined by

$$H = \int \mathrm{d}^3 \vec{x} \mathscr{H}. \tag{3.66}$$

With help of (3.67) we find for the functional derivative of H, where t is to be taken as a parameter,

$$\frac{\delta H}{\delta \phi} = \frac{\partial \mathscr{H}}{\partial \phi} - \nabla \frac{\partial \mathscr{H}}{\partial(\nabla\phi)}, \quad \frac{\delta H}{\delta \Pi} = \frac{\mathscr{H}}{\partial \pi} \tag{3.67}$$

and using the definition of $\mathscr{H}$ together with the equations of motion (3.32), we find the Hamilton equations of motion

$$\partial_t \Pi = -\frac{\delta H}{\delta \phi}, \quad \partial_t \phi = \frac{\delta H}{\delta \Pi}. \tag{3.68}$$

For an arbitrary observable, which is a functional of ϕ and Π and may depend explicitly on t, we find

$$\partial_t O = \int \mathrm{d}^3 \vec{x} \left[\frac{\delta O}{\delta \phi(t,\vec{x})} \frac{\delta H}{\delta \Pi(t,\vec{x})} - \frac{\delta O}{\delta \Pi(t,\vec{x})} \frac{\delta H}{\delta \phi(t,\vec{x})} \right] + \partial_t O := \{O, H\}_{\text{pb}}. \tag{3.69}$$

Particularly we have the following fundamental Poisson brackets:

$$\{\phi(t,\vec{x}), \Pi(t,\vec{y})\}_{\text{pb}} = \delta^3(\vec{x}-\vec{y}), \quad \{\Pi(t,\vec{x}), \Pi(t,\vec{y})\}_{\text{pb}} = 0, \quad \{\phi(t,\vec{x}), \phi(t,\vec{y})\}_{\text{pb}} = 0. \tag{3.70}$$

It should be kept in mind that the Poisson brackets are only defined for functionals at one instant of time. That means that a Poisson bracket makes only sense if the quantities entering have equal time arguments.

Now we can use the recipe of canonical quantisation. The fields become operators and build together with the unit operator the algebra of observables. As we know from the nonrelativistic case, we can quantise fields with help of commutators describing bosonic particles or anti-commutators describing fermionic particles. We shall see soon that in the relativistic case, assuming some simple physically motivated axioms, we can quantise the scalar field only in terms of bosons. The direct translation of the Poisson bracket rules in commutation rules are the fields in the Heisenberg picture, which we shall give capital Greek symbols. Then the canonical commutation relations read

$$\frac{1}{\mathrm{i}}\left[\mathbf{\Phi}(t,\vec{x}),\mathbf{\Pi}(t,\vec{y})\right]=\delta^{(3)}(\vec{x}-\vec{y}),\quad \frac{1}{\mathrm{i}}\left[\mathbf{\Phi}(t,\vec{x}),\mathbf{\Phi}(t,\vec{y})\right]=\frac{1}{\mathrm{i}}\left[\mathbf{\Pi}(t,\vec{x}),\mathbf{\Pi}(t,\vec{y})\right]=0. \tag{3.71}$$

The classical Lagrangian for the free case is given by (3.61). Here ϕ and ϕ^* have to be seen to represent two independent real field degrees of freedom. Now we like to quantise these free fields. The first step is to define the canonical field momenta:

$$\Pi(x)=\frac{\partial\mathscr{L}}{\partial[\partial_t\phi(x)]}=\partial_t\phi^*(x),\quad \Pi^*(x)=\frac{\partial\mathscr{L}}{\partial[\partial_t\phi^*(x)]}=\partial_t\phi(x). \tag{3.72}$$

The canonical non-vanishing commutation relations (3.71) for these field operators therefore read

$$\mathbf{\Phi}(t,\vec{x})\overset{\leftrightarrow}{\partial}_t\mathbf{\Phi}^\dagger(t,\vec{y})=i\delta^{(3)}(\vec{x}-\vec{y}), \tag{3.73}$$

where the symbol $\overset{\leftrightarrow}{\partial}_t$ is defined as

$$f(t,\vec{x})\overset{\leftrightarrow}{\partial}_t g(t,\vec{y})=f(t,\vec{x})\partial_t g(t,\vec{y})-[\partial_t f(t,\vec{x})]g(t,\vec{y}). \tag{3.74}$$

The physical field operators have to fulfill the equations of motion, i.e.

$$(\Box+m^2)\mathbf{\Phi}=0. \tag{3.75}$$

In terms of a Fourier decomposition the field can be written as

$$\begin{aligned}\boldsymbol{\phi}(x)&=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}}\left[\mathbf{A}_+(\vec{p})\exp(-\mathrm{i}\omega(\vec{p})t+\mathrm{i}\vec{p}\vec{x})+\mathbf{A}_-(\vec{p})\exp(+\mathrm{i}\omega(\vec{p})t+\mathrm{i}\vec{p}\vec{x})\right]\\ &\text{with}\quad \omega(\vec{p})=+\sqrt{\vec{p}^2+m^2},\end{aligned} \tag{3.76}$$

where the normalisation of the fields will be explained later on. Now the second part does not look physical since it seems to describe a particle with a time-like momentum in the *negative* light-cone, i.e., a particle which moves in the "direction of the past", which is evidently not consistent with causality. We can reinterpret this term with by using a creation operator of another sort of particles with help of $\mathbf{A}_-(\vec{p})=\mathbf{b}^\dagger(-\vec{p})$ and substitution of $\vec{p}\to-\vec{p}$ in the integral of the "negative-frequency" piece,

$$\boldsymbol{\phi}(x)=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}}\left[\mathbf{a}(\vec{p})\exp(-\mathrm{i}px)+\mathbf{b}^\dagger(\vec{p})\exp(\mathrm{i}px)\right]_{p^0=\omega(\vec{p})}. \tag{3.77}$$

This is the solution of the problem of negative energy states, the so called *Feynman-Stueckelberg interpretation*. This trick is made possible, because we have introduced a multi-particle description with help of the field quantisation, where the annihilation of a negative energy state

corresponding to the motion of the particle backward in time can be seen as the creation of a state of positive energy moving forward in time with a momentum in the opposite direction. This reinterpretation is not possible with c-number fields. This is the first place we see explicitely that the relativistic quantum theory is necessarily a multi-particle theory. Now we define

$$\varphi_{\vec{q}}(x) = \frac{1}{\sqrt{2\omega(\vec{q})(2\pi)^3}} \exp[-\mathrm{i}\omega(\vec{q})t + \mathrm{i}\vec{q}\vec{x}]. \tag{3.78}$$

A simple direct calculation shows that

$$\mathrm{i}\int \mathrm{d}^3\vec{x}\varphi^*_{\vec{q}}(t,\vec{x})\overset{\leftrightarrow}{\partial}_t\mathbf{\Phi}(t,\vec{x}) = \mathbf{a}(\vec{p}), \quad \mathrm{i}\int \mathrm{d}^3\vec{x}\varphi^*_{\vec{q}}(t,\vec{x})\overset{\leftrightarrow}{\partial}_t\mathbf{\Phi}^\dagger(t,\vec{x}) = \mathbf{b}(\vec{p}). \tag{3.79}$$

With help of the canonical commutation relations (3.73) we find

$$\left[\mathbf{a}(\vec{p}), \mathbf{a}^\dagger(\vec{q})\right] = \delta^{(3)}(\vec{p}-\vec{q}), \quad \left[\mathbf{b}(\vec{p}), \mathbf{b}^\dagger(\vec{q})\right] = \delta^{(3)}(\vec{p}-\vec{q}). \tag{3.80}$$

All other commutators between the a- and b-operators vanish. These relations show that the free complex scalar field describes two distinct sorts of particles and that the Hilbert space, the annihilation and creation operators operate in, is the Fock space of a- and b-particles. For example the one-particle states are given by

$$|a\vec{p}\rangle = \mathbf{a}^\dagger(\vec{p})\,|0\rangle\,, \quad |b\vec{p}\rangle = \mathbf{b}^\dagger(\vec{p})\,|0\rangle\,, \tag{3.81}$$

where $|0\rangle$ is the *vacuum* of the theory uniquely defined by $\mathbf{a}(\vec{p})\,|0\rangle = \mathbf{b}(\vec{p})\,|0\rangle = 0$ for all $\vec{p}\in\mathbb{R}^3$.

We now want to calculate the total energy and momentum operators. We just remember that due to Noether's theorem for the classical field theory these quantities are given by

$$P_\nu = \int \mathrm{d}^3\vec{x}\Theta^0{}_\nu = \int \mathrm{d}^3\vec{x}\left[\frac{\partial\mathscr{L}}{\partial(\partial_t\phi)}\partial_\nu\phi + \frac{\partial\mathscr{L}}{\partial(\partial_t\phi^*)}\partial_\nu\phi^* - \delta^0_\nu\mathscr{L}\right], \tag{3.82}$$

which was obtained by using (3.49) and (3.50) from translation invariance of the action.

Now we take instead of the c-number fields their operator counterparts in a very naive way. Here arises the problem of operator ordering since the operators multiplied at the same space-time point do not commute. Let us start with three-momentum with just an arbitrary ordering of the operators:

$$\vec{\mathbf{P}} = \int \mathrm{d}^3\vec{x}[\mathbf{\Pi}(x)\nabla\boldsymbol{\phi}(x) + \nabla\boldsymbol{\phi}^\dagger(x)\mathbf{\Pi}^\dagger(x)]. \tag{3.83}$$

Using the plain wave representation of the field operators (3.77) we find after some algebra

$$\vec{\mathbf{P}} = \frac{1}{2}\int \mathrm{d}^3\vec{p}\,\vec{p}\,[\mathbf{a}^\dagger(\vec{p})\mathbf{a}(\vec{p}) + \mathbf{a}(\vec{p})\mathbf{a}^\dagger(\vec{p}) + \mathbf{b}^\dagger(\vec{p})\mathbf{b}(\vec{p}) + \mathbf{b}(\vec{p})\mathbf{b}^\dagger(\vec{p})]. \tag{3.84}$$

Now $\mathbf{n}_a(\vec{p}) = \mathbf{a}^\dagger(\vec{p})\mathbf{a}(\vec{p})$ is the operator for the number of a-particles per momentum volume with momentum $\vec{p}$ (and the same is true for the analogous expression for the b-particles).

Now we fix the ordering of the fields at the same space-time point by the definition that the vacuum expectation value of the momentum should vanish, because if there is no particle the total momentum should be zero. This cancels the infinite vacuum expectation value of the terms with the creation operator on the right side. This is the first time we have "renormalised" a physical quantity, namely the vacuum expectation value of the three-momentum.

This definition of the vacuum expectation value can be defined by the prespription, just to change the order of the creation and annihilation operators in operator products such that all annihilation operators come to the right and all creation operators to the left. The order of the annihilation or creation operators among themselves is not important since they commute. This is called *normal ordering* and is denoted by enclosing the field operators which are intended to be normal ordered in colons. One should keep in mind that this normal-ordering operation on field-operator products in space-time is by no means a trivial procedure since the fields are a sum of annihilation and creation operators in contrast to the nonrelativistic case, where we have only annihilation operators in the mode decomposition of the field operator. So the final result for the total momentum operator is

$$\tilde{\mathbf{P}} = \int \mathrm{d}^3\vec{x}\vec{e}_a : \mathbf{\Theta}^{0a}(x) := \int \mathrm{d}^3\vec{p}\,\vec{p}[\mathbf{n}_a(\vec{p}) + \mathbf{n}_b(\vec{p})]. \tag{3.85}$$

Herein the field operator ordering in the canonical energy-momentum tensor is well-defined with help of the normal-ordering colons.

Applying this procedure to the zero component of the total four-momentum, which is the total energy of the system, we find the canonically quantised Hamiltonian density which has to be normal ordered as the momentum operator to get rid of the operator-ordering problem and to ensure that the vacuum expectation value of the energy vanishes:

$$\mathbf{H} = \int \mathrm{d}^3\vec{x} : \mathscr{H} := \int \mathrm{d}^3\vec{p}\,\omega(\vec{p})[\mathbf{n}_a(\vec{p}) + \mathbf{n}_b(\vec{p})]. \tag{3.86}$$

Since the density operators are positive semi-definite, the Hamiltonian is bounded from below (it is a positive semi-definite operator), and the vacuum state is the state with the lowest energy.

So we have found a physically sensible interpretation for the free quantised scalar field, if we can show that this scalar field obeys the equation of motion of quantum mechanics. For this purpose one has simply to show that the equations of motion for the Heisenberg picture leads to the correct equations of motion for the field operators, namely

$$\partial_t \boldsymbol{\phi}(x) = \frac{1}{\mathrm{i}}\left[\boldsymbol{\phi}(\mathbf{x}), \mathbf{H}(t)\right], \tag{3.87}$$

which is simply proven by inserting the plain-wave expansion for the field operator (3.77) and (3.86) by use of the commutator relations for the annihilation and creation operators.

Now we have another additional symmetry for our free-field Lagrangian (3.61). This is the invariance under *global phase transformations* of the fields, given by

$$\phi'(x) = \exp(-\mathrm{i}e\alpha)\phi(x), \quad \phi'^*(x) = \exp(+\mathrm{i}e\alpha)\phi^*(x), \quad x' = x \quad \text{with} \quad \alpha \in [0, 2\pi]. \tag{3.88}$$

This is the most simple example for an internal symmetry, i.e., a symmetry of the internal structure of the fields which has nothing to do with the symmetries of space and time. It is called a *global* internal symmetry since the transformation is the same at all space-time points, because the transformation law is independent of space and time variables. We shall see that the assumption of a *local* internal symmetry, which is called a "gauge symmetry" has far reaching important implications for the structure of the theory. Since the standard model is a gauge theory, these type of quantum field theories are the most successful theory in elementary particle physics. Quantum electrodynamics, which we shall study in detail in Chapter 6 is the local gauge theory of the here shown global phase invariance.

Now we apply Noether's theorem to this symmetry. At first we take the infinitesimal version of (3.88):

$$\delta\phi = -\mathrm{i}e\delta\alpha\phi, \quad \delta\phi^* = +\mathrm{i}e\delta\alpha\phi^*, \quad \delta x = 0 \Rightarrow \delta\mathscr{L} = \delta \mathrm{d}^4x = 0 \Rightarrow \Omega^\mu = 0. \tag{3.89}$$

The conserved Noether current from this symmetry is then immediately given by (3.44):

$$j^\mu(x) = -\mathrm{i}e\phi^* \overset{\leftrightarrow}{\partial}{}^\mu \phi. \tag{3.90}$$

In the case of a real field, which means $\phi^* = \phi$ this current vanishes identically. This is understandable since in the real field case there is no phase invariance!

The quantised version is again given a unique meaning by normal-ordering:

$$\mathbf{j}^\mu = -\mathrm{i}e : \boldsymbol{\phi}^\dagger(x) \overset{\leftrightarrow}{\partial}{}^\mu \boldsymbol{\phi}(x) : . \tag{3.91}$$

The corresponding conserved quantity is given by integration over the three-dimensional space (see eq. 3.47) and a simple calculation with use of (3.77) yields

$$\mathbf{Q} = \int \mathrm{d}^3\vec{x}\, \mathbf{j}^0(x) = -e \int \mathrm{d}^3\vec{p}\, \left[\mathbf{n}_a(\vec{p}) - \mathbf{n}_b(\vec{p})\right], \tag{3.92}$$

which shows that the a-particles and b-particles have the same electric charge e with opposite signs. It should be emphasised that it is alone this phase symmetry which makes this pair of particles special, namely to be particle and the corresponding antiparticle. Without this symmetry there would be no connection between the two independent sorts of particles we have called a and b-particles.

At the same time this shows that the normalisation of the plain waves is chosen such that we have the simple representation $\mathbf{n}_a = \mathbf{a}^\dagger\mathbf{a}$ (and analogous for the b-particles) for the density operators. Eq. (3.92) shows that not the particle or antiparticle number are conserved for themselves, but only the difference, the *net-particle number*, which we thus have reinterpreted as the charge-quantum number.

Now we have described the most important physical symmetries and the related quantities for the most simple case of a relativistic quantum field theory. There are more general mathematical, but also very important topics left out. They are shown from a more general point of view in Appendix B, which should be read at the end of this chapter. It describes systematically all representations of the Poincaré group (more precisely we should say its covering group), which have been found to be important in physics so far.

3.5 The Most Simple Interacting Field Theory: ϕ^4

Now we have studied a lot of quantities of the free scalar field. But such fields are a little boring since we never can observe the corresponding particles because they do not interact with any measurement apparatus. But the developed mathematical tools are very important since we have seen in Section 1.9 on the example of nonrelativistic potential-scattering theory that the initial and final states are asymptotically free states fulfilling the free equations of motion. We note that it is not possible to interpret interacting relativistic quantum fields as particles so easily since this interpretation for the free fields rested on the plane-wave mode decomposition in terms of creation and annihilation operators of the field operator and the definition of a vacuum state as

the lowest-energy state. Such a decomposition is generally not possible for interacting quantum fields. Thus, the particle interpretation of the quantum-field theoretical formalism makes only sense in terms of *aymptotically free particles*.

In this Section we shall discuss the most simple interacting quantum field theory. It is not of much importance in the sense of a real physical model but an ideal playground to learn all important mathematical methods used for quantum field theory. We use it here as the most convenient example to derive the Feynman rules from the operator formalism of quantum field theory. That was first done by F. Dyson in 1949 for quantum electrodynamics [Dys49b, Dys49a]. But for this very important theory the path integral technique is much more convenient, so that we use it in only in Chapter 6 to treat QED with many physically important examples for low order perturbation theory. Here we use the operator technique for the much simpler case of ϕ^4-theory to learn all concepts in the operator formalism, which is quite useful when using the path integral formalism too.

The classical ϕ^4 theory is the theory of a single *real scalar field* with the Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m}{2}\phi^2}_{\mathscr{L}_0} - \frac{\lambda}{4!}\phi^4. \tag{3.93}$$

Herein $\mathscr{L}_0$ is the free Lagrangian. The ϕ^4 term is called the perturbation. We now use the *interaction picture*, which is defined such that the time evolution of the operators is generated by the free Hamiltonian. Then the time evolution of the states is necessary generated by the perturbation (see Section 1.1).

The fields, which are the set of fundamental operators generating the algebra of observables, are now self-adjoint. In this case the antiparticles are the same as the particles. The interaction-picture field operators obey the free field equations of motion. Thus the plane-wave decomposition in terms of annihilation and creation operators is given by

$$\boldsymbol{\phi}(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}} \left[\mathbf{a}(\vec{p})\exp(-\mathrm{i}px) + \mathbf{a}^\dagger(\vec{p})\exp(\mathrm{i}px)\right]_{p^0=\omega(\vec{p})}. \tag{3.94}$$

The free Hamiltonian and the perturbation potential are given by

$$\mathbf{H}_0(t) = \int \mathrm{d}^3\vec{x} : \boldsymbol{\Pi}(x)\partial_t\boldsymbol{\phi}(x) - \mathscr{L}_0 :, \quad \mathbf{V}(t) = \frac{\lambda}{4!}\int d^3\vec{x} : \boldsymbol{\phi}^4 : \tag{3.95}$$

where we have again applied the normal-ordering description to give the local operator products a precise meaning.

Now we want to solve the problem of finding the S-matrix elements for scattering processes. In principle that is the same as we did for the case of potential scattering in the nonrelativistic quantum theory.

In the remote past we think about the system as prepared in the distant past an initial state of asymptotically free particles $|i\rangle$ and ask for the transition amplitude to a given final state $|f\rangle$ of also asymptotically free particles in the distant future. These states need to be connected by a time evolution operator, which we obtain in the same manner as we did for the potential scattering problem. In the interaction picture the state $|i\rangle$ of the system is prepared at the time $t_i \to -\infty$ and then evolves with time due to (1.30):

$$|i,t\rangle = T_c \exp\left[-\mathrm{i}\int_{t_i}^{t} d\tau \mathbf{V}(\tau)\right]|i,t_i\rangle. \tag{3.96}$$

For the scattering operator we find

$$\mathbf{S} = T_c \exp\left[-\mathrm{i}\int \mathrm{d}t \mathbf{V}(t)\right] = T_c \exp\left[\int \mathrm{d}^4 x \frac{-\mathrm{i}\lambda}{4!} :\boldsymbol{\phi}^4(x):\right]. \tag{3.97}$$

So the *unitary* operator $\mathbf{S}$ maps the initial state from the remote past, i.e. the in-state, to the final state in the ultimate future, i.e. to an out-state. So up to an indefinite phase factor one obtains the S-matrix elements by "sandwiching" the $\mathbf{S}$-operator with free multi-particle states which can in turn be written as operating with creation and annihilation operators to the vacuum state:

$$|i\rangle = |\vec{p}_1 \vec{p}_2 \ldots \vec{p}_{n_i}\rangle = \prod_{k=1}^{n_i} \mathbf{a}^\dagger(\vec{p}_k)\,|0\rangle\,, \quad \langle f| = \langle \vec{q}_1 \vec{q}_2 \ldots \vec{q}_{n_f}| = \langle 0| \prod_{k=1}^{n_f} \mathbf{a}(\vec{q}_k). \tag{3.98}$$

Thus the S-matrix element for a process with n_i particles with given momenta $\vec{p}_k$ in the initial state scattered to the final state with n_f particles with certain momenta $\vec{q}_k$ is given by

$$S_{fi} = \left\langle 0 \left| \prod_{j=1}^{n_f} \mathbf{a}(\vec{q}_j) \mathbf{S} \prod_{k=1}^{n_1} \mathbf{a}^\dagger(\vec{p}_k) \right| 0 \right\rangle. \tag{3.99}$$

So our problem of calculating S-matrix elements is formulated as the calculation of vacuum expectation values. Perturbation theory is then the expansion of the $\mathbf{S}$-operator given by (3.97):

$$\mathbf{S} = 1 + \frac{-\mathrm{i}\lambda}{4!} T_c \int \mathrm{d}^4 x_1 :\boldsymbol{\phi}^4(x_1): + \frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{4!}\right)^2 T_c \int \mathrm{d}^4 x_1 d^4 x_2 :\boldsymbol{\phi}^4(x_1)::\boldsymbol{\phi}^4(x_2): + \cdots. \tag{3.100}$$

The small parameter of the expansion is the coupling constant λ.

Now the task is formulated: To calculate the S-matrix elements in perturbation theory we need rules to evaluate vacuum expectation values of annihilation an creation operators times time-ordered products of normal-ordered interaction operators. The final result will be the *Feynman-diagram rules*. To obtain them there is some work to be done.

3.6 The LSZ Reduction Formula

The first step is to reduce the problem to that of calculating vacuum expectation values of time-ordered products. This has been achieved by Lehmann, Symanzik and Zimmermann and is one of the most important results of axiomatic quantum field theory [LSZ57, LSZ55]. On the one hand it shows that the mathematical meaning of the asymptotic limits $t_i \to -\infty$ and $t_f \to \infty$ is to be seen as a so-called weak limit, which means that first one has to take vacuum expectation values for the matrix elements and then take the limit. On the other hand it shows for practical purposes that all is done if one can calculate the so called *n-point Green's functions* of the theory, at least in a certain approximation. Our aim is the perturbation theory for calculating this functions. The LSZ reduction formula is deduced most easily with help of the Heisenberg picture. From now on we denote the Heisenberg picture field operators by capital Greek letters. The exact n-point function is defined as

$$\mathrm{i}G^{(n)}(x_1, x_2, \ldots, x_n) = \langle 0\,|T_c \boldsymbol{\Phi}(x_1)\boldsymbol{\Phi}(x_2)\ldots\boldsymbol{\Phi}(x_n)|\,0\rangle\,. \tag{3.101}$$

Since in the Heisenberg picture the states are constant in time, the time evolution of the field operators is generated by the full Hamiltonian, which means that they obey the equation of motion

$$(\Box_x + m^2)\mathbf{\Phi}(x) = -\frac{\lambda}{3!} : \mathbf{\Phi}^3(x) : \quad . \tag{3.102}$$

Now we use the asymptotic condition that in the remote past and future the Heisenberg operators become a free in-field or out-field, respectively. This means

$$\underset{t\to-\infty}{\text{w-lim}}\, \mathbf{\Phi}(x) = \sqrt{Z}\boldsymbol{\phi}_{\text{in}}(x), \quad \underset{t\to\infty}{\text{w-lim}}\, \mathbf{\Phi}(x) = \sqrt{Z}\boldsymbol{\phi}_{\text{out}}(x), \tag{3.103}$$

where w-lim is the symbol for the above explained weak limit and $\sqrt{Z}$ is a normalisation constant. Now an S-matrix element is given by (3.98-3.99). For deduction of the reduction formula it is more convenient to reformulate it step by step out of the overlap

$$S_{fi} = \langle \vec{q}_1 \dots \vec{q}_k; \text{out} \,|\, \vec{p}_1 \dots \vec{p}_l; \text{in} \rangle \,. \tag{3.104}$$

Using the asymptotic condition (3.103) together with the definition of the free field wave functions (3.78) we can write with help of (3.79):

$$\begin{aligned} S_{fi} &= \left\langle \vec{q}_1 \dots \vec{q}_k; \text{out} \left| \mathbf{a}^\dagger_{\text{in}}(\vec{p}_1) \right| \vec{p}_2 \dots \vec{p}_l; \text{in} \right\rangle = \\ &= -\mathrm{i}Z^{-1/2} \lim_{t\to-\infty} \int \mathrm{d}^3\vec{x} \varphi_{\vec{p}_1}(x) \overset{\leftrightarrow}{\partial}_t \langle \vec{q}_1 \dots \vec{q}_k; \text{out} \,|\mathbf{\Phi}(x)|\, \vec{p}_2 \dots \vec{p}_l; \text{in} \rangle \,. \end{aligned} \tag{3.105}$$

Now we can write for an arbitrary function

$$\lim_{t\to\infty} f(t) - \lim_{t\to-\infty} f(t) = \lim_{t_{f/i}\to\pm\infty} \int_{t_i}^{t_f} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} f(t). \tag{3.106}$$

So we find

$$\begin{aligned} S_{fi} &= \underbrace{\left\langle \vec{q}_1 \dots; \text{out} \left| \mathbf{a}^\dagger_{\text{out}}(\vec{p}_1) \right| \vec{p}_2 \dots; \text{in} \right\rangle}_{\text{disc.}} + \\ &+ \mathrm{i}Z^{-1/2} \int \mathrm{d}^4x \partial_t \left[\varphi_{\vec{p}_1}(x) \overset{\leftrightarrow}{\partial}_t \langle \vec{q}_1 \dots; \text{out} \,|\mathbf{\Phi}(x)|\, \vec{p}_2 \dots; \text{in} \rangle \right]. \end{aligned} \tag{3.107}$$

Here we have applied the asymptotic condition (3.103) again. The matrix element in the first line is a δ function times a reaction with one particle less in the out state, which can be seen immediately when we let act the out-creation operators to the out-bra as an out annihilation operator. This is called a *disconnected part*. It corresponds to the situation that one particle is not scattered in the reaction. This is not a very interesting part of the S-Matrix. So we treat only the connected part further. For this we do the time derivatives under the integral:

$$S_{fi} = \text{disc.} + \mathrm{i}Z^{-1/2} \int \mathrm{d}^4x \varphi_{\vec{q}_1}(x)[(q_1^0)^2 + \partial_t^2] \langle \vec{q}_1 \dots; \text{out} \,|\mathbf{\Phi}(x)|\, \vec{p}_2 \dots; \text{in} \rangle \,. \tag{3.108}$$

Now we use the on-shell condition $q_1^2 = m^2$ and the fact that we can integrate by parts with respect to the spatial coordinates (which is not allowed for the time coordinates due to the non-vanishing asymptotic boundary conditions!). With help of these manipulations we obtain the result for the first reduction:

$$S_{fi} = \text{disc.} + \mathrm{i}Z^{-1/2} \int \mathrm{d}^4x_1 \varphi_{\vec{p}_1}(x_1)[\Box_{x_1} + m^2] \langle \vec{q}_1 \dots; \text{out} \,|\mathbf{\Phi}(x_1)|\, \vec{p}_2 \dots; \text{in} \rangle \,. \tag{3.109}$$

The next step is done the same way as before. But now we have to be careful with the operator ordering. As an example look at the out-state with momentum $\vec{q}_1$. Since we like to act with the corresponding in-state to the in-ket by using (3.106) again and this is given for $t \to -\infty$ we have to time-order the Heisenberg field operators. With this ordering we can do all calculations we have done in the first reduction step. This calculations are repeated until there is no in- and out-state left but the vacuum expectation value of the time-ordered Heisenberg operator product, which is the exact n-point Green's function defined above (3.101). The final result is

$$\begin{aligned} S_{fi} &= \text{disc.} + (\mathrm{i}Z^{-1/2})^{n+l} \int \prod_{a=1}^{k} \varphi^*_{\vec{q}_a}(y_a) \prod_{b=1}^{l} \varphi_{\vec{p}_b}(x_b) \prod_{c=1}^{k} (\Box_{y_c} + m^2) \prod_{d=1}^{l} (\Box_{x_d} + m^2) \times \\ &\times \underbrace{\langle 0 \,|T_c \mathbf{\Phi}(y_1) \dots \mathbf{\Phi}(y_k) \mathbf{\Phi}(x_1) \dots \mathbf{\Phi}(x_1) \dots \mathbf{\Phi}(x_l)| \, 0 \rangle}_{\mathrm{i}G^{(k+l)}(y_1,\dots,y_k,x_1,\dots y_l)} . \end{aligned} \tag{3.110}$$

This is called the *LSZ reduction formula* where "LSZ" stands for the physicists Lehmann, Symanzik and Zimmermann who found this theorem when investigating the mathematical meaning of the asymptotic condition [LSZ57, LSZ55]. It describes the connection between the n-point functions (3.101) and the S-matrix elements. The formula tells us to *truncate* the n-point functions with help of the Klein-Gordon operators and multiplying it with free wave functions φ^* for the out- and φ for the in- states. The name *truncated Green's functions* will be explained when we have introduced the formulation in terms of Feynman diagrams!

3.7 The Dyson-Wick Series

Now we want to describe how to calculate the Green's functions order by order perturbation theory. For this we need the transformation from the Heisenberg to the interaction picture and vice versa. We shall show that the transformation from one general picture to another is given by a common time dependent unitary operator for the states and the operators.

As shown in Section 1.1 an operator, which is not explicitly time dependent, has the time dependence

$$\mathbf{O}^{(j)}(t) = \mathbf{A}^{(j)}(t, t_0) \mathbf{O}^{(j)}(t_0) \mathbf{A}^{(j)\dagger}(t, t_0). \tag{3.111}$$

Herein j, running over 1 and 2, labels the two pictures which are assumed to coincide at $t = t_0$. So we have immediately

$$\mathbf{O}^{(1)}(t) = \mathbf{B}^{(12)}(t, t_0) \mathbf{O}^{(2)}(t) \mathbf{B}^{(12)\dagger}(t, t_0) \quad \text{with} \quad \mathbf{B}^{(12)} = \mathbf{A}^{(1)}(t, t_0) \mathbf{A}^{(2)\dagger}(t, t_0). \tag{3.112}$$

In this way we have a unitary operator transforming the operators of the second picture to the operators of the first one. The same argument leads to a unitary operator transforming the states from the second to the first picture:

$$\left| \psi^{(1)}, t \right\rangle = \mathbf{B}'^{(12)}(t, t_0) \left| \psi^{(2)}, t \right\rangle \quad \text{with} \quad \mathbf{B}'^{(12)} = \mathbf{C}^{(1)}(t, t_0) \mathbf{C}^{(2)\dagger}(t, t_0). \tag{3.113}$$

Now we have to prove that the picture transformations are the same for both the states and the operators, i.e. we have to show that

$$\mathbf{B}'^{(12)} = \mathbf{B}^{(12)} \Leftrightarrow \mathbf{A}^{(1)\dagger} \mathbf{C}^{(1)} = \mathbf{A}^{(2)\dagger} \mathbf{C}^{(2)}. \tag{3.114}$$

So in an arbitrary picture we define

$$\mathbf{U}(t,t_0) = \mathbf{A}^\dagger(t,t_0)\mathbf{C}(t,t_0). \tag{3.115}$$

By using (1.10) and (1.12) we have

$$\mathrm{i}\partial_t \mathbf{U}(t,t_0) = \mathbf{A}^\dagger \mathbf{H}[\mathbf{f}(t),t]\mathbf{A}(t,t_0)\mathbf{U}(t,t_0) = \mathbf{H}[\mathbf{f}(t_0),t]\mathbf{U}(t,t_0). \tag{3.116}$$

Herein $\mathbf{f}$ stands for a complete set of fundamental operators generating the algebra of observables.

Since by definition the pictures coincide for $t = t_0$ the operator $\mathbf{U}$ obeys the same equation of motion for all pictures and the same initial condition $\mathbf{U}(t_0,t_0) = \mathbf{1}$, $\mathbf{U}$ is the same for all pictures, i.e., it is a picture independent operator which proves (3.114).

We apply this to the Heisenberg and interaction picture. Since in the Heisenberg picture the states are constant in time, we have

$$|\Psi,t\rangle = |\Psi\rangle \Rightarrow \mathbf{C}^{(H)}(t,t_0) = \mathbf{1} \tag{3.117}$$

and in the interaction picture the time evolution of the states is generated by the interaction part of the Hamiltonian:

$$|\psi,t\rangle = T_c\left[-\mathrm{i}\int_{t_0}^{t} \mathrm{d}\tau \mathbf{V}^{(I)}(\tau)\right]|\psi,t_0\rangle\,. \tag{3.118}$$

So we have for the picture transformation operator

$$\mathbf{B}^{HI} = \mathbf{C}^{(H)}\mathbf{C}^{(I)\dagger} = \mathbf{C}^{(I)\dagger} \tag{3.119}$$

and

$$\mathbf{\Phi}(x) = \mathbf{C}^{(I)\dagger}(t,t_0)\boldsymbol{\phi}(x)\mathbf{C}^{(I)}(t,t_0). \tag{3.120}$$

Since the Heisenberg field operators are defined to obey the asymptotic condition, the two pictures coincide for $t_0 \to -\infty$, so we define

$$\mathbf{C}(t) = \mathbf{C}^{(I)}(t,-\infty) = T_c \exp\left[-\mathrm{i}\int_{-\infty}^{t} \mathrm{d}\tau \mathbf{V}^{(I)}(\tau)\right]. \tag{3.121}$$

Now with this relation we go into (3.101):

$$\mathrm{i}G^{(n)}(x_1,x_2,\ldots,x_n) = \left\langle 0\left|T_c\mathbf{C}^\dagger(t_1)\boldsymbol{\phi}(x_1)\mathbf{C}(t_1)\mathbf{C}^\dagger(t_2)\boldsymbol{\phi}(x_2)\mathbf{C}(t_2)\cdots\mathbf{C}^\dagger(t_n)\boldsymbol{\phi}(x_n)\mathbf{C}(t_n)\right|0\right\rangle. \tag{3.122}$$

Now from the composition rule for the time evolution operator we have

$$\mathrm{i}G^{(n)}(x_1,\ldots,x_n) = \left\langle 0\left|T_c\mathbf{C}^\dagger(t_1)\boldsymbol{\phi}(x_1)\mathbf{C}(t_1,t_2)\boldsymbol{\phi}(x_2)\mathbf{C}(t_2,t_3)\cdots\mathbf{C}(t_{n-1,t_n})\boldsymbol{\phi}(x_n)\mathbf{C}(t_n)\right|0\right\rangle. \tag{3.123}$$

Introducing a time instant $t > \max(|t_1|,\ldots,|t_n|)$ we can write

$$\mathrm{i}G^{(n)}(x_1,\ldots,x_n) = \left\langle 0\left|\mathbf{C}^\dagger(t)T_c\mathbf{C}(t,t_1)\boldsymbol{\phi}(x_1)\mathbf{C}(t_1,t_2)\cdots\mathbf{C}(t_n,-t)\mathbf{C}(-t)\right|0\right\rangle. \tag{3.124}$$

Since the Heisenberg vacuum coincides for $t \to -\infty$ with the free one and is stable under time evolution we have

$$\lim_{t\to\infty}\mathbf{C}(-t)\,|0\rangle = |0\rangle\,,\quad \lim_{t\to\infty}\mathbf{C}(t)\,|0\rangle = |0\rangle\,\langle 0\,|\mathbf{S}|\,0\rangle\,, \tag{3.125}$$

where the vacuum vacuum transition element is just a phase factor. Inserting these relations to (3.124) and taking $t \to \infty$ we find

$$\langle 0\,|\mathbf{S}|\,0\rangle\,\mathrm{i}G^{(n)}(x_1,x_2,\ldots,x_n) = \left\langle 0\left|T_c\boldsymbol{\phi}(x_1)\boldsymbol{\phi}(x_2)\cdots\boldsymbol{\phi}(x_n)\exp\left[-\mathrm{i}\int \mathrm{d}\tau \mathbf{V}^{(I)}(\tau)\right]\right|0\right\rangle. \tag{3.126}$$

3.8 Wick's Theorem

Now we obtain the perturbation series, also known as the Dyson Wick series, by expanding the exponential in (3.126). It is a (formal) expansion in orders of interactions, i.e., in powers of the coupling constant(s). To evaluate this in practice we prove Wick's theorem which expresses time-ordered products in terms of normal-ordered products and vacuum expectation values of time-ordered products of operator pairs.

It is clear that this can be done by using the commutation relations of field operators. By use of the plane-wave expansion (3.77) we find indeed a c-number

$$[\boldsymbol{\phi}(x), \boldsymbol{\phi}(y)] = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})} \left\{\exp[\mathrm{i}p(y-x)] - \exp[\mathrm{i}p(x-y)]\right\}. \tag{3.127}$$

We note that it is a Lorentz-scalar function, because we can write

$$\frac{\mathrm{d}^3\vec{p}}{2\omega(\vec{p})} = \mathrm{d}^4 p\, \Theta(p^0)\delta(p^2 - m^2). \tag{3.128}$$

Obviously $\mathrm{d}^4 p$ and the "on-shell δ distribution" are Lorentz invariant. Further the support of the δ distribution is any time-like momentum on the mass shell, $p^2 = m^2 > 0$, and thus the sign of p^0 is invariant under orthochronous Lorentz transformations. Thus we can write a time-ordered product of interaction-picture field operators with help of the commutator functions as a sum of normal-ordered products. Our aim is to apply this on time-ordered products as they appear in the S-Matrix.

More precisely we can say we need the reformulation of a time-ordered operator product with interaction operators which are composed out of local normal-ordered field operator products coming from the interaction part of the Hamiltonian. In the case of ϕ^4 theory these products are of the form

$$T_c \boldsymbol{\phi}(x_1) \cdots \boldsymbol{\phi}(x_n) : \boldsymbol{\phi}^4(y_1) : \cdots : \boldsymbol{\phi}^4(y_k) : . \tag{3.129}$$

This problem is solved by *Wick's theorem*.

We start with the definition of the contraction of a pair of operators to be

$$T_c \mathbf{UV} = \mathbf{U}^\bullet \mathbf{V}^\bullet + : \mathbf{UV} : . \tag{3.130}$$

The two dots at the left summand of the expression on the right hand side of the equation is what is called a *contraction of two operators*. The contraction is a c-number, i.e., it is an operator proportional to the unit operator. To prove this we split the field operator in a creation and an annihilation part:

$$\boldsymbol{\phi}(x) = \boldsymbol{\phi}_+(x) + \boldsymbol{\phi}_-(x) \tag{3.131}$$

with

$$\begin{aligned} \boldsymbol{\phi}_+(x) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}} \mathbf{a}(\vec{p}) \exp(-\mathrm{i}px)\Big|_{p^0=\omega(\vec{p})}, \\ \boldsymbol{\phi}_-(x) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}} \mathbf{a}^\dagger(\vec{p}) \exp(+\mathrm{i}px)\Big|_{p^0=\omega(\vec{p})}. \end{aligned} \tag{3.132}$$

With this notation we obtain from the definition of time and normal-ordering respectively

$$\begin{aligned} T_c\boldsymbol{\phi}(x_1)\boldsymbol{\phi}(x_2) =: \boldsymbol{\phi}(x_1)\phi(x_2) : &+\Theta(t_1-t_2)\left[\boldsymbol{\phi}_+(x_1),\boldsymbol{\phi}_-(x_2)\right]+ \\ &+\Theta(t_2-t_1)\left[\boldsymbol{\phi}_+(x_2),\boldsymbol{\phi}_-(x_1)\right]. \end{aligned} \tag{3.133}$$

The commutators are indeed c-numbers as shown in eq. (3.127). Since the vacuum expectation value of the normal-ordered product vanishes, a much more convenient equivalent definition is thus

$$\boldsymbol{\phi}(x)^\bullet\boldsymbol{\phi}^\bullet(y) = \langle 0\,|T_c\boldsymbol{\phi}(x)\boldsymbol{\phi}(y)|\,0\rangle = \mathrm{i}\Delta_F(x-y). \tag{3.134}$$

The scalar function Δ_F is the *Feynman propagator for scalar fields* and will be calculated below in terms of its Fourier transform. Now we start with our proof of Wick's theorem by stating a simple lemma. Let $\mathbf{U}$, $\mathbf{V}\ldots\mathbf{X}$ be field operators with arbitrary time arguments and $\mathbf{Z}$ a field operator with a time earlier than any of the times in the former ones. Then we shall prove the following formula

$$\begin{aligned} :\mathbf{UV}\cdots\mathbf{XY}:\mathbf{Z} \;=\; &:\mathbf{UV}\cdots\mathbf{XY}^\bullet\mathbf{Z}^\bullet: + :\mathbf{UV}\cdots\mathbf{X}^\bullet\mathbf{YZ}^\bullet: +\cdots \\ &+ :\mathbf{U}^\bullet\mathbf{V}\cdots\mathbf{XYZ}^\bullet: + :\mathbf{UV}\cdots\mathbf{XYZ}:\,. \end{aligned} \tag{3.135}$$

Since arbitrarily ordered products obey the distribution law of algebra we can assume without loss of generality that all operators are pure annihilation or creation operators.

The statement is trivial, if $\mathbf{Z}$ is an annihilation operator since then the left hand side is a normal-ordered product and all contractions on the right hand side vanish, so that in this case the equation is indeed valid.

Now we assume $\mathbf{Z}$ to be a creation operator. Since one can change the order of the operators under the normal-ordering sign without changing the meaning we assume that $\mathbf{U}\cdots\mathbf{Y}$ is normal-ordered. Now it is trivial to multiply the product from the right with an arbitrary number of creation operators without changing the normal-ordering on both sides of the equation. So we can assume that all operators except $\mathbf{Z}$ are annihilation operators. Then we have just to interchange $\mathbf{Z}$ until it comes to the very left of the product to obtain the normal-ordering of the whole product. The commutators are identical with the contractions, because it is assumed that Z is a time argument less than all other time arguments in the product, and the annihilation operators commute and are thus normal-ordered. So the lemma is proved.

Now we are ready for proving *Wick's theorem*: A time-ordered product of field operators with different time arguments can be transformed into a sum of normal-ordered products multiplied with c-number contractions as follows:

$$\begin{aligned} T_c\mathbf{UV}\cdots\mathbf{XYZ} = \; & \underbrace{:\mathbf{UV}\cdots\mathbf{XYZ}:}_{\text{normal-ordered product without contractions}} \\ & + \underbrace{:\mathbf{U}^\bullet\mathbf{V}^\bullet\mathbf{W}\cdots\mathbf{XYZ}: + :\mathbf{U}^\bullet\mathbf{VW}^\bullet\cdots\mathbf{XYZ}:\cdots}_{\text{sum of all normal-ordered products together with one contraction}} \\ & + \vdots \\ & + \underbrace{\mathbf{U}^\bullet\mathbf{V}^{\bullet\bullet}\mathbf{W}^{\bullet\bullet\bullet}\cdots\mathbf{X}^{\bullet\bullet\bullet}\mathbf{Y}^{\bullet\bullet}\mathbf{Z}^\bullet+\cdots}_{\text{sum over all possible total contracted operator pairs}}\,. \end{aligned} \tag{3.136}$$

It is clear that in the case that we have a product with an odd number of operators in the last line there is just one operator left, in the case of an even number of operators the last line is the product of contracted pairs, which is a c-number.

The proof of the theorem is not difficult and as the reader may have guessed is done by induction. There is indeed nothing to prove if there is just one field operator in the game. It is clear that for two operators the formula is identical with the definition of the contraction of an operator pair (3.130). So suppose the theorem is true for a product with k operators. Now we multiply the formula from the right with an operator $\mathbf{\Omega}$ with a time earlier than all the times in the other operators. Then the left hand side stays time-ordered, and on the right hand side one can apply the lemma proved above. The result is evidently Wick's theorem for $(k+1)$ operators.

If $\mathbf{\Omega}$ has not the most early time argument of all operators, just take the operator with the earliest time out of the product and apply Wick's theorem to the k remaining operators. Then when multiplying with the out-taken operator from the right, which has now the earliest time argument, the argument given above holds showing that Wick's theorem for time-ordered products of field operators with different time arguments hold.

Now it is easy to see that Wick's theorem holds for products containing normal-ordered local products of field operators. Due to normal-ordering one does not have to contract normal-ordered operators, because such contractions vanish.

One should realize that nothing will change in the case of fermionic field operators if we include the following sign conventions to the time and normal-ordering and to contractions: The time and normal-ordering of a given field operator products includes the sign of the permutation needed to reorder the operators into the corresponding sequence of the ordering prescription. In the case of contractions one has to multiply by the sign needed to bring the field operators in the contracted pairs together. The order of the two operators in the pair is the same as in the original product as it is understood above for the bosonic case too.

3.9 The Feynman Diagrams

Now we have all the techniques in our hands to evaluate the perturbation series for the n point Green's functions. But as one can imagine this is not an easy task even for the here discussed vacuum theory, where we have to take vacuum expectation values of time-ordered products. There remain only the totally contracted parts of the right hand side of Wick's theorem (vacuum expectation values of an odd number of fields is always vanishing). Fortunately there is a very nice tool at hand, namely the Feynman diagrams. These give not only a systematic technique for calculating the perturbation series but also very suggestive physical pictures of the scattering processes. We have explained this interpretation in the case of non-relativistic potential scattering in Section 1.9.

To derive the diagram rules we start with the expression (3.126) and expand the exponential inserting the ϕ^4 interaction, $\mathbf{V}^{(I)}(t) = \int d^4x : \boldsymbol{\phi}^4(x) :$

$$\begin{aligned} &\mathrm{i}G^{(n)}(x_1,\ldots,x_n)\,\langle 0\,|\mathbf{S}|\,0\rangle \\ &\quad = \left\langle 0 \left| T_c \boldsymbol{\phi}(x_1)\cdots\boldsymbol{\phi}(x_n) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\mathrm{i}\lambda}{4!}\right)^k \int \mathrm{d}^4y_1\cdots\mathrm{d}^4y_k : \boldsymbol{\phi}^4(y_1):\cdots:\boldsymbol{\phi}^4(y_k): \right| 0 \right\rangle . \end{aligned} \tag{3.137}$$

Now we apply Wick's theorem (3.136). In our case of vacuum quantum field theory we have to sum over all *fully contracted expressions* leaving out those which contain contractions over normal-ordered operator pairs. Although this can be done for low orders by hand, it is a quite complicated combinatorial business. he finding of all contractions of a given order for the n-point Green's function can be systemised with help of the diagrammatic rules invented by Feynman [Fey49] for QED (before Dyson's field theoretical derivation). In this chapter we want to find the rules for ϕ^4-theory.

Each interaction contains four normal-ordered field operators. The diagrammatic element, which describes this, is a so-called vertex, drawn as a point representing the space-time variables entering the normal-ordered field operators with four legs standing for the field factors contained in the interaction. A contraction is depicted as the link between two space-time points. The

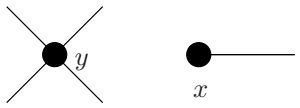


Figure 3.2: *Diagrammatic elements for ϕ^4-theory. The vertex contains an inner space-time point entered by four legs representing the four fields in the interaction. An outer point is attached to one leg representing the field operator with this external point as its argument.*

contraction of two operators at the same space-time point, a so called *tadpole diagram*, makes the whole expression to vanish, because it represents the contraction of two field operators coming from one interaction term of the Hamiltonian and these are normal ordered by definition.

Now we can give the *Feynman rules* in the space-time representation.

For calculating the contributions of the Dyson Wick series to the n-point function

$$G^{(n)}(x_1, \ldots, x_n) \langle 0 | \mathbf{S} | 0 \rangle \tag{3.138}$$

of kth order draw all topologically distinct diagrams with n external space-time points $x_1, \ldots, x_n$ containing k vertices with inner space-time points $y_1, \ldots, y_k$.

To find the analytical expression depicted by a certain diagram

1. Write down a factor $-\mathrm{i}\lambda/4!$ for each vertex.
2. Link the legs of the vertices and external field operators together to obtain the given diagram. At the same time count the number of ways you can do this. Hereby the factor $1/k!$ in the series is cancelled by the permutations of the vertices. So one can count the possible links to obtain the desired diagram with fixed vertices. Multiply the whole diagram with this number.
3. For each line connecting two space-time points (internal or external) multiply with a free Feynman propagator
$$\mathrm{i}D_F(x_1 - x_2) = \langle 0 | T_c \boldsymbol{\phi}(x_1) \boldsymbol{\phi}(x_2) | 0 \rangle . \tag{3.139}$$
4. Integrate over all internal space-time points.

Now we have a nice tool for calculating the n-point functions in perturbation theory. Before we give examples for this we have to find out what is about the vacuum to vacuum transition factor.

This is given by

$$\langle 0\,|\mathbf{S}|\,0\rangle = \sum_{j=0}^{\infty} \frac{1}{j!} \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_j \left\langle 0 \left| T_c : \boldsymbol{\phi}^4(y_1) : \cdots : \boldsymbol{\phi}^4(y_j) : \right| 0 \right\rangle . \tag{3.140}$$

In diagrammatic language this is the sum over all diagrams without external points, the closed diagrams. One has to take into account that in this case the $1/j!$ is not cancelled completely. Thus one has to count the possibilities of pairing the fields to contract them without fixing the vertices. It comes out that there is always a factor $1/j$ left compared to diagrams with external points.

Now we define diagrams of the first class to be such which do not contain any vacuum subdiagram. A vacuum subdiagram is a diagram which does not contain at least one of the external points.

Now we can write the k^{th}-order term of the Dyson Wick series as the sum over all first class diagrams times the vacuum subdiagrams with together k vertices. The result is

$$\begin{aligned} \mathrm{i}G^{(n)}(x_1, x_2, \ldots, x_n)\,\langle 0\,|\mathbf{S}|\,0\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \times \\ \times \left\langle 0 \left| T_c \boldsymbol{\phi}(x_1) \cdots \boldsymbol{\phi}(x_n) \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_j \left(\frac{-\mathrm{i}\lambda}{4!} \right)^j : \boldsymbol{\phi}^4(y_1) : \cdots : \boldsymbol{\phi}^4(y_j) : \right| 0 \right\rangle^{(1)} \\ \times \left\langle 0 \left| T_c \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_k \left(\frac{-\mathrm{i}\lambda}{4!} \right)^{k-j} : \boldsymbol{\phi}^4(y_{j+1}) : \cdots : \boldsymbol{\phi}^4(y_k) : \right| 0 \right\rangle . \end{aligned} \tag{3.141}$$

The binomial is the combinatorial number of possibilities to pick j interaction factors out of k connected with external points, denoted by $\langle \cdots \rangle^{(1)}$. Now we interchange the order of the two summations

$$\begin{aligned} \mathrm{i}G^{(n)}(x_1, x_2, \ldots, x_n)\,\langle 0\,|\mathbf{S}|\,0\rangle = \\ \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=j}^{\infty} \frac{1}{(k-j)!} \left\langle 0 \left| T_c \int \mathrm{d}^4 y_{j+1} \cdots \mathrm{d}^4 y_k \left(\frac{-\mathrm{i}\lambda}{4!} \right)^{k-j} : \boldsymbol{\phi}^4(y_{j+1}) : \cdots : \boldsymbol{\phi}^4(y_k) : \right| 0 \right\rangle \\ \times \left\langle 0 \left| T_c \boldsymbol{\phi}(x_1) \cdots \boldsymbol{\phi}(x_n) \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_j \left(\frac{-\mathrm{i}\lambda}{4!} \right)^j : \boldsymbol{\phi}^4(y_1) : \cdots : \boldsymbol{\phi}^4(y_j) : \right| 0 \right\rangle^{(1)} . \end{aligned} \tag{3.142}$$

Now substituting $k' = k - j$ in the inner sum and renaming the integration variables of the vacuum to vacuum diagrams one sees that this expression factorises such that the vacuum to vacuum expectation value on the left side of the equation can be cancelled:

$$\begin{aligned} \mathrm{i}G^{(n)}(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{\infty} \frac{1}{j!} \\ \times \left\langle 0 \left| T_c \boldsymbol{\phi}(x_1) \cdots \boldsymbol{\phi}(x_n) \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_j \left(\frac{-\mathrm{i}\lambda}{4!} \right)^j : \boldsymbol{\phi}^4(y_1) : \cdots : \boldsymbol{\phi}^4(y_j) : \right| 0 \right\rangle^{(1)} . \end{aligned} \tag{3.143}$$

So one has only to sum over all diagrams, connected and disconnected, where all sub-diagrams are connected to at least one external point. From Wick's theorem we know that only $G^{(n)}$ with n even are different from 0.

To complete the Feynman rules, we only have to calculate the Feynman propagator (3.139). With help of the unit-step function we can rewrite it as

$$\mathrm{i}D_f(x_1-x_2)=\Theta(t_1-t_2)\,\langle 0\,|\boldsymbol{\phi}(x_1)\boldsymbol{\phi}(x_2)|\,0\rangle+\Theta(t_2-t_1)\,\langle 0\,|\boldsymbol{\phi}(x_2)\boldsymbol{\phi}(x_1)|\,0\rangle\,. \tag{3.144}$$

Using (3.94) we find after a little calculation

$$\begin{aligned}\mathrm{i}D_F(x_1-x_2) &= \int\frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3}\{\exp[-\mathrm{i}k(x_1-x_2)]\Theta(t_1-t_2)+\\ &+ \Theta(t_2-t_1)\exp[-\mathrm{i}k(x_2-x_1)]\}_{k^0=\omega(\vec{k})}.\end{aligned} \tag{3.145}$$

Now it is easy to show with help of the residue theorem that we can write this as

$$\mathrm{i}D_F(x_1-x_2)=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\exp[-\mathrm{i}k(x_1-x_2)]}{k^2-m^2+\mathrm{i}\epsilon}. \tag{3.146}$$

For this purpose one has to close the path of the p^0 integration running along the real axes with an infinite half circle in the upper plane for $t_1<t_2$ or in the lower half plane for $t_2<t_1$ (the half circles are then negligible because the contribution coming from them is exponentially damped). This is depicted in figure 3.3.

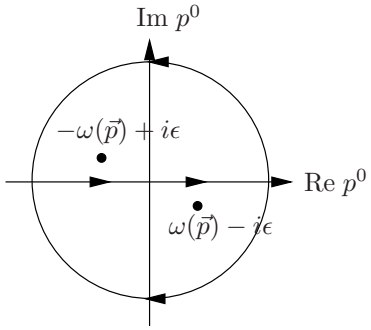


Figure 3.3: *The p^0-plane for the evaluation of the Fourier integral (3.146). The poles of the integrand are shown. The path of integration is to be closed in the upper half plane for $t_1<t_2$ and in the lower one for $t_2<t_1$. In this way the $\mathrm{i}\epsilon$-description in the p^0 plane gives the correct causal boundary conditions for the Feynman propagator which is defined by the time ordering of the field operators, i.e., the Θ-functions in (3.145).*

This Fourier representation for the Feynman propagator shows that it is a Lorentz-scalar translation-invariant field. Its explicit form in space-time representation is involved and not needed further on, since we shall now derive the Feynman rules in four-momentum representation. For this purpose we introduce the Fourier transform of the n-point Green's function as

$$G^{(n)}(x_1,\ldots,x_n)=\int\frac{\mathrm{d}^4p_1}{(2\pi)^4}\cdots\frac{\mathrm{d}^4p_n}{(2\pi)^4}\exp\left(\mathrm{i}\sum_{j=1}^{n}p_jx_j\right)\tilde{G}^{(n)}(p_1,\ldots,p_n). \tag{3.147}$$

Because of four-momentum conservation, i.e., space-time translation invariance, there is an energy-momentum conserving δ-function contained in $\tilde{G}$:

$$\tilde{G}^{(n)}(p_1,\ldots,p_n)=(2\pi)^4\delta^{(4)}(p_1+\cdots+p_n)G^{(n)}(p_1,\ldots,p_n). \tag{3.148}$$

We have used the same symbol $G^{(n)}$ for both the space-time as well as for the momentum representation of the n-point Green's function. There is no danger to mix these functions up since it is clear from the context and by assigning the arguments which one is meant.

Putting in this definition to the Feynman rules, writing all Feynman propagators in the Fourier representation (3.146) one realises that the integration over the internal vertex point y_j gives a factor

$$\int \mathrm{d}^4 y_j \exp(-\mathrm{i} y_j q_j) = (2\pi)^4 \delta^{(4)}(q_j), \tag{3.149}$$

where q_j is the sum of all momenta flowing into the vertex. These momenta can come from propagators (in which case one has to integrate over them) as well as from the external momenta flowing into the whole diagram. All the integrations over the external x_k yield momentum space propagators for the external lines, containing the over-all energy-momentum conserving δ distribution as given in (3.147) at any order of perturbation theory separately.

Now we write down the Feynman rules in momentum space. The diagrams are the same as in the space-time representation. Each vertex stands for $-\mathrm{i}\lambda/4!$. Each propagator line carries an internal momentum k_j or an external momentum p_j and represents a factor $\mathrm{i}D_F(k_j) = \mathrm{i}/(p^2 - m^2 + \mathrm{i}\epsilon)$. The calculation of the symmetry factor for the diagram is the same as in the space-time representation.

For calculating the $G^{(n)}(p_1, \ldots, p_n)$ the incoming momenta have to be conserved, which means $p_1 + \cdots + p_n = 0$. Further on one has to fulfill momentum conservation at each vertex. Then there is just to integrate over the internal momenta with $\int \mathrm{d}^4 k/(2\pi)^4$ which are not yet fixed by momentum conservation. They are all related with *closed loops within the Feynman diagram.*

As the most simple nontrivial example let us give the expression for the diagram shown in figure 3.4 which is a second order two loop contribution to the two point function:

$$\begin{aligned} \mathrm{i}G_2^{(2)}(p,-p) &= \left(\frac{-\mathrm{i}\lambda}{4!}\right)^2 \cdot 4 \cdot 4! \int \frac{d^4k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{\mathrm{i}}{k_1^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{k_2^2 - m^2 + \mathrm{i}\epsilon} \times \\ &\times \frac{\mathrm{i}}{(k_1 + k_2 + p)^2 - m^2 + \mathrm{i}\epsilon} \left(\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}\right)^2. \end{aligned} \tag{3.150}$$

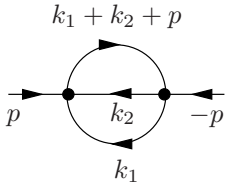


Figure 3.4: *second order two-loop Contribution to the 2-point Green's function in ϕ^4 theory.*

Here for the first time we encounter a very serious problem which made quantum field theory look pretty obscure to the physicists for many years. The integral given in (3.150) is divergent from integrating over all $\mathbb{R}^4$ for both loop momenta, k_1 and k_2. The divergence orginates from the high-momentum region. This can be seen by a simple power-counting argument: The integrand is of order k_1^{-4} and the four-volume element is of order k_1^3, so that the already the k_1-integral alone, keeping k_2 fixed for this consideration, is logarithmically divergent. The experience shows that the most diagrams with loops are divergent for high momenta. This is known as ultraviolet divergence or *UV divergence.* If there are particles with zero masses there are also infrared divergences or *IR-divergences*, i.e., divergences for small momenta.

On the other hand we have seen that there is the problem of normalisation in the LSZ reduction formalism. We shall see in Chapter 5, how to get rid of the divergent integrals and solve the normalisation problem in the reduction formula. Here we just give the idea of this so called *renormalisation program* of quantum field theory: The first step is to find a way to make the integrals finite in a way which keeps the symmetries of the procedure unchanged. This is called the *regularisation*. The regularised integrals of the theory contain a parameter (for instance ϵ), where the physical case is given for some limit of this parameter (e.g. $\epsilon \to 0$). In this limit the results are divergent.

On the other hand there is a certain class of quantum field theories, called *renormalisable* which can be made finite by assuming that the parameters entering the Lagrangian (like masses, coupling constants, wave function normalisation factors etc.) are not the physical observed quantities but the so-called *bare* quantities. Since we do not observe free particles but the fully interacting ones, we can never see the bare parameters but only the physical ones. A theory is now called renormalisable if one has a finite set of conditions on the n-point Green's functions defining the physical parameters of the theory at certain points in momentum space, and these conditions can be fulfilled by adding counter terms to the Lagrangian which are of the same form as the given ones, making the theory finite for $\epsilon \to 0$. Then the infinities are absorbed by the bare quantities which cannot be observed. The physical parameters are to be found by fitting to observed measurements like cross sections, and at the end all quantities have to be expressed in terms of these finite physical parameters.

Although this concept does not look very convincing at the first glance it is very successful in describing the subatomic world. The most precise single theory is QED, which predicts quantities like the *anomalous magnetic moment of the electron* or the radiative corrections to the energy levels of the Hydrogen atom (the so-called *Lamb shift*) with a phantastic precision of several significant digits.

Also our theoretical picture about "elementary particles", i.e., the *Standard model of elementary particles* is a quantum field theory (more precisely a gauge theory) where there is up to now no experimental result which is against this theory. To the contrary, the standard model proves to describe all data at any available bombarding energy of leptons and protons to an astonishing high accuracy, including the radiative corrections of the electroweak interaction. As I edit these notes (November 2012), all the particles and their properties predicted by consistency arguments of the theory, are found (except of the famous Higgs boson, for which however very strong evidences have just been found at the Large Hadron Collider (LHC) at CERN very recently).

Chapter 4

Relativistic Quantum Fields

In this chapter we shall exploit the work we have done in appendix B and use it to construct the quantum field theory for fields of higher spin. In chapter 3 we have treated from a more physical point of view the scalar field. From this we learnt the concepts we try to extend to the more general case. These concepts can be summarised in the following axioms

1. A system consisting of elementary particles can be described by local quantum fields. The Hilbert space is given as the Fock space with asymptotic one particle states building a irreducible representation of the universal covering group of the proper orthochronous Poincaré group $\mathrm{P}_+^\uparrow$.

2. The Hamiltonian of the system is bounded from below, and it exists a normalisable ground state. This is called "the vacuum" further on and denoted by $|0\rangle$.

3. Let $\mathscr{O}_1(x)$ and $\mathscr{O}_2(x)$ operators, which represent local observables (e.g. energy-momentum density, charge density etc.). Then they commute at space-like separated points, i.e.

$$[\mathscr{O}_1(x), \mathscr{O}_2(y)] = 0 \text{ for } (x-y)^2 < 0. \tag{4.1}$$

   This assumption is called the *microcausality condition*.

4. The Hamiltonian can be formulated as an integral over a normal-ordered Hamiltonian density $\mathscr{H}(x)$.

At first we shall construct the representation spaces of $\mathrm{P}_+^\uparrow$ in terms of one-particle configuration space wave functions with spins ≤ 1. These will be formulated as appropriate c-number fields which are restricted to the representation states by deriving certain equations of motion and constraints. It will come out that we can find in this way the free particle wave functions which can be *also* described with help of a Lagrangian density.

After this we are using the method of canonical field quantisation to construct the Fock space of states. It is well known that there are problems with quantising fields with constraints, such that it is more convenient to use formal path integral and functional methods for quantising this kind of theories.

The first part of the the chapter has the aim to prove the two most general theorems of local relativistic quantum field theory namely the *connection between spin and statistics*, i.e., the fact

that particles with half-integer spins are fermions and particles with integer spins are bosons, and the CPT *theorem* which tells us that a theory with the above mentioned features has to be also invariant under the transformations which simultaneously reflects space and time and interchanges particles with their corresponding antiparticles.

It should be emphasised that both theorems are connected only in the way that the CPT needs the assumption that fields building a tensor representation of $\mathrm{SO}(1,3)^{\uparrow}$ are quantised as bosons and such which build a "half integer" representation of the covering group $SL(2,\mathbb{C})$ are quantised as fermions. This assumption for the proof of the CPT theorem is consistent with the spin-statistics theorem while the latter is logically independent from the former.

4.1 Causal Massive Fields

In appendix B we have constructed the one particle irreducible Hilbert spaces which are consistent with relativistic invariance. This means that these spaces build irreducible representations of the proper orthochronous Poincaré group.

Now we want to find the configuration space representation of these Hilbert spaces, i.e. the representation in L^2 with the functions defining the free fields.

At first we have to find the momentum eigenfunctions with definite mass and spin which build a complete set for the given irreducible Hilbert space. The momentum operators are immediately found by the fact that they generate translations:

$$\phi'_\sigma(x') = \phi_\sigma(x),\ x' = x + \delta x \Rightarrow \mathbf{p}_\mu = i\partial_\mu. \tag{4.2}$$

From this we have the eigenstate of $\mathbf{p}$ in $\mathscr{H}(m,s,+)$

$$\phi_{p\sigma}(x) = N\exp(-\mathrm{i}px)u_\sigma(p). \tag{4.3}$$

The fact that $p^2 = m^2$ is described in configuration space by the field equation

$$\mathbf{p}^2\phi_{p\sigma}(x) = m^2\phi_{p\sigma}(x) \Rightarrow (\Box + m^2)\phi_{p\sigma}(x) = 0. \tag{4.4}$$

Thus we have the simple result that each field component has to obey the *Klein-Gordon equation* we have found by the same (but a little bit more hand-waving presented argument in chapter 3 for the scalar field which is of course a special case included here as the L^2-representation for $\mathscr{H}(m, s=0, \pm)$.

Since we are in the space with positive energy the momenta fulfil the on-shell condition in the form

$$p^0 = +\sqrt{m^2 + \vec{p}^2} := \omega(\vec{p}). \tag{4.5}$$

The transformation law under Lorentz transformations is given with help of the representation $D^{(s)}$ of the $SU(2)$ which is the little group of the standard momentum $p_0 = (m,0,0,0)$ (see appendix B):

$$\mathbf{U}(\hat{L})\phi_{p\sigma}(x) = D^{(s)}_{\sigma\sigma'}[\hat{K}(\hat{L},p)]\phi_{p\sigma'}(\hat{L}^{-1}x). \tag{4.6}$$

In our space with definite mass m we have

$$\phi_{p\sigma}(x)\overset{\leftrightarrow}{\Box}\phi_{p'\sigma'}(x) = \partial_\mu\phi_{p\sigma}\overset{\leftrightarrow}{\partial^\mu}\phi_{p'\sigma'}(x) = 0. \tag{4.7}$$

As discussed in connection with Noether's theorem in chapter 3 we know that this implies that

$$F_{\sigma\sigma'}(p,p') = \int \mathrm{d}^3\vec{x} N^*(p') N(p) u^*_{\sigma'}(p') u_\sigma(p) \exp[\mathrm{i}\omega(\vec{p}') - \mathrm{i}\vec{p}\vec{x}] \overset{\leftrightarrow}{\partial}_t \exp[-\mathrm{i}\omega(\vec{p}) + \mathrm{i}\vec{p}x] = \\ = -(2\pi)^3 \mathrm{i} 2\omega(\vec{p}) u'_\sigma(\vec{p})^* u_\sigma(\vec{p}) |N(p)|^2 \delta^{(3)}(\vec{p} - \vec{p}') \tag{4.8}$$

is a $SO(1,3)^\uparrow$-invariant quantity. Since it is positive definite, it is the invariant scalar product on the Hilbert space $\mathscr{H}(m,s,+)$. The same is true with an additional sign for the corresponding space with negative energy $\mathscr{H}(m,s,+)$. As in nonrelativistic quantum theory $N(p)$ is defined up to an unimportant phase factor by the assumption that the generalised energy-momentum eigen-solutions are normalised to a δ function in the continuous independent quantum numbers $\vec{p}$ and a Kronecker-δ with respect to the discrete spin projection quantum numbers σ.

Since we have treated the scalar fields in a more intuitive way in chapter 3 we can go further and apply our representation theory on the fields with $s \geq 1/2$. The reader is invited to do the scalar case in our more systematic way as a simple exercise!

4.1.1 Massive Vector Fields

We start with $s = 1$ since this is the most simple case besides the scalar field to get the idea of the whole story.

But it is clear from our discussion that all types of fields can be built as field products of spinors since all representations are given by tensor products of spinors (eventually reduced to its irreducible parts). This makes it possible to restrict the rest of the chapter to the spin-1/2 Dirac fields explained below without loss of generality in respect to the transformation properties of the fields under the Poincaré groups.

But now we treat first the case of $\mathscr{H}(m,1,+)$. The standard momentum for the little group is again chosen to be $p_0 = (m,0,0,0)^t$. The little group is the $s = 1$ representation of $SU(2)$ which is equivalent to the fundamental $SO(3)$-representation for the rotation group. We will use this representation.

We shall not apply the complicated method of constructing the representation out of the little group representation. For p_0, i.e., if the particle is at rest we define $u_k(p_0) = e_k$. Then the operation of the little group on the field for this p_0 is given by the rotations which leave the 0-component invariant. For the general case of arbitrary on shell momenta the *polarisation vectors* u_k can be determined by $pu_k(p) = p_\mu u_k^\mu = 0$. Thus these are three space-like vectors transverse to the four-momentum.

In configuration space the fields are thus determined by the on-shell condition and the transversality constraint

$$-\Box A^\mu = m^2 A^\mu,\ \partial_\mu A^\mu = 0. \tag{4.9}$$

The scalar product is given with help of (4.8) together with the space-like character of the u_k

$$\left\langle A^1_\nu \,\middle|\, A^2_\mu \right\rangle = -\mathrm{i} \int \mathrm{d}^3\vec{x} A^{(1)*}_\nu \overset{\leftrightarrow}{\partial}_0 A^{(2)}_\nu. \tag{4.10}$$

4.1.2 Massive Spin-1/2 Fields

We start to describe Dirac spinors which are defined in the direct sum $(1/2,0)\oplus(0,1/2)$ of the two fundamental $SL(2,\mathbb{C})$ representations. This are the most often needed spinor fields and the Weyl

spinor fields can be obtained by restriction of the field on one summand of this representation. In appendix B it is shown that we can look on this as the irreducible representation space for the $O(1,3)^{\uparrow}$ which contains the parity operator but this topic is treated in a later section in this chapter. In this section we concentrate on the fields itself and develop a rather convenient notation for calculations with spinors.

Since the space of Dirac spinors is given by the direct sum of $SL(2,\mathbb{C})$ representations the Dirac spinors are four-component quantities of the following form

$$\psi = \begin{pmatrix} \xi^{\alpha} \\ \eta_{\dot{\beta}} \end{pmatrix},\ \alpha,\beta \in \{1,2\} \tag{4.11}$$

where ξ^{α} and $\eta_{\dot{\beta}}$ are Weyl spinors of the two different kinds which are explained in detail in appendix B.1. Here we just note how to raise and lower indices of the Weyl spinors

$$\xi_{\alpha} = \epsilon_{\alpha\beta}\xi^{\beta},\ \eta_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\eta^{\dot{\beta}},\ \xi^{\beta} = \epsilon^{\alpha\beta}\xi_{\alpha},\ \eta^{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}\eta_{\dot{\alpha}}, \tag{4.12}$$

where $\epsilon_{\alpha\beta}$ is the skew symmetric symbol in two dimensions:

$$(\epsilon_{\alpha\beta}) = (\epsilon^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.13}$$

In appendix B we have also shown how to represent a four vector as mixed second rank spinor. Now we want formulate this with help of the Pauli matrices. Let p a four-vector. Then the unique mapping to a mixed second rank spinor is given as

$$(p_{\alpha\dot{\beta}}) = \begin{pmatrix} p^0 + p^3 & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & p^0 - p^3 \end{pmatrix} = p^0 1 + \vec{\sigma}\vec{p}. \tag{4.14}$$

Herein $\vec{\sigma}$ are the generators of $SU(2)$, i.e., the three linearly independent 2×2 hermitian traceless matrices, given here in the standard form where σ_3 is diagonal. These matrices are known as the *Pauli-matrices*:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.15}$$

With help of (4.12) we find

$$(p_{\alpha\dot{\beta}}) = \epsilon(p^{\alpha\dot{\beta}})\epsilon^t = p^0 1 - \vec{p}\vec{\sigma}. \tag{4.16}$$

Now the action of the momenta on the dotted and undotted Weyl spinors is given by

$$p^{\alpha\dot{\beta}}\eta_{\dot{\beta}},\ p_{\dot{\beta}\alpha}\xi^{\alpha} \tag{4.17}$$

respectively. Together with (4.14) and (4.16) this reads for acting on the Dirac spinor (4.11) in matrix representation

$$p_{\mu}\gamma^{\mu} = \begin{pmatrix} 0 & p^0 1 + \vec{p}\vec{\sigma} \\ p^0 1 - \vec{p}\vec{\sigma} & 0 \end{pmatrix}. \tag{4.18}$$

Herein the left hand side is a formal Minkowski product of the four-vector p with the 4×4 matrices (operating on the Dirac spinors) γ^{μ}. These are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^m = \begin{pmatrix} 0 & -\sigma_m \\ \sigma_m & 0 \end{pmatrix}. \tag{4.19}$$

One proves immediately by direct calculation the *anti commutator* relations of the γ-matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{4.20}$$

This is known in mathematics as the *Clifford algebra* of $\mathbb{R}^{(1,3)}$. With help of this we find

$$\not{p}^2 = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = p^2, \tag{4.21}$$

where we have used the *Feynman slash* $\not{p} = p_\mu \gamma^\mu$ for convenience. Since we have $\not{p}^2 = p^2$ we obtain an invariant equation for projecting the spinors to $\mathscr{H}(m, 1/2, \pm)$:

$$\mathrm{i}\not{\partial}\Psi = m\Psi, \tag{4.22}$$

where we have set the phase factor of the energy-momentum eigenfunctions to be $\exp(\pm ipx)$ for $\Psi \in \mathscr{H}(m, 1/2, \pm)$ which is convenient for the Feynman-Stueckelberg formalism when quantising the theory.

This is the *Dirac equation*. Taking energy-momentum eigen-solutions of this equation for the standard vectors $p_0 = (\pm m, 0, 0, 0)^t$ for the positive and negative energy eigenvalues respectively we obtain the coefficient functions

$$\begin{aligned} u_+(p_0, 1/2) &= N \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; u_+(p_0, -1/2) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \\ u_-(p_0, 1/2) &= N \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; u_-(p_0, -1/2) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \end{aligned} \tag{4.23}$$

Here we have introduced the notation $u_\pm(p, \sigma)$ which will be used from now on, with p the energy-momentum eigenvector fulfilling the on-shell condition in the form $p^0 = \pm\omega(\vec{p}$ and $\sigma = \pm 1/2$ the eigenvalue of the spin-3-component in the rest frame of the particle.

Since the little group is given by the rotations leaving the zero component of the four-vectors unchanged the spin operator is given by

$$\tilde{\mathbf{S}} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{4.24}$$

Applying these matrices to the solutions (4.23) of the Dirac equation we find that they are simultaneously eigenvectors of $\mathbf{S}_3$ with eigenvalues $\pm 1/2$.

In appendix B we have shown that one obtains the solution for general momenta $(\pm\omega(\vec{p}), \vec{p})^t$ on the mass shell by applying the appropriate boost $\Lambda(p)$ to the solutions to the solution for the standard vectors p_0. For this purpose we show that all boosts lay in the image of the exponential function $\exp sl(2, \mathbb{C}) \to SL(2, \mathbb{C})$. Taking $\vec{n} = \vec{p}/\|\vec{p}\|$, i.e. the direction of the boost $\Lambda(p)$, the boost transformation in the Dirac spinor space is given by

$$B(\vec{n}, \lambda) = \begin{pmatrix} \exp\left(\frac{\lambda}{2}\vec{n}\vec{\sigma}\right) & 0 \\ 0 & \exp\left(-\frac{\lambda}{2}\vec{n}\vec{\sigma}\right) \end{pmatrix}, \tag{4.25}$$

where λ is the boost parameter which is related with the relative velocity β of the two frames by $\beta = \tanh\lambda$.

Clearly it is enough to prove this claim for Weyl spinors. To this end we realize by direct calculation the following properties of the Pauli matrices

$$\{\sigma_a, \sigma_b\} = 2\delta_{ab},\ \mathrm{tr}\,\vec{\sigma} = 0 \Rightarrow \mathrm{tr}(p^0 + \vec{p}\vec{\sigma}) = 2p^0,\ \mathrm{tr}[(\vec{p}\vec{\sigma})\vec{\sigma}] = 2\vec{a}. \tag{4.26}$$

Let T denote a boost transformation which operates by definition on the second rank spinor (4.14) is given by

$$p'^0 + \vec{p}'\vec{\sigma} = T(p^0 + \vec{p}\vec{\sigma})T^\dagger \tag{4.27}$$

and for a infinitesimal transformation $T = 1 + \delta$ we have up to first order in δ:

$$\delta(p^0 + \vec{p}\vec{\sigma}) = \delta(p^0 + \vec{p}\vec{\sigma}) + (p^0 + \vec{p}\vec{\sigma})\delta^\dagger. \tag{4.28}$$

On the other hand an infinitesimal boost with velocity $\delta\vec{v}$ is given by

$$p'^0 = p^0 - \vec{p}\delta\vec{v},\ \vec{p}' = \vec{p} - p^0\delta\vec{v}. \tag{4.29}$$

Comparing this with (4.28) we have the properties

$$\vec{\sigma}\delta + \delta^\dagger\vec{\sigma} = -\delta\vec{v},\ \delta + \delta^\dagger = -\vec{\sigma}\delta\vec{v}. \tag{4.30}$$

Using (4.26) we find immediately that this is fulfilled by the choice

$$\delta = -\frac{1}{2}\vec{\sigma}\delta\vec{v}. \tag{4.31}$$

Since the boosts in an arbitrary fixed direction is a one-parameter subgroup of $SO(1,3)^\uparrow$ this gives (4.25) by adding the representations.

Together with the definition of $\Lambda(p)$ (B.63) and (B.66) we have

$$u(p) = B[\Lambda(p)]u(p_0), \tag{4.32}$$

where $u(p_0)$ stands for one of the four possible solutions of the Dirac equation which is the same time an eigenfunction of the momentum operator with eigenvalues given by the standard momentum p_0 and $u(p)$ for the same with eigenvalues given by arbitrary on-shell momentum p.

With help of (4.26) we obtain by applying the series expansion of the exponential function

$$\exp\left(\frac{\lambda}{2}\vec{n}\vec{\sigma}\right) = \cosh\left(\frac{\lambda}{2}\right) + \vec{n}\vec{\sigma}\sinh\left(\frac{\lambda}{2}\right). \tag{4.33}$$

Using the addition theorems for the hyperbolic functions together with $\beta = \tanh\lambda$ we find the relation between this functions and the physical boost parameter

$$\cosh\left(\frac{\lambda}{2}\right) = \sqrt{\frac{1+\cosh\lambda}{2}} = \sqrt{\frac{1+\gamma}{2}},\ \sinh\left(\frac{\lambda}{2}\right) = \sqrt{\frac{\cosh\lambda - 1}{2}} = \sqrt{\frac{\gamma-1}{2}}, \tag{4.34}$$

where we have used the usual abbreviations

$$\gamma = \sqrt{\frac{1}{1-\beta^2}},\ \beta = \frac{\|\vec{p}\|}{\omega(\vec{p})}. \tag{4.35}$$

Thus putting this together we find

$$\cosh\left(\frac{\lambda}{2}\right) = \sqrt{\frac{\omega(\vec{p})+m}{2m}},\ \sinh\left(\frac{\lambda}{2}\right) = \sqrt{\frac{\omega(\vec{p})-m}{2m}} \tag{4.36}$$

and now applying (4.25) together with (4.19) we find as the desired result

$$u(p) = \sqrt{\frac{1}{2m[m+\omega(\vec{p})]}}(m + p_\mu\gamma^\mu\gamma^0)u(p_0). \tag{4.37}$$

Thus a general spin 1/2 field in the orthogonal sum of the Hilbert spaces $\mathscr{H}(m, 1/2, \pm)$ in configuration space is given by

$$\psi(x) = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega(\vec{p})}} \left[A_+^\sigma(\vec{p})u_+^\sigma(p)\exp(-\mathrm{i}px) + A_-^\sigma(\vec{p})u_-^\sigma(-p)\exp(\mathrm{i}px)\right]_{p^0=\omega(\vec{p})}. \tag{4.38}$$

The normalisation of the eigen-spinors of momentum is chosen in an invariant way by

$$\bar{u}_+(p,\sigma)u_+(p,\sigma') = 2m\delta_{\sigma\sigma'},\ \bar{u}_-(-p,\sigma)u_-(-p,\sigma') = -2m\delta_{\sigma\sigma'} \text{ with } \bar{u} = u^\dagger\gamma^0, \tag{4.39}$$

which will be convenient for the quantised fields. The spinor invariance of this form follows directly from the definition (4.11) of the Dirac spinors and the transformations properties of the dotted and undotted representations.

Since the $u_\pm(p,\sigma)$ are solutions of the Dirac equation in momentum space we find together with the normalisation (4.39)

$$\left.\begin{array}{c} p_\mu\bar{u}_\pm(\pm p,\sigma)\gamma^\mu u_\pm(\pm p,\sigma) = \pm 2p^2 = \pm 2m^2 \Rightarrow \\ \bar{u}_\pm(\pm p,\sigma)\gamma^\mu u_\pm(\pm p,\sigma) = \pm 2p^\mu \end{array}\right\} \text{ with } p^0 = \pm\omega(\vec{p}). \tag{4.40}$$

Thus we find for the invariant scalar product in our case the following form in configuration space

$$\langle\phi\,|\,\psi\rangle = \int \mathrm{d}^3\vec{x}\bar{\phi}(x)\gamma^0\psi(x). \tag{4.41}$$

The projection operators on the subspaces $\mathscr{H}(m, 1/2, \pm)$ is given by the projection operators

$$P_\pm(p) = \frac{m \mp \not{p}}{2m}, \tag{4.42}$$

which can directly be generalised from the solution for standard momentum (4.23).

4.2 Causal Massless Fields

Since there is nothing new to say about massless scalar fields compared to their massive counterpart, we come immediately to the important case of a massless vector field, which is realized in nature by the gauge bosons of the standard model especially the photon.

4.2.1 Massless Vector Field

As we know from the discussions in appendix B the little group in the massless case is isomorphic to $ISO(2)$. We chose the standard vectors for both cases $\mathscr{H}(0,\lambda,\pm)$ as $p_0 = (\pm 1, 0, 0, 1)^t$. Then the little group is generated by the rotations around the 3-axis and two independent null rotations. This null rotations correspond to the translations in the isomorphic mapping of the little group to $ISO(2)$ and has to be represented trivially in the physical representations since there is no particle known which has a continuous number of intrinsic degrees of freedom.

Since the Casimir operator $\mathbf{p}^2$ has eigenvalue zero in the case of massless fields we have

$$\Box A^\mu(x) = 0. \tag{4.43}$$

If we represent the whole little group trivially we have a scalar field and the vector field is given as the gradient of this scalar field.

Since we want to give a representation with helicity $\lambda = 1$ we impose the constraint

$$\partial_\mu A^\mu = 0 \tag{4.44}$$

which was enough to fix the field to be of spin one in the massive case. We shall show now that this constraint is not enough to ensure the field to be of helicity one. This can be seen by calculating the fields with $\lambda = 1$ and $\lambda = -1$ and operating with the null rotations on this fields. One finds at $\mathrm{Eig}(\mathbf{p}, p_0)$ that the null rotations are not represented trivially but, using the parameterisation (B.90)

$$A'^\mu(p_0) \to A^\mu(p_0) + (\mathrm{Re}bA^1 - \mathrm{Im}bA^2)p_0^\mu. \tag{4.45}$$

We find that the null rotations are trivially represented if and only if $A_\mu \propto p_\mu$ but this corresponds to the $\lambda = 0$, i.e., the scalar representation.

In other words we don't find a function Hilbert space which is complementary to $\lambda = 0$ realized with vector fields.

Instead we have necessarily to construct the Hilbert space as a quotient vector space. We show that it is given algebraically by

$$\mathscr{H}(0,1,\pm) = \{A^\mu | \Box A^\mu = 0,\ \partial_\mu A^\mu = 0\}/\{A^\mu |\ \partial_\mu A_\nu - \partial_\nu A_\mu = 0\}. \tag{4.46}$$

At first we find that on this space the null rotations are trivially represented. From the action on the momentum eigenspace with eigenvector p_0 we learnt that this null rotations are given by $A'_\mu = A_\mu + \partial_\mu \chi$ with an appropriate scalar function χ. But since $[\partial_\mu, \partial_\nu] = 0$ on the space of continuously differentiable functions, which is a dense subspace in L^2, the transformed field A'_μ is identified in the quotient vector space (4.46).

It is also important to notice that the definition of the quotient space can be expressed in the form

$$\mathscr{H}(0,1,\pm) = \{A^\mu | \Box A^\mu - \partial_\nu \partial^\mu A^\nu = 0\}/\{A^\mu |\ \partial_\mu A_\nu - \partial_\nu A_\mu = 0\}. \tag{4.47}$$

The reason for that is that the equation

$$\Box A^\mu - \partial_\nu \partial^\mu A^\nu = 0 \tag{4.48}$$

is invariant under the transformation

$$A'_\mu = A_\mu + \partial_\mu \chi \tag{4.49}$$

with an arbitrary smooth scalar function χ and for any solution one can find a scalar field χ such that

$$\partial_\mu A'^\mu = 0. \tag{4.50}$$

This means we can express the fact that we want to construct a field equation for a massless spin-1-field such that the null-rotations contained in the little group are represented trivially and project out the Spin-0-components by using the field equation (4.48) for a representative of the vector field in the quotient space. This equation is invariant under the *gauge transformation* (4.49) which ensures the triviality of the representation of the null-rotations and taking this modulo the pure gauge fields which are gradients of (massless) scalar fields. This ensures in addition that the spin-0-component vanishes.

As our analysis shows, both conditions together ensure also the masslessness of the representation because there is always a representative A'_μ of each A_μ (connected to A_μ by a gauge transformation (4.49) such that $\partial_\nu A'^\nu = 0$. This means that a massless spin-1-field is necessarily described as a *gauge field*, i.e. a theory which is invariant under transformations which operates on the vector field in the form (4.49). The application of the gauge transformation to other fields contained in the theory has to be chosen appropriately in order to leave the action invariant. The necessity of gauge invariance for massless vector fields together with Poincaré invariance restricts the possibility for building theories of such fields with other fields drastically. This is a very important guide to build theories for the strong and electro-weak interactions within the standard model of elementary particles and is called *gauge principle*. There the gauge group is extended to non-abelian groups. We shall come back to these important ideas in chapter 7.

To finish the proof that this quotient space is indeed the correct representations space for the helicity 1 representation of $\mathrm{P}^\uparrow_+$ we show that in this quotient space the scalar product (4.10), given above for the massive case, is positive definite also for the massless case. To this end we have to show that for a real massless transversal vector field A_μ which is a eigenvector for $\mathbf{p}$ with light-like eigenvalue p which fulfils

$$\langle A \,|\, A \rangle = \int \mathrm{d}^3\vec{x} A_\nu \partial_t A^\nu = 0 \tag{4.51}$$

fulfils $\partial_\mu A_\nu - \partial_\nu A_\mu$. To this end we use the momentum representation of the invariant scalar product

$$\langle A \,|\, A \rangle = -\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})} A^*_\nu(\vec{p}) A^\nu(\vec{p}) \text{ with } \omega(\vec{p}) = \|\vec{p}\|. \tag{4.52}$$

Since p is light-like and $pA = 0$ the real as well as the imaginary part of $A_\nu(p)$ is light-like or space-like (it is a nice exercise to show that a four-vector which is orthogonal to a light-like vector is space-like or light-like. In the latter case it is proportional to the given light-like vector!). From this we learn that the invariant scalar product is positive semi-definite also for the light-like case. If it vanishes than we have necessarily $A^*(p)A^*(p) = 0$ since the integrand is positive semi-definite. This means that real and imaginary part of $A(p)$ are light-like and since this vector is orthogonal to the light-like momentum p, it is necessarily proportional to p, i.e., $A_\nu(p) = p\tilde{A}(p)$. In momentum space this means $A_\nu(x) = \partial_\nu \tilde{A}(x)$ and thus the field equivalent to zero in the quotient space (4.46). Q.E.D.

Thus we have obtained the correct physical space for representing the Hilbert space as the quotient space (4.46). Since $\mathbb{R}^{(1,3)}$ is a simply connected space two vector fields A'_μ and A_μ are identified in this space if and only if $\exists \chi : \Box\chi(x) = 0, A'_\mu - A_\mu = \partial_\mu \chi$. This is a restricted form

of *gauge invariance*, namely under gauge transformations with gauge fields χ which respect the Lorentz condition $\partial_\mu A^\mu = 0$ which is expressed in form of the constraint $\Box\chi = 0$.

At last we want to give the fields of helicity 1 and -1. As shown in appendix B to this end we need the Pauli-Lubanski vector. To calculate it we first look for the generators of $SO(1,3)^\uparrow$-transformations:

$$\begin{aligned} A'^\mu(x') &= (\delta^\mu_\nu + \delta\omega^\mu{}_\nu)A^\nu(x) = (\delta^\mu_\nu + \delta\omega^\mu{}_\nu)A(x'^\sigma) - \delta\sigma^\rho{}_\rho x^\rho = \\ &= A^\mu(x') - \frac{\mathrm{i}}{2}\delta\omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu A^\nu(x'). \end{aligned} \tag{4.53}$$

Comparison of the first with the second line of this equation shows that the generators of the $SO(1,3)^\uparrow$-transformations in the spin one representation of fields are given by

$$(M^{\rho\sigma})^\mu{}_\nu = \mathrm{i}\left[(x^\rho\partial^\sigma - x^\sigma\partial^\rho)\delta^\mu_\nu + g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu\right]. \tag{4.54}$$

For the Pauli-Lubanski vector we have

$$(W_\mu)_{\alpha\beta} = \mathrm{i}\epsilon_{\alpha\beta\nu\mu}\partial^\nu = \epsilon_{\alpha\beta\nu\mu}\mathbf{p}^\nu. \tag{4.55}$$

So from (B.98) we see that the states of helicity $\lambda = \pm 1$ are given by the constraint

$$\epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma = \lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) = \lambda F_{\mu\nu}, \tag{4.56}$$

i.e., a massless vector field has helicity 1 (-1) if its *Faraday tensor* $F_{\mu\nu}$ is self-dual (antiself-dual). One can show that these are the right (left) circular polarised wave solutions of the free Maxwell equations.

4.2.2 Massless Helicity 1/2 Fields

The case of massless helicity 1/2 fields can be found immediately by setting $m = 0$ in the corresponding massive case. The projector on the positive or negative energy eigenstates are given by the projectors (4.42). The standard vectors of the little group are chosen to be $p_0 = (\pm 1, 0, 0, 1)$.

The only thing we have to show is that the null rotations are represented trivially. To this end we use the fact that the Dirac equation (4.22) for $m = 0$ separates in equations for the two Weyl spinors, namely

$$(p^0 - \vec{p}\vec{\sigma})\xi = 0. \tag{4.57}$$

This is the so called *Weyl equation*. Now we look on the subspace $\mathrm{Eig}(\mathbf{p}, p_0)$ of $\mathscr{H}(0, 1/2, +)$. The Weyl equation is then given by

$$(1 - \sigma_3)\xi_+ = 0, \tag{4.58}$$

which is solved by $\xi^2 = 0$. The space of solutions is one-dimensional and is due to (B.90) not changed by null rotations (which means for $\alpha = 0$). So indeed the null rotations are represented trivially in our representation. It shows also that this spinor has helicity $\lambda = +1/2$. It remains as an exercise for the reader to show that the analogue calculations for the dotted spinors give helicity $\lambda = -1/2$.

4.3 Quantisation and the Spin-Statistics Theorem

Now we want to quantise the fields above. We have to fulfil the postulates given in the beginning of this chapter. The one-particle wave functions are in general not the solution of the problem to quantise relativistic particles. The reasons are explained already in chapter 3: A mathematical reason is that we cannot restrict the fields to the Hilbert spaces of positive energy since the interactions "scatter" the wave functions from the positive energy space to the negative one which means that the Hilbert spaces are in general given as the orthogonal sum of the positive and the negative energy space leading to a closed Hilbert space under time evolution. The physical reason is nowadays obvious since the production and annihilation of particle antiparticle pairs are a well known fact. Thus the relativistic quantum mechanics is necessarily a *many-particle theory.*

To quantise the theory we take the case of scalar particles as a guideline. There we have found the following "recipes" to solve the problems of the negative energy states and the zero point energy:

- The negative energy states are identified with the anti-causal propagation of particles which can be reinterpreted as the causal propagation of an antiparticle with positive energy. This is known as the *Feynman-Stueckelberg formalism.*
- To renormalise the vacuum to energy 0 we have to normal-order the local operators representing densities of additive observables. This densities are expressed with help of the fields by quantising the corresponding Noether currents of symmetries of the classical action.

Since we have done this quantisation procedure for scalar fields in great detail in chapter 3 we come immediately to the important case of Dirac fields.

4.3.1 Quantisation of the spin-1/2 Dirac Field

As shown in section 4.1. in the massive case these fields are completely determined by the Dirac equation which is is the field equation of motion

$$\mathrm{i}\partial\!\!\!/_\mu \psi = m\psi. \tag{4.59}$$

Since there are no constraints necessary to obtain the spin-1/2 field the theory is completely determined by the action, which is given by the Lagrangian

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi. \tag{4.60}$$

From this we read off the canonical energy-momentum (3.49) tensor

$$\Theta^\mu{}_\nu = \bar{\psi}\mathrm{i}\gamma^\mu\partial_\nu\psi - \delta^\mu_\nu\mathscr{L}. \tag{4.61}$$

We like now to quantise this theory. For this purpose we need the total energy of the field configuration to be sure to fulfil postulate 2, namely that the energy is bounded from below. The Hamiltonian is given with help of (4.51):

$$H = \int \mathrm{d}^3\vec{x}\Theta^0{}_0 = \int \mathrm{d}^3\vec{x}\bar{\psi}\gamma^0(i\vec{\gamma}\nabla + m)\psi. \tag{4.62}$$

For fields fulfilling the free Dirac equation (4.59) we can write this in the shorter form

$$H = \int \mathrm{d}^3\vec{x}\bar{\psi} i\partial_t \psi. \tag{4.63}$$

Now we use the fact that there are known only bosons and fermions in nature which means that the multi-particle Fock space states are totally symmetric or antisymmetric respectively under exchange of single-particle states contained in this multi-particle states. In chapter 1 we have given a nice path integral argument for this.

Since in the case of free fields the Lagrangian is bilinear in the field operators the local observables are represented by bilinear forms of field operators too. Thus the microcausality condition (4.1) can be fulfilled for bosonic as well as fermionic quantisation. This can be seen from the simple formula

$$[\mathbf{AB}, \mathbf{X}] = \mathbf{A}\,[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{B}]\,\mathbf{C} = \mathbf{A}\,\{\mathbf{B}, \mathbf{C}\} - \{\mathbf{A}, \mathbf{B}\}\,\mathbf{C}. \tag{4.64}$$

From nature we know that spin-1/2 particles. Thus we try to quantise the Dirac fields as a fermionic field, i.e., we introduce particle annihilation operators $\mathbf{a}(\vec{p}, \sigma)$ and antiparticle annihilation operators $\mathbf{b}(\vec{p}, \sigma)$ fulfilling the anti-commutator relations

$$\left\{\mathbf{a}(\vec{p}_1, \sigma_1), \mathbf{a}^\dagger(\vec{p}_2, \sigma_2)\right\} = \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\delta_{\sigma_1\sigma_2},\ \left\{\mathbf{b}(\vec{p}_1, \sigma_1), \mathbf{b}^\dagger(\vec{p}_2, \sigma_2)\right\} = \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\delta_{\sigma_1\sigma_2}, \tag{4.65}$$

with all other *anti-commutators* vanishing.

Now we express the field operators in terms of the energy-momentum eigenstates (4.23):

$$\boldsymbol{\psi}(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}} \left[\mathbf{a}(\vec{p}, \sigma) u_+(p, \sigma) \exp(-\mathrm{i}px) + \mathbf{b}^\dagger(\vec{p}, \sigma) u_-(-p, \sigma) \exp(\mathrm{i}px)\right]_{p^0=\omega(\vec{p})}, \tag{4.66}$$

where we have incorporated the Feynman-Stueckelberg formalism.

Since from (4.40) we have

$$\bar{u}_\pm(\pm p, \sigma)\gamma^0 u_\pm(\pm p, \sigma) = u_\pm^\dagger(\pm p, \sigma) u_\pm(\pm p, \sigma) = 2\omega(\vec{p}) \tag{4.67}$$

we find by inserting this ansatz the *equal-time* anti-commutator relation

$$\left\{\boldsymbol{\psi}(t, \vec{x}), \boldsymbol{\psi}^\dagger(t, \vec{y})\right\} = \delta^{(3)}(\vec{x} - \vec{y})\hat{1}, \tag{4.68}$$

where $\hat{1}$ is the 4×4 unity matrix in the Dirac-spinor space.

The Hamiltonian can also be calculated in a straightforward way. With application of normal-ordering the quantised form of (4.63) reads

$$\mathbf{H} = \sum_\sigma \int \mathrm{d}^3\vec{p}\omega(\vec{p})[\mathbf{N}_a(\vec{p}, \sigma) + \mathbf{N}_b(\vec{p}, \sigma)], \tag{4.69}$$

where the normal-ordering implies signs from interchanging two fermionic field operators due to the anti-commutators used for fermionic quantisation. This change of sign is crucial for the energy to be positive definite. Thus we have a very important result: The spin-1/2 field is *necessarily* to be quantised in terms of fermions to get a positive definite energy for the free particles. We shall show below that the observables built with help of the invariant bilinear forms

of the field operators, especially the energy and momentum densities, fulfil the requirement of micro causality.

The number operators are given as usual defined by

$$\mathbf{N}_a(\vec{p},\sigma) = \mathbf{a}^\dagger(\vec{p},\sigma)\mathbf{a}(\vec{p},\sigma),\ \mathbf{N}_b(\vec{p},\sigma) = \mathbf{b}^\dagger(\vec{p},\sigma)\mathbf{b}(\vec{p},\sigma). \tag{4.70}$$

The momentum operator is given by

$$\mathbf{p}_k = \int \mathrm{d}^3\vec{x}\boldsymbol{\Theta}^0{}_k = \sum_\sigma \int \mathrm{d}^3\vec{p}\, p_k[\mathbf{N}_a(\vec{p},\sigma) + \mathbf{N}_b(\vec{p},\sigma)]. \tag{4.71}$$

At last we calculate the electrical charge which is given by the Noether charge of the symmetry under $U(1)$ transformations

$$\delta\psi = \mathrm{i}q\psi\delta\alpha,\ \delta\bar{\psi} = -\mathrm{i}q\bar{\psi}\delta\alpha,\ \delta x = 0 \text{ with } q, \delta\alpha \in \mathbb{R}. \tag{4.72}$$

The Noether current is given by (3.44):

$$j^\mu = q\bar{\psi}\gamma^\mu\psi. \tag{4.73}$$

Its quantised version is given with help of normal-ordering

$$\mathbf{j}^\mu = q : \bar{\psi}\gamma^\mu\psi : \tag{4.74}$$

leading to the operator representing the electric charge

$$\mathbf{Q} = q\int \mathrm{d}^3\vec{x} : \bar{\psi}\gamma^0\psi := q\sum_\sigma \int \mathrm{d}^3\vec{p}[\mathbf{N}_a(\vec{p},\sigma) - \mathbf{N}_b(\vec{p},\sigma)] \tag{4.75}$$

which shows that the a-particles have electric charge $+q$ and the b-particles $-q$. Thus from the anti-commutator relations we find that particles and antiparticles have the same quantum numbers except for the electric charge which has opposite sign.

Now we want to show that the microcausality condition is fulfilled. To this end we calculate the anti-commutator of the Dirac field operators for arbitrary space-time points:

$$\begin{aligned}\big\{\boldsymbol{\psi}(x_1), \bar{\boldsymbol{\psi}}(x_2)\big\} = \sum_{\sigma_1,\sigma_2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p_1}}{\sqrt{2\omega(\vec{p_1})(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p_2}}{\sqrt{2\omega(\vec{p_2})(2\pi)^3}} \\ \big[\mathbf{a}(p_1,\sigma_1)u_+(p_1,\sigma_1)\exp(-\mathrm{i}p_1x_1) + \mathbf{b}^\dagger(p_1,\sigma_1)u_-(-p_1,\sigma_1)\exp(+\mathrm{i}p_1x_1), \\ \mathbf{a}^\dagger(p_2,\sigma_2)\bar{u}_+(p_2,\sigma_2)\exp(\mathrm{i}p_2x_2) + \mathbf{b}(p_2,\sigma_2)\bar{u}_-(-p_2,\sigma_2)\exp(-\mathrm{i}p_2x_2)\big]_+.\end{aligned} \tag{4.76}$$

Using the anti-commutator relations for the creation and annihilation operators (4.65) we find

$$\begin{aligned}\sum_\sigma \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}} \big\{ u_+(p,\sigma)\bar{u}_+(p,\sigma)\exp[\mathrm{i}p(x_1-x_2)] + \\ + u_-(-p,\sigma)\bar{u}_-(-p,\sigma)\exp[-\mathrm{i}p(x_1-x_2)]\big\}.\end{aligned} \tag{4.77}$$

Herein we have to calculate the matrix

$$2\hat{\rho}_+(p,0) = \sum_\sigma u_+(p,\sigma)\bar{u}_+(p,\sigma) \tag{4.78}$$

which is two times the polarisation matrix for particles. We know that this matrix can only depend on the on-shell momentum p. Further we know that u is a solution of the free Dirac equation in momentum representation (4.22) with the upper sign on the right hand side.

By direct calculation we verify

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger} \tag{4.79}$$

and together with (4.22) we find

$$\bar{u}_+(p,\sigma)(\not{p} - m) = 0. \tag{4.80}$$

The end of the story is that the polarisation matrix fulfils the equation

$$(\not{p} - m)\hat{\rho}_+(p,0) = \hat{\rho}_+(p,0)(\not{p} - m) \tag{4.81}$$

which has the unique solution

$$\hat{\rho}_+(p,0) = N(\not{p} + m) \tag{4.82}$$

where the normalisation constant N is given with help of our normalisation convention (4.39)

$$2\,\mathrm{tr}\,\hat{\rho}_+(p,0) = \sum_\sigma \bar{u}_+(p,\sigma)u_+(p,\sigma) = 4m \Rightarrow N = \frac{1}{2}. \tag{4.83}$$

So our final result is

$$\hat{\rho}_+(p,0) = \frac{1}{2}(\not{p} + m). \tag{4.84}$$

In the same line of arguing one obtains for the polarisation matrix for unpolarised antiparticles

$$\hat{\rho}_-(p,0) = \frac{1}{2}\sum_\sigma u_-(p,\sigma)\bar{u}_-(p,\sigma) = \frac{1}{2}(\not{p} - m). \tag{4.85}$$

With help of this we can calculate the non-vanishing anti-commutator between the field operators at arbitrary space-time points. To this end we use again (4.66) and the anti-commutator relations (4.65). The result is

$$\left\{\boldsymbol{\psi}(x_1), \bar{\boldsymbol{\psi}}(\mathbf{x_2})\right\} = (\mathrm{i}\partial\!\!\!/_1 - m)\mathrm{i}\Delta_-(x_1 - x_2), \tag{4.86}$$

where Δ_- is given by

$$i\Delta_-(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})}[\exp(-\mathrm{i}px) - \exp(\mathrm{i}px)]. \tag{4.87}$$

This can be written as

$$i\Delta_-(x) = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{sign}\,p^0 2\pi\delta(p^2 - m^2)\exp(-\mathrm{i}px). \tag{4.88}$$

This is an odd solution of the Klein-Gordon equation and invariant under $SO(1,3)^\uparrow$ transformations, i.e., a Lorentz-invariant function.

To show the microcausality condition it is convenient to write the time-dependent part out:

$$\mathrm{i}\Delta_-(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\{\exp[-\mathrm{i}\omega(\vec{p})t] - \exp[\mathrm{i}\omega(\vec{p})t]\}\exp(\mathrm{i}\vec{p}\vec{x}). \tag{4.89}$$

Since the integrand vanishes for $t = x^0 = 0$ and we can transform any space-like vector x with $x^2 < 0$ with help of a $SO(1,3)^\uparrow$- transformation to a vector x' with $x'^0 = 0$ this means together with the Lorentz invariance of Δ_-

$$\Delta_-(x) = 0 \text{ for } x^2 < 0. \tag{4.90}$$

Applied this to (4.86) we find that the anti-commutator of the Dirac field operators on space-like related space-time points vanishes. Together with (4.64) this shows that the operators representing observables which are given as bilinear forms of the Dirac operators fulfil the microcausality condition.

The same time we have found the restriction for interaction terms that these have to be built with a *even number of Dirac-field operators* in order to fulfil the microcausality condition.

4.4 Discrete Symmetries and the CPT Theorem

Now we come to the important investigation of discrete symmetries. From space-time we know the reflections of space (*parity*) and time reflections (*time reversal*).

Since we have shown that there are necessarily to each sort of particles its corresponding antiparticles (for sake of simplicity we include the case that the antiparticles may be identical with the particles (e.g. the *real* particles), we can exchange the particles and the antiparticles and look whether the system is invariant under this transformation. In the following we will clarify this idea. But it is clear that this transformation which is called *charge conjugation* builds the group $\mathbb{Z}_2$, i.e., $\hat{C}^2 = 1$ and that a charge-neutral system (it might be an elementary particle also denoted as strictly neutral systems or a bound state like the positronium or hydrogen atom) can be even or odd under charge conjugation.

We will show the famous CPT theorem which states that any Lorentz-invariant theory, i.e., a theory which is invariant under $\mathrm{SO}(1,3)^\uparrow$-transformations is automatically invariant under the combined discrete transformation which reverses time, reflects space and interchanges particles with antiparticles, in short notation the CPT transformation. There is no necessity for C, P or T alone to be a symmetry transformation. We know since the famous experiment by Wu that the weak interaction violates the P invariance and from Kaon decay that also the combined CP transformation is violated. But there is yet nothing known about a violation of CPT invariance which would be a strong hint that local relativistic quantum field theory is not the right picture about nature.

4.4.1 Charge Conjugation for Dirac spinors

We shall consider in this section again only Dirac particles since with help of them we can compose all other representations with help of Kronecker multiplication and reduction to irreducible parts.

Now let us consider the particle solutions in momentum space $u_+(p,\sigma)$ and look how to transform them to a corresponding antiparticle solution. From our treatment of antiparticles in terms of the Feynman-Stueckelberg formalism it is easy to guess that it could have something to do with complex conjugation. Complex conjugation in the context of Dirac spinors is better done as the Dirac conjugation $\bar{u} = u^\dagger \gamma^0$ which leaves the formalism invariant. As we have shown above (4.80) u_+ fulfils the following equations

$$(\not{p} - m)u_+(p,\sigma) = 0,\ \bar{u}_+(p,\sigma)(\not{p} - m). \tag{4.91}$$

Transposing this equation leads back to a column spinor

$$(\not{p}^t - m)\bar{u}_+^t(p,\sigma) = 0. \tag{4.92}$$

Now we seek a unitary transformation which makes this an equation for an antiparticle wave function in momentum representation, called the charge conjugated wave function of u and denoted with u^c

$$u_+^c(p,\sigma) = \hat{C}\bar{u}_+^t(p,\sigma) \text{ with } (-\not{p} + m)u_+^c(p,\sigma) = 0. \tag{4.93}$$

Comparing this equation with (4.92) we have the condition

$$\gamma^{\mu t}\hat{C} = -\gamma^\mu\hat{C}. \tag{4.94}$$

In our representation of the γ-matrices (4.19) together with the representation (4.15) of the Pauli spin matrices we have

$$\gamma^{\mu t} = (-1)^\mu\gamma^\mu \tag{4.95}$$

and a solution of (4.94) for $\hat{C}$ is given up to a factor

$$\hat{C} = \eta_C\gamma^2\gamma^0, \tag{4.96}$$

where η_C is an arbitrary factor. From the unitarity condition for $\hat{C}$

$$\hat{C}^\dagger = \hat{C}^{-1} \tag{4.97}$$

we find that $|\eta_C| = 1$ which shows that η_C is only a phase factor which may be set to 1. We have so far

$$u_+^c(p,\sigma) = \eta_C\gamma^2\gamma^0\bar{u}_+^t(p,\sigma). \tag{4.98}$$

Since σ is the eigenvalue of the spin 3-component in the rest frame of the particle we look for the action of the spin operator (4.24) on the charge conjugated state for $p = p_0$:

$$\mathbf{S}_3 u_+^c(p_0,\sigma) = -\sigma u_+^c(p_0,\sigma). \tag{4.99}$$

Since $\mathbf{S}_3 = i\left[\gamma^1,\gamma^2\right]$ this is proven by using the anti-commutator relations of the γ-matrices (4.20). Choosing $\eta_C = i$ we have

$$u_+^c(p,\sigma) = u_-(p,-\sigma). \tag{4.100}$$

At last we want to calculate the action of the *unitary* charge conjugation operator $\mathbf{C}$ in the Fock space. It is defined to operate on the field operators the same way as the charge conjugation on the c-number fields:

$$\boldsymbol{\psi}^c(x) = \mathbf{C}\boldsymbol{\psi}\mathbf{C}^\dagger = \hat{C}\bar{\boldsymbol{\psi}}^t(x). \tag{4.101}$$

Using (4.100) we find

$$\boldsymbol{\psi}^c(x) = \sum_\sigma \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega(\vec{p})(2\pi)^3}} \left[\exp(\mathrm{i}px)u_-(p,-\sigma)\mathbf{a}^\dagger(p,\sigma) + \exp(-\mathrm{i}px)u_+(p,-\sigma)\mathbf{b}(p,\sigma)\right]. \tag{4.102}$$

Comparing the charge conjugated annihilation and creation operators with this we have

$$\mathbf{C}\mathbf{a}(p,\sigma)\mathbf{C}^\dagger = \mathbf{b}(p,\sigma),\ \mathbf{C}\mathbf{b}(p,\sigma)\mathbf{C}^\dagger = \mathbf{a}(p,\sigma). \tag{4.103}$$

The action of the charge conjugation operation on the bilinear local or global observables shows that energy and momentum keep unchanged while the electromagnetic charges as well as the local electromagnetic currents change sign as one would expect from charge conjugation.

This charge conjugation interchanging particles with their antiparticle counterparts is another definition (beside the phase invariance of the action) of the connection between these sorts of particles.

4.4.2 Time Reversal

Now we come to the time-reversal operation. Mathematically it is nothing else than the time reflection, i.e., the change $t \to -t$. This looks simple, but this is not the case in physics. The point is that the time manifold is not simply $\mathbb{R}$ but *oriented* $\mathbb{R}$ which means nothing else than *causality*, a principle which we have pronounced throughout these whole lectures. On my point of view there is much confusion about time reversal since it is misunderstood. The point is not the mathematical theory of the symmetry but the physical interpretation and thus there are some words at place about it before treating it mathematically.

As we have seen above in the introductory chapters 1 and 2 *causality* is just an axiom in physics, i.e., physics as we know would not make sense if we would not assume the existence of causal natural laws. Causality is not broken by quantum theory but quantum theory is a completely causal theory, it only modifies the meaning of physically sensible observables, it just defines what we can know about the system here and now and if we were clever enough to solve the time-evolution equations for the states and operators exactly it would perfectly tell us what this possible knowledge tells us about our possible knowledge for later instants of time.

As we have seen in chapter 1 and 2 time is directed in quantum theory. For instance it is expressed in the fact that we have to use the time-ordering for calculating the (global) time-evolution kernel out of the (local) generators and this gave us the proper meaning for the propagator in momentum representation, namely to be retarded in the nonrelativistic and consistent with the Feynman-Stueckelberg formalism in the relativistic case.

Now we come to the correct interpretation of time-reversal operation. The idea can be seen in any physical theory, so we look at first on classical Newtonian mechanics. The most simple system is a point particle moving in an external potential. A "pure state" of such a point particle is given as a point in six dimensional phase space, namely by giving the position and the momentum of the point particle at the initial time t_0. Now suppose we were able to solve the Newtonian equation of motion exactly (which is of course the case for many textbook examples). Then we know the trajectory of the particle in phase space for all instants of time $t > t_0$. Now suppose we read off the position of the particle in phase space at some instant of time $t_1 > t_0$, reverse it is momentum leaving it at exactly the same position as it was and start now the calculation again. Then time-reversal invariance means that the particle will be after a time interval $t_1 - t_0$ at the time-reversed initial point we started with at $t = t_0$.

Thus the time-reversal operation has to be defined on the *states* of the system. This can be done also in quantum mechanics with the statistical operator for the general case or with the state vectors in Hilbert space for the pure states. Now we remember scattering theory in chapters 1 and 2 where we have defined very carefully the initial and final asymptotic states. Now from our classical example it is clear that time reversal in quantum mechanics *cannot* simply means a unitary transformation of the asymptotic states but we have to take into account the causality expressed in the orientedness of time. This means that the time-reversal operator takes an asymptotic final state to an asymptotic initial state and vice versa.

Now to express this idea mathematically we use the S-matrix which defined the *causal* transition from the initial to the final state forced by the dynamics of the system which is locally described by the Hamiltonian and globally by the time-evolution operator. The S-matrix is known if we have all matrix elements with respect to a complete set of initial and final asymptotic states $|i\rangle$ and $\langle f|$ which we have denoted with $S_{fi} = \langle f \,|\, i \rangle$. If we act now with the time-reversal operator on $\mathbf{S}$ which gives the S-matrix for the time-reversed asymptotic states this means to interchange

initial and final states and thus

$$S^t_{fi} = \langle \mathbf{T}f \,|\, \mathbf{T}i \rangle = S_{if} = \langle i \,|\, f \rangle = \langle f \,|\, i \rangle^* = S^*_{fi}. \tag{4.104}$$

This means that the time-reversal operator is necessarily *anti-unitary* rather than unitary.

The treatment of time reversal for the Dirac fields is analogue to that of the charge conjugation in the last section. The only difference is that we start here in the configuration space representation. The Dirac equation reads

$$(\mathrm{i}\partial\!\!\!/ - m)\psi = 0. \tag{4.105}$$

The equation for the Dirac conjugated spinor is given by

$$(\mathrm{i}\partial\!\!\!/^t + m)\bar{\psi}^t = 0. \tag{4.106}$$

If time reversal is a symmetry of the free Dirac equation there must be, since it is given by an anti-unitary operator, a unitary transformation $\hat{T}$ such that

$$\psi^T(x) = \hat{T}\bar{\psi}^t(\hat{T}x) \text{ with } \hat{T}(t, \vec{x}) = (-t, \vec{x}) \tag{4.107}$$

fulfils the Dirac equation. From now on we use the same symbol $\hat{T}$ for the operation on the Dirac spinors and the action on space-time vectors. This should not cause any confusion. Changing the sign of t in (4.106) gives

$$(-\mathrm{i}\gamma^{0t}\partial_t + \mathrm{i}\vec{\gamma}^t\vec{\nabla} + m)\bar{\psi}^t(-t, \vec{x}) = 0. \tag{4.108}$$

Multiplying this from the left with $\hat{T}$ we find that the spinor field (4.107) fulfils the Dirac equation if $\hat{T}$ fulfils the equation

$$\hat{T}\gamma^{0t} = \gamma^0\hat{T}, \hat{T}\vec{\gamma}^t = -\vec{\gamma}\hat{T}. \tag{4.109}$$

In our representation of the Dirac matrices we have

$$\gamma^{0t} = \gamma^0,\ \gamma^{at} = (-1)^a\gamma^a \text{ for } a = 1 \ldots 3 \tag{4.110}$$

which shows that we may chose

$$\hat{T} = \eta_T\gamma^3\gamma^1\gamma^0, \tag{4.111}$$

where η_T is an arbitrary phase factor. Thus the time-reversed state $\psi^T(x)$ is given by

$$\psi^T(t, \vec{x}) = \hat{T}\bar{\psi}(-t, \vec{x}) = \hat{T}\gamma^0\psi^*(-t, \vec{x}) = \eta_T\gamma^3\gamma^1\psi^*(-t, \vec{x}). \tag{4.112}$$

In our convention (4.23) for the momentum eigen-solutions of the Dirac equation the time-reversal matrix $\hat{T}$ acts on these as

$$u^T_\pm(p, \sigma) = -\operatorname{sign}\sigma u_\pm(\hat{P}p, -\sigma), \tag{4.113}$$

where $\hat{P}(p^0, \vec{p}) = (p^0, -\vec{p})$ which shows that time reversal operates as reflection of the momenta.

We find now the *anti-unitary* time-reversal operator $\mathbf{T}$ operating in the Fock space by the analogue calculation as we have done in the case of charge conjugation but here we have to take account on the anti-unitarity of $\mathbf{T}$. First we see that $\mathbf{T}$ has to fulfil

$$\mathbf{T}\boldsymbol{\psi}(\mathbf{x})\mathbf{T}^{-1} = \hat{T}\bar{\boldsymbol{\psi}}^t(\hat{T}x). \tag{4.114}$$

Using the expansion in annihilation and creation operators (4.66) and (4.113) we find by comparing the operator valued factors in front of the exponential functions

$$\mathbf{T}\mathbf{a}(p,\sigma)\mathbf{T}^{-1} = \operatorname{sign}\sigma\, \mathbf{a}(\hat{P}p,-\sigma),\ \mathbf{T}\mathbf{b}(p,\sigma)\mathbf{T}^{-1} = \operatorname{sign}\sigma\, \mathbf{b}(\hat{P}p,-\sigma). \tag{4.115}$$

Applying this to (4.69) and (4.71) we see that the energy is unchanged under time-reversal transformation while momentum changes sign. With (4.75) we find that the electrical charge operator $\mathbf{Q}$ is unchanged. It is also straight forward to show that we have for the local electromagnetic current

$$\mathbf{j}^{\mu T}(x) = \hat{P}\mathbf{j}(\hat{T}x) \text{ with } \hat{T}(t,\vec{x}) = (-t,\vec{x}). \tag{4.116}$$

This is also consistent with our naive understanding of the electromagnetic four-current for a moving charged medium known from classical electrodynamics as "charge times velocity".

4.4.3 Parity

Parity can be treated very short since we thought about it in appendix B. This was the reason why we introduced Dirac spinors in favour of Weyl spinors, namely to have the possibility to build parity invariant theories.

Using the covering (B.45) of the $\mathrm{O}(1,3)^{\uparrow}$ and rewriting it in our Dirac spinor convention we find

$$\psi^P(x) = \hat{P}\psi(\hat{P}x) = \mathrm{i}\gamma^0\psi(\hat{P}x). \tag{4.117}$$

A direct calculation yields

$$\hat{P}u_{\pm}(p,\sigma) = \pm\mathrm{i}u_{\pm}(\hat{P}p,\sigma) \tag{4.118}$$

which shows that the energy and spin of the state is unchanged while the momenta are reflected by the parity transformation as we expect from the meaning of space reflection.

In the same kind of calculation as we did at length for the case of charge conjugation we obtain (remembering that $\mathbf{P}$ is *unitary*)

$$\mathbf{P}\mathbf{a}(p,\sigma)\mathbf{P}^{\dagger} = \mathrm{i}\mathbf{a}(\hat{P},\sigma),\ \mathbf{P}\mathbf{b}(p,\sigma)\mathbf{P}^{\dagger} = \mathrm{i}\mathbf{b}(\hat{P},\sigma). \tag{4.119}$$

This shows that the field energy is unchanged and the momentum changes sign, the electromagnetic charge is unchanged and the three-current changes sign.

4.4.4 Lorentz Classification of Bilinear Forms

As we have seen studying the examples of the energy-momentum vector and the electromagnetic current density, we can build physical important quantities out of the Dirac spinor field which have certain tensor structure with respect to $\mathrm{SO}(1,3)^{\uparrow}$-transformations.

Now the spinor has four linearly independent complex components and thus we can build $4\times 4 = 16$ linearly independent local bilinear forms. These can be written as

$$B_{\hat{O}}(x) = \bar{\psi}_a(x)\hat{O}\psi_b(x), \tag{4.120}$$

where we preferred to take $\bar{\psi}$ rather than $\psi\dagger$ as the left argument of the bilinear form since this makes the definition relativistically invariant.

Now we can find sixteen linearly independent matrices which have certain transformation properties under $\mathrm{SO}(1,3)^{\uparrow}$-transformations. This is important when we construct interaction terms for the Lagrangian including Dirac spinor fields. Since our discussion has shown that all local observables have to be built with help of *pairs* of Dirac spinors, i.e., a monomial contained in the Lagrangian has to contain an even number of spinor field factors in order to obtain local observables fulfilling the microcausality condition with commutators rather than with anti-commutators.

The matrices are given as

$$\hat{1},\ \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,\ \gamma^\mu,\ \gamma^\mu\gamma^5,\ \hat{\Sigma}^{\mu\nu} = \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]. \tag{4.121}$$

The last matrix we have identified above with the generators of the Dirac spinor representation of the $SL(2,\mathbb{C})$. Together we have $1+1+4+4+6=16$ matrices which can be shown to be linearly independent.

With these matrices we can build the following local observables

$$S_{ab} = \bar{\psi}_a\psi_b,\ P_{ab} = \mathrm{i}\bar{\psi}_a\gamma^5\psi_b,\ V^\mu_{ab} = \mathrm{i}\bar{\psi}_a\gamma^\mu\psi_b,\ = \mathrm{i}\bar{\psi}_a\gamma^\mu\gamma^5\psi_b,\ T^{\mu\nu}_{ab} = \mathrm{i}\bar{\psi}_a\hat{\Sigma}^{\mu\nu}\psi_b, \tag{4.122}$$

which are scalars, vectors and an antisymmetric tensor with respect to $\mathrm{SO}(1,3)^{\uparrow}$-transformations. Herein we have left out the space-time arguments. This is evident from the meaning of the γ-matrices and the first part of appendix B.

Now we want to classify these objects further with respect to parity transformations. To this end we write down the Dirac spinor with help of the Weyl-spinors $\psi = \begin{pmatrix}\xi\\ \eta\end{pmatrix}$. Since

$$S = \xi^\dagger\eta + \eta^\dagger\xi,\ P = \mathrm{i}(\xi^\dagger\eta - \eta^\dagger\xi) \tag{4.123}$$

and due to (4.117) the parity transformation interchanges ξ and η we find that S is unchanged while P changes sign. Thus S is a *scalar* and P a *pseudo-scalar*. From the Dirac equation we learn that V^μ is a *vector* and A^μ an *axial vector* or pseudo-vector. $T^{\mu\nu}$ builds an antisymmetric tensor of second rank which changes sign under parity transformations. A second-rank symmetric tensor reduces to a scalar since $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$.

Here we summarise again the behaviour under parity transformation

$$S^P_{ab} = S_{ab},\ P^P_{ab} = -\hat{P}P_{ab},\ V^P_{ab} = \hat{P}V_{ab},\ A^P_{ab} = -\hat{P}A_{ab},\ T^P_{ab} = -\hat{P}T_{ab}\hat{P}. \tag{4.124}$$

Herein we left out the space-time arguments of the fields. It is understood that on the right hand side the space-time arguments have to be taken as the parity transformed ones of the left hand side: $x \to \hat{P}x = (t,-\vec{x})$.

All these properties hold also true for the normal-ordered field operator counterparts of these bilinear forms.

But now let us look on charge conjugation. We define

$$\mathbf{B}_O =: \bar{\boldsymbol{\psi}}_a\hat{O}\boldsymbol{\psi}_b : . \tag{4.125}$$

From (4.96) we learn

$$\boldsymbol{\psi}^C = \hat{C}\bar{\boldsymbol{\psi}}^t \Rightarrow \bar{\boldsymbol{\psi}}^C = \boldsymbol{\psi}^t\hat{C}^* = -\boldsymbol{\psi}^t\hat{C}. \tag{4.126}$$

Inserting this into 4.125 we find

$$\mathbf{B}^C_O =: \bar{\boldsymbol{\psi}}^C_a\hat{O}\boldsymbol{\psi}^C_b := + : \bar{\boldsymbol{\psi}}_b\hat{C}\hat{O}U^\dagger_C\boldsymbol{\psi}_a :, \tag{4.127}$$

where we have used the fact that we can interchange the field operators under the normal product *with an extra sign*. Had we used not the field operators but the classical fields we would have obtained the *opposite* sign in (4.127). We shall see in the context of the path integral treatment for fermionic fields that the correct classical field counterparts are not the well known commuting c-numbers but the anti-commuting *Grassmann numbers*. This does not play any role concerning observables since the fermionic fields are not observable at all since they obey not the commutator but the anti-commutator microcausality condition. As we have seen the reason why the charge conjugated observable bilinear forms have a changed sign compared to the commuting classical counterparts is that this transformation is anti-unitary (we had to take a transformation from ψ to $\bar{\psi}$ and then a unitary one). Since the same is true for the *time-reversal transformation* we have for this case also a sign change compared to the classical commuting fields.

Putting in the different matrices for $\hat{O}$ we find

$$\mathbf{S}^C_{ab} = \mathbf{S}_{ba},\ \mathbf{P}^C_{ab} = \mathbf{P}_{ba},\ \mathbf{V}^{C\mu}_{ab} = -\mathbf{V}^{\mu}_{ba},\ \mathbf{A}^{C\mu}_{ab} = \mathbf{A}^{\mu}_{ba},\ \mathbf{T}^{C\mu\nu}_{ab} = -\mathbf{T}^{\mu\nu}_{ba}. \tag{4.128}$$

The time reversal is due to (4.111) given by

$$\boldsymbol{\psi}^T = \hat{T}\bar{\psi}^t, \bar{\boldsymbol{\psi}}^T = -\bar{\psi}\hat{T}. \tag{4.129}$$

We have also to take into account that the time reversal includes a change of order of field operators since it interchanges initial and final states. For the scalar we have for example:

$$(\bar{\boldsymbol{\psi}}_a\boldsymbol{\psi}_b)^T = -(\bar{\boldsymbol{\psi}}_b\hat{T}^t)(\hat{T}^\dagger\boldsymbol{\psi}_a) = \bar{\boldsymbol{\psi}}_{\mathbf{b}}\boldsymbol{\psi}_a. \tag{4.130}$$

For the different bilinear forms we find

$$\mathbf{S}^T_{ab} = \mathbf{S}^T_{ba},\ \mathbf{P}^T_{ab} = -\mathbf{P}_{ba},\ \mathbf{V}^T_{ab} = \hat{P}\mathbf{V}_{ba},\ \mathbf{A}^T_{ab} = \hat{P}\mathbf{A}_{ba},\ \mathbf{T}^T_{ab} = -\hat{P}T_{ba}\hat{P}. \tag{4.131}$$

4.4.5 The CPT Theorem

Now we show the famous theorem that each local quantum field theory with a $\mathrm{SO}(1,3)^\uparrow$-invariant real action and being quantised according to the spin-statistics theorem is also invariant under the discrete transformation which applies instantaneously the time reversal, the parity and the charge conjugation transformation, shortly denoted as the CPT *transformation*.

We want to show that the Hamilton density obeys

$$(\mathbf{CPT})\mathscr{H}(x)(\mathbf{CPT})^{-1} = \mathscr{H}(-x). \tag{4.132}$$

This proves the theorem since then it is clear that the Hamiltonian is unchanged under CPT transformations because the change of sign in the last equation has no effect on the three-space integral $\mathbf{H} = \int \mathrm{d}^3\vec{x}\mathscr{H}(x)$.

The main message of this theorem is that there is *no necessity* for the discrete transformations P, T, C or CP alone from the principle of Poincaré invariance, at least not for local quantum field theory as we consider here. We shall see that indeed the weak interaction does break these discrete symmetries. This is directly proven from experiments for parity and CP (β-decay for parity- and K-meson decay for CP non-conservation) and if one assumes that the elementary particles are described correctly by a local quantum field theory we know from the CPT theorem that also the time reversal is not a symmetry of nature.

Now we like to prove the theorem. We summarise the combined action of the three discrete transformations by applying (4.124), (4.128) and (4.131) to the bilinear forms (4.122).

$$\mathbf{S}_{ab}^{CPT}(x) = \mathbf{S}_{ab}(-x),\ \mathbf{P}_{ab}^{CPT}(x) = \mathbf{P}_{ab}(-x), \tag{4.133}$$

$$\mathbf{V}_{ab}^{CPT}(x) = -\mathbf{V}_{ab}(-x),\ \mathbf{A}_{ab}^{CPT}(x) = -\mathbf{A}_{ab}(-x),\ \mathbf{T}_{ab}^{CPT}(x) = \mathbf{T}_{ab}(-x). \tag{4.134}$$

Since we can consider all fields as built from the Dirac spinors, with help of the sixteen matrices (4.121) and reduction to irreducible parts of the representation we see that all *real* $\mathrm{SO}(1,3)^{\uparrow}$-tensors which can be build are even or odd if they have an even or odd rank number. There is *no difference* between tensors and pseudo-tensors for the whole CPT transformation. This holds also true for tensors for which one or more indices are obtained by applying ∂_μ. Finally for a elementary Dirac spinor field we find in our bispinor representation

$$\psi^{CPT}(x) = \mathrm{i}\gamma_5\psi(-x). \tag{4.135}$$

Since only the time-reversal transformation $\mathbf{P}$ is anti-unitarian a general even (odd) complex tensor transforms to it is hermitian conjugate (negative hermitian conjugate) and any complex valued constants are complex conjugated.

Now any *Lagrangian* which is $\mathrm{SO}(1,3)^{\uparrow}$-invariant is build as a contraction of tensors (which may be built out of spinor bilinear forms described above). Since it has to be the sum of scalars and pseudo-scalars the sum of all ranks in a monomial contained in the Lagrangian has to be *even*. Thus the Lagrangian behaves like any scalar or pseudo-scalar. Since it is assumed to be a *real local quantity* we have

$$\mathscr{L}^{CPT}(x) = \mathscr{L}(-x). \tag{4.136}$$

Now the Hamiltonian density is given by

$$\mathscr{H} =: \mathbf{\Pi}_r\dot{\boldsymbol{\phi}}_r : -\mathscr{L}, \tag{4.137}$$

where the sum over r runs over all fields contained in the theory.

It is easy to show that the canonical field momenta transform under CPT with the negative conjugate complex phase factor compared to the original fields since the Lagrangian is *hermitian*. The time derivative of the field gets only an additional sign compared to the CPT-transformed field. Thus the canonical commutator relations or anti-commutator relations for tensor and spinor fields respectively are invariant under CPT transformations and thus together with (4.136) we have proven (4.132) which completes the proof of the CPT theorem also known as the *Pauli-Lüders Theorem*.

4.4.6 Remark on Strictly Neutral Spin–1/2–Fermions

This subsection completes the investigation of the discrete symmetry transformations. Especially it shows that except for the case of strictly neutral Dirac fermions, which we do not have found in nature yet, the two inequivalent representation of the parity transformation, one with $\hat{P}^2 = 1$ and one with $\hat{P}^2 = -1$, are indeed physically equivalent .

We start with the investigation of the so called *relative parity of particles and their corresponding antiparticles* which is in principle observable. As we have seen in the beginning of the chapter the intrinsic properties of particles are most easily observed when looking on them in their rest system. In our description of C, P and T we have chosen the parity representation with

$\hat{P}^2 = -1$ in the bispinor representation and splitting up the so far used Dirac spinor in it's Weyl components explicitely we have

$$\underbrace{\hat{P}}_{\mathrm{i}\gamma^0} \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{pmatrix} = \mathrm{i} \begin{pmatrix} \eta_{\dot{\alpha}} \\ \xi^\alpha \end{pmatrix}. \tag{4.138}$$

Using the charge conjugation transformation (4.96) we find that the charge conjugated spinor transforms in the same way. In the rest frame we have

$$\xi^\alpha = \eta_{\dot{\alpha}}, \xi\alpha C = \eta_{\dot{\alpha}} \tag{4.139}$$

which is the reduction to a Weyl spinor which is appropriate for the rest frame of the particle. As we read off from (4.138) these Weyl spinors for the particle and its corresponding antiparticle are both multiplied with a factor i. This means that the Weyl spinor product $\xi^\alpha \xi^C_\alpha$ changes sign under parity transformations, i.e., it is a pseudo-scalar. By definition this means that particle and the corresponding antiparticle have *opposite intrinsic parity*. As one can show explicitly in the same way this is also the case if we chose the representation with $\hat{P}^2 = +1$. The only difference is that in this latter case the parity transformation has a 1 instead of the i in 4.138 while the charge conjugated state has -1. But the net effect for the relative intrinsic parity is the same as before, namely to be of opposite sign. Thus there seems to *no physical difference in the two coverings of the* $O(1,3)^\uparrow$.

Now we look on a particle which is identical with its antiparticle. This means that $\boldsymbol{\psi} = \boldsymbol{\psi}^C$. In bispinor representation this means

$$\boldsymbol{\xi}^\alpha = -\mathrm{i}\boldsymbol{\eta}^{\dot{\alpha}}, \ \boldsymbol{\eta}_{\dot{\alpha}} = -\mathrm{i}\boldsymbol{\xi}^\dagger_\alpha. \tag{4.140}$$

Such spinors are called *Majorana–spinors*. One can show by direct calculation that this condition is invariant under CPT if we define the C, P and T transformations as described above. But with our description where $\hat{P}^2 = -1$ this definition is also CP-invariant since particle and antiparticle transform with the same transformation, while under the other representation with $\hat{P}^2 = -1$ the sign of the charge conjugated state changes under parity transformations. Thus the different descriptions of parity transformations are *physically distinguishable* if one can find a strictly neutral spin 1/2 particle in nature, because then one could investigate if the neutral particles would be CP-invariant or not. Since up to now there is no strictly neutral spin-1/2 particle observed by experiment there is no physical important difference in the different parity transformation concepts so far.

4.5 Path Integral Formulation

Now we come to the path integral formulation of relativistic quantum field theory which follows the same line of arguments as the analogous path integral for nonrelativistic quantum theory as shown in chapter 1.

Here we derive the path integral formalism for fields from the so far used operator methods. First we consider a real scalar field ϕ with its operator counterpart $\boldsymbol{\phi}$ and the conjugated momentum operator $\boldsymbol{\pi}$.

At time $t = 0$ we define the generalised eigenvectors of the field operators as $|\varphi\rangle$ with

$$\boldsymbol{\phi}(0, \vec{x}) \, |\varphi\rangle = \varphi(\vec{x}) \, |\phi\rangle \, . \tag{4.141}$$

As in the case of ordinary quantum mechanics these generalised states in the Fock space build a complete orthogonal set

$$\int \mathrm{D}\varphi(\vec{x})\,|\varphi\rangle\,\langle\varphi| = \mathbf{1},\ \langle\varphi_a\,|\,\varphi_b\rangle = \delta[\varphi_a - \varphi_b], \tag{4.142}$$

where both the integral and the δ-distribution have to be taken in a formal functional sense. One can think about this as the following limit: First one takes a finite volume in three-space, let us say a cube, and applies periodic boundary conditions to the fields, then discretises this space and then does the same calculations as in the quantum mechanical case. After this one goes to an infinite continuous space. We shall show this procedure later on explicitely if necessary.

The same line of arguments can be done with the canonical field momentum operator:

$$\begin{aligned} \boldsymbol{\pi}(0,\vec{x})\,|\pi\rangle &= \pi(\vec{x})\,|\pi\rangle \\ \int \frac{\mathrm{D}\pi(\vec{x})}{2\pi}\,|\pi\rangle\,\langle\pi| &= \mathbf{1} \\ \langle\pi_a\,|\,\pi_b\rangle &= \delta[\pi_a - \pi_b]. \end{aligned} \tag{4.143}$$

Herein the introduction of the 2π denominator in the measure of the path integral is just for later convenience. As in ordinary quantum mechanics in the case of position and momentum operators one can prove from the canonical commutation relations

$$\frac{1}{\mathrm{i}}\,[\boldsymbol{\phi}(0,\vec{x}),\boldsymbol{\pi}(0,\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y}) \tag{4.144}$$

the important overlap between field and momentum eigenvectors as follows

$$\langle\varphi\,|\,\pi\rangle = \exp\left[\mathrm{i}\int \mathrm{d}^3\vec{x}\pi(\vec{x})\varphi(\vec{x})\right]. \tag{4.145}$$

Now we can do the same as in the case of quantum mechanics in chapter 1 to get rid of field and momentum operators with help of the path integral formalism. The transition amplitude with respect to the field eigenvectors can be written with help of the Schrödinger picture time-evolution as

$$C_{ba}(t_f,t_i) = \langle\varphi_b\,|\exp[-\mathrm{i}\mathbf{H}(t_f-t_i)]|\,\varphi_a\rangle\,. \tag{4.146}$$

Of course the result is picture independent! Herein $\mathbf{H} = \int \mathrm{d}^3\vec{x}\mathscr{H}$ is the Hamiltonian which is assumed time-independent, which is the case iff $\mathscr{H}$ does not depend explicitely on time (Noether's theorem for symmetry under time translation). Now we divide the time interval (t_i,t_f) in N equidistant pieces and insert complete sets of momentum and field eigen-kets as we did in the path integral formulation in chapter 1:

$$\begin{aligned} C_{ba}(t_f,t_i) = \lim_{N\to\infty}\int \prod_{k=1}^{N} \frac{\mathrm{D}\pi_k\mathrm{D}\varphi_k}{2\pi}\,\langle\varphi_a\,|\,\pi_N\rangle\,\times \\ \times\,\langle\pi_N\,|\exp(-\mathrm{i}\mathbf{H}\Delta t)|\,\varphi_N\rangle\ldots\langle\pi_1\,|\exp(-\mathrm{i}\mathbf{H}\Delta t)|\,\varphi_1\rangle\,\langle\varphi_1\,|\,\varphi_a\rangle\,. \end{aligned} \tag{4.147}$$

Now we assume that $\mathbf{H}$ seen as an operator-valued functional of $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$ is ordered in a way that all canonical momenta are on the left of all fields, which is called *Weyl ordering*. If there are no derivative couplings in the theory, as for instance in ϕ^4-theory, then we can write

$$\langle\pi_k\,|\exp(-\mathrm{i}\mathbf{H}\Delta t)|\,\varphi_k\rangle \cong_{\Delta t\to 0} (1-\mathrm{i}H_k\Delta t)\exp[-\mathrm{i}\int \mathrm{d}^3\vec{x}\pi_k\varphi_k], \tag{4.148}$$

where we have used (4.145) and the abbreviation $H_k = \int \mathrm{d}^3\vec{x}\mathscr{H}(\pi_k, \varphi_k)$. Now putting all this together one obtains

$$C_{ba}(t_f, t_i) = \lim_{N\to\infty} \int \prod_{k=1}^{N} \frac{\mathrm{d}\pi_k \mathrm{d}\varphi_k}{2\pi} \times \exp\left\{-\mathrm{i}\Delta t \sum_{j=1}^{N} \int \mathrm{d}^3\vec{x}[\mathscr{H}(\pi_j, \phi_j) - \pi_j(\varphi_{j+1} - \varphi_j)/\Delta t]\right\} \delta(\varphi_1 - \varphi_a). \tag{4.149}$$

In the continuum limit this may be written as

$$C_{ba}(t_f, t_i) = \int \mathrm{D}\pi \int_{\phi(t_i,\vec{x})=\varphi_a(x)}^{\phi(t_f,\vec{x})=\varphi_b(x)} \mathrm{D}\phi \exp\left\{\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}^4x \left[\pi(x)\frac{\partial\phi(x)}{\partial t} - \mathscr{H}(\pi,\phi)\right]\right\}. \tag{4.150}$$

Now we can calculate the vacuum-to-vacuum transition amplitude by using the iϵ description introduced at the end of chapter 1. As we have shown there explicitely, the iϵ-description projects out the vacuum state and makes the Green's function solutions unique which will be used later on for perturbation theory. It selects out of the possible Green's functions the causal one, i.e., the Feynman Green's function. In the operator formalism this came out of the proper Fourier transform of the vacuum expectation value of the time-ordered product of two field operators which we called a contraction.

All this has not to be repeated here at length, because it is the same story as for the nonrelativistic case in chapter 1. We end with the *Feynman-Kac Formula* for the calculation of vacuum expectation values of time-ordered functionals:

$$\langle 0\,|T_c F[\boldsymbol{\phi}, \boldsymbol{\pi}]|\,0\rangle = N \int \frac{\mathrm{D}\pi}{2\pi} \int \mathrm{D}\phi \exp\{\mathrm{i}S[\phi,\pi]\}\, F[\phi,\pi]. \tag{4.151}$$

Herein $F[\boldsymbol{\phi}, \boldsymbol{\pi}]$ has to be written in Weyl-ordered form, i.e., with all canonical field momenta π to the left of all fields ϕ to obtain the correct c-number functional. Also we have to take the action functional in its canonical form $S[\phi,\pi] = \int \mathrm{d}^4x\{\partial_t\phi(x)\pi(x) - \mathscr{H}[\phi,\pi]\}$. We shall see that in many cases it is possible to integrate out the canonical field momenta resulting in the Lagrangian form of the path integral.

Especially we can calculate the generating functional for disconnected Green's functions by demanding

$$Z[J] = N \int \frac{\mathrm{D}\pi}{2\pi} \int \mathrm{D}\phi \exp\left\{\mathrm{i}S[\phi,\pi] + \mathrm{i}\int \mathrm{d}^4x J(x)\phi(x)\right\}, \tag{4.152}$$

where the normalisation constant has to be chosen such that $Z[J=0] = \langle 0\,|\,0\rangle = 1$ holds.

Then according to (4.151) the n-point Green's function is given by the nth order functional derivative of Z with respect to J:

$$\mathrm{i}G^{(n)}(x_1, x_2, \ldots, x_n) = \langle 0\,|T_c\boldsymbol{\phi}(x_1)\boldsymbol{\phi}(x_2)\cdots\boldsymbol{\phi}(x_n)|\,0\rangle = \frac{1}{\mathrm{i}^n}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}. \tag{4.153}$$

Before we come to the calculation of path integrals for physically relevant examples we want to treat also fermions with the path integral formalism. As the first step we introduce the generalised eigenvectors of the fermionic field operator $\boldsymbol{\psi}$ by

$$\boldsymbol{\psi}(t_i, \vec{x})\,|\psi\rangle = \psi(\vec{x})\,|\psi\rangle\,. \tag{4.154}$$

But here arises a problem, because the fermionic fields are anti-commuting rather than commuting which means

$$\{\boldsymbol{\psi}(t_i,\vec{x}),\boldsymbol{\psi}(t_i,\vec{y})\}=0. \tag{4.155}$$

This means that (4.154) can only be consistent with (4.155) if the $\psi(\vec{x})$ are anti-commuting numbers. The correct mathematical concept was given by *Grassmann* in the 19th century.

Here we have to define how to calculate with such anti-commuting *Grassmann numbers*, especially we have to define how to integrate and differentiate with these objects.

Nevertheless we first have to look on the algebraic properties of the Grassmann numbers. In the mathematical sense it is an *algebra*, i.e. a vector space over the number fields $\mathbb{R}$ or $\mathbb{C}$ with a multiplication we have to define now. First we look on a finite-dimensional Grassmann algebra. We start with the G-basis of the Grassmann algebra, which should not be interchanged with the basis of the vector space the Grassmann algebra is constructed with. The G-basis is a set of n linear independent basic Grassmann elements g_k with $k=1\ldots n$. Now we define the product of basis elements to be anti-commutative, associative and distributive with respect to vector addition. At the same time it is commutative with the multiplication with ordinary numbers. Then the algebra is given as the algebraic closure of this structure.

With help of this ideas one can build a basis of the vector space underlying the Grassmann algebra with help of the basis elements. By definition we include the number 1 as a basis element of that space. Thus we have

$$1,\ g_j,\ g_{jk}=g_jg_k=-g_kg_j,\ g_{jkl}=g_jg_kg_l,\ldots,\ g_{k_1k_2\ldots k_n}=g_{k_1}g_{k_2}\cdots g_{k_n}. \tag{4.156}$$

Since the product of two g_k is anti-commuting in any of the higher products each basic element can appear at most one time. Two such products which are only different by the interchange of some elements are the same up to a sign given by the permutation of the one order of indices to the other. A standard choice for the basis is that we order the indices from left to right. Thus the dimension of the vector space is

$$\sum_{k=0}^{n}\binom{n}{k}=2^n. \tag{4.157}$$

Now we define the integral over the Grassmann numbers to be a linear mapping from the algebraic functions $\mathscr{A}(G_n)$, where G_n is the Grassmann algebra. We assume also that it should be translationally invariant. Let us start with the most simple case of the Grassmann algebra G_1. The general algebraic function is

$$f(g)=\alpha+\beta g \text{ with } \alpha,\beta\in\mathbb{R} \text{ or } \mathbb{C}. \tag{4.158}$$

Using translational invariance and linearity of the integral we find

$$\int \mathrm{d}g f(g+\gamma)=(\alpha+\beta\gamma)\int \mathrm{d}g+\beta\int \mathrm{d}g g=\alpha\int \mathrm{d}g+\beta\int \mathrm{d}g g \tag{4.159}$$

and since this has to hold for all $\alpha,\beta,\gamma\in\mathbb{R}$ or $\mathbb{C}$ we have

$$\int \mathrm{d}g=0,\ \int \mathrm{d}g g\neq 0. \tag{4.160}$$

For convenience we chose

$$\int \mathrm{d}g g=1. \tag{4.161}$$

For a Grassmann algebra of rank n we define $\int \mathrm{d}g_1 \mathrm{d}g_2 \cdots \mathrm{d}g_n f(g_1, \ldots, g_n)$ as the iterated integral over the basis elements. The only thing we have to do is to define a sign convention for a monomial. If we like to integrate a monomial over g_k we bring at first this element to the very right and integrate after that over it using the rule defined above:

$$\int \mathrm{d}g_k f(g_1, \ldots, \hat{g}_k, \ldots, g_n) g_k = f(g_1, \ldots, \hat{g}_k, \ldots, g_n). \tag{4.162}$$

The hat over g_k as the element of f means that f is independent of it. By this sign convention now the integral over $g_1 \ldots g_n$ is well defined as an iterated integral. At the same time we see, that also the "differentials" $\mathrm{d}g_1 \ldots \mathrm{d}g_n$ are *anti-commuting.*

Since the most general element of $\mathscr{A}(G_n)$ is of the form

$$f(g_1, \ldots, g_n) = \alpha + \beta_{ij} g_i g_j + \cdots + \omega g_1 g_2 \cdots g_n. \tag{4.163}$$

With the definition above we have the simple rule

$$\int \mathrm{d}g_1 \mathrm{d}g_2 \cdots \mathrm{d}g_n f(g_1, g_2, \ldots, g_n) = \omega. \tag{4.164}$$

Now if we define analytic functions of Grassmann algebra of finite rank by their formal series expansion with respect to the basis elements we find that this series breaks down after n summands. Thus the set of analytic functions of Grassmann numbers is identical with $\mathscr{A}(G_n)$. So we have defined the integral over Grassmann numbers completely by Eq. (4.164).

Now we prove the crucial theorem for the quantum field theory of fermions. Let $\{\eta_k, \eta_k^*\}_{n=1\ldots n}$ the basis of a Grassmann algebra of rank $2n$ and $\hat{A}$ an arbitrary $n \times n$-matrix. Then

$$\int \mathrm{d}\eta_1^* \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n^* \mathrm{d}\eta_n \exp(\eta^* \hat{A} \eta) = \det \hat{A} \text{ where } \eta = (\eta_1, \ldots, \eta_n)^t,\ \eta^* = (\eta_1^*, \cdots, \eta_n^*). \tag{4.165}$$

To prove (4.164) we have to find the coefficient of $\eta_1^* \eta_1 \cdots \eta_n^* \eta_n$ in the series expansion of $\exp(\eta^*) \hat{A} \eta$ and show that this is $\det \hat{A}$. Now the expansion is

$$\exp(\eta^* \hat{A} \eta) = 1 + A_{kl} \eta_k^* \eta_l + \cdots + \frac{1}{n!} \prod_{\nu=1}^{n} A_{k_\nu l_\nu} (\eta_{k_\nu}^* \eta_{l_\nu}). \tag{4.166}$$

Now we rewrite the last summand in this expansion. Since the Grassmann basis elements are anti-commutative the sum runs over all possible permutations of the indices k and l. Any term with equal indices is zero. So the summand is

$$\frac{1}{n!} \sum_{k \in S_n} \sum_{l \in S_n} \prod_{\nu=1}^{n} A_{k(\nu) l(\nu)} (\eta_{k(\nu)}^* \eta_{l(\nu)}). \tag{4.167}$$

Now we have to bring the Grassmann numbers in the lexical order to integrate over them all using the definition of the integral. The only problem is to get the correct sign. First of all we change the pairs of Grassmann numbers in a given permutation such that the starred numbers are in lexical order. Now we have to bring the non-starred numbers to the correct position right from the starred one with the same index. First we bring η_1 to the right of the pair containing η_1^*. This gives no sign change, since η_1 commutes with the paired objects it has to be interchanged with. After this we interchange the non-starred number which was paired with η_1^* previously

with the next starred number in order to get the correct ordered pair $\eta_1^*\eta_1$ on the very left. This procedure gives an additional sign and the rest contains $(n-1)$ pairs with the starred objects in lexical order. We have to repeat this procedure with this rest of the product as long as all is in the correct order. The sign is thus the sign of the permutation of the non-starred objects times the sign of the starred ones. So our product is

$$\frac{1}{n!}\sum_{k\in S_n}\sum_{l\in S_n}\prod_{\nu=1}^{n}\sigma(l)\sigma(k)A_{k(\nu)l(\nu)}(\eta_1^*\eta_1)\cdots(\eta_n^*\eta_n). \tag{4.168}$$

But now from the elementary theory of determinants we know that

$$\sum_{l\in S_n}\prod_{\nu=1}^{n}\sigma(l)A_{k(\nu)l(\nu)}=\sigma(k)\det\hat{A}. \tag{4.169}$$

This shows that the sum over l is always the same result, namely $\det\hat{A}$ and the sum over k cancels the factorial in the denominator. This proves (4.165).

Now we like to define the derivative with respect to Grassmann numbers. There are two sorts of derivatives called *left and right derivatives*. The derivative operation is by definition a linear mapping from $\mathscr{A}(G_n)$ to itself and thus it is enough to define the left (right) derivative applied to a monomial. To obtain the left (right) derivative of a monomial with respect to g_k one has bring g_k to the very left (right) of the monomial and cancel it from the product. If g_k is not contained in the monomial the derivative is set to zero. This shows the funny fact that in the case of Grassmann numbers the integration is the same as right derivative.

Now we want to show another important result. For this we look on a Grassmann algebra of rank $4n$ with basis elements $g_1,\cdots,g_n$ and $g_1^*,\cdots g_n^*$, $J_1,\cdots,J_n$ and $J_1^*,\ldots,J_n^*$. Now we define a function of the J and J^* of the following form

$$f(J^*,J)=\exp[C(g^*,g)+J^*g+g^*J]. \tag{4.170}$$

If C contains only monomials with an even number of g-elements, then

$$\frac{\partial^R f}{\partial J_k}=g_k^*f,\ \frac{\partial^L f}{\partial J_k^*}=g_kf. \tag{4.171}$$

The proof is simple: Since C contains only monomials with an even number of Grassmann basis elements it commutes with all basis elements. Thus we can apply Cauchy's law of multiplication for (formal) power series to show that the exponential factorises

$$f(J^*,J)=\exp[C(g^*,g)]\exp[J^*g]\exp[g^*J]. \tag{4.172}$$

Let us consider the right derivative with respect to J_k. From the definition of derivatives we have to apply it only to the right factor. This exponential factorises by the same reason as before. So we have

$$\frac{\partial^R\exp[g^*J]}{\partial J_k}=\frac{\partial^R}{\partial J_k}\prod_{j=1}^{n}\exp(g_j^*J_j)=g_k^*\prod_{k\neq j=1}^{n}exp(g_j^*J_j)=g_k^*\exp(g^*J), \tag{4.173}$$

where we have repeatedly used the fact that g_k commutes with functions containing only monomials with an even number of Grassmann basis elements. In the last step we have further used

the identity $g_k^* \exp(g_k^* g_k) = g_k^*$. Plugging all this together we have proven the first equation (4.171). With the same argument we can also show the second one to be true.

We shall also use the simple result

$$\int \mathrm{d}g_1 \dots \mathrm{d}g_n \frac{\partial^{L/R}}{\partial g_k} f(g_1, \dots g_n) = 0. \tag{4.174}$$

To see this we have only to remember that integration with respect to g_k is the same as to take the right derivative of g_k. In any monomial g_k can show up at most one time. If this is the case, the (left or right) derivative will cancel it and the integration over g_k makes the whole monomial vanish. If g_k is not contained in a monomial the (left or right) derivative cancels it. Thus the net result of the whole procedure is zero in any case as we have claimed.

In conclusion we may say that the "analysis" of Grassmann numbers (which is more algebra than real analysis) is in great parts analogue to the corresponding operations in real or complex analysis. The only thing one has to take care of are the signs arising from the anti-commuting structure of the Grassmann algebra.

4.5.1 Example: The Free Scalar Field

Here we like to derive an important example for the path integral. As we have seen in the previous chapter in detail from the point of view of the operator formalism, to calculate transition amplitudes in a perturbative expansion we need the time-ordered n-point functions. Now we can directly apply what we have learnt in section 1.10. studying the example of $1+0$-dimensional ordinary quantum mechanics, namely to write down the generating functionals for time-ordered n-point functions. The rest of this section is used to calculate this generating functionals for the case of a free scalar field.

The Lagrangian is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2. \tag{4.175}$$

Now we like to apply the path-integral formula for calculating of the generating functional for n-point Green's functions:

$$Z_0[J] = \langle 0 | \exp\left[\mathrm{i} \left\langle \boldsymbol{\phi}_1 J_1 \right\rangle_1\right] | 0 \rangle = N \int \frac{\mathrm{D}\pi}{2\pi} \int \mathrm{D}\phi \exp[\mathrm{i}S[\phi, \pi] + \mathrm{i} \langle J\phi \rangle]. \tag{4.176}$$

Herein we use the abbreviation $\langle f_{12\dots n} \rangle_{12\dots k}$ for the integral of a space-time-dependent function over the space-time variables $x_1, x_2, \dots x_k$ resulting in another space-time-dependent function of the variables $x_{k+1}, \dots, x_n$.

The canonical field momentum is given by its definition:

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} = \partial_t \phi. \tag{4.177}$$

Thus the Hamiltonian is

$$\mathscr{H} = \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} \partial_t \phi - \mathscr{L} = \frac{1}{2}[\pi^2 + (\nabla\phi)^2 + m^2] \tag{4.178}$$

and the canonical action is given by

$$S[\phi, \pi] = \int \mathrm{d}^4 x \left\{ \pi \partial_t \phi - \frac{1}{2}[\pi^2 + (\nabla\phi)^2 + m^2] \right\}. \tag{4.179}$$

Herein the canonical momentum π has to be seen independent from the fields ϕ!

But since (4.179) is a quadratical form in π, the π-integration in (4.176) is the value of the exponential at the momentum for which the exponential becomes stationary (with fixed ϕ) times a J-independent constant. The stationary point is given by

$$\frac{\delta S}{\delta \pi} = 0 \Rightarrow \pi = \partial_t \phi \tag{4.180}$$

and this is the value the momentum has in the Lagrangian formalism. Thus we have (including the J-independent constant in the overall normalisation N)

$$Z_0[J] = N \int \mathrm{D}\phi \exp[\mathrm{i}S[\phi] + \mathrm{i}\,\langle J\phi\rangle]. \tag{4.181}$$

It should be clear that this calculation is also valid in the interacting case as long as the interaction does not contain any derivative couplings and thus no canonical field momenta in the Hamiltonian formulation. So we can start in these cases with the Lagrangian version of the path integral formula, which has the advantage of being *manifestly covariant*. This is not the case in the canonical formulation, because there we have split explicitly in space and time variables in an arbitrary fixed reference frame!

Now we can rewrite the action with help of a integration by parts using the boundary conditions included in the path integral, namely that the fields have to vanish in space-time infinity:

$$S[\phi] = -\frac{1}{2} \int \mathrm{d}^4 x \phi(\Box + m^2)\phi. \tag{4.182}$$

Since this is a quadratic functional in ϕ we can invoke the same trick as we used for integrating over the field momentum. The path integral is given by the stationary point of the exponential in (4.181). The corresponding field will be denoted with φ and is given by the equations of motion

$$\frac{\delta S[\varphi]}{\delta \varphi} = -(\Box + m^2 - \mathrm{i}\epsilon)\varphi = -J. \tag{4.183}$$

For the path integral we have included a regularisation reading m^2 as $m^2 - \mathrm{i}\epsilon$ to project out the vacuum state as explained at length in chapter 1. We find immediately the solution of (4.183) with help of the Green's function of the Klein-Gordon equation with the boundary conditions given by the iϵ-description. We find the same as in chapter 1 in the case of nonrelativistic path integral formulation that projecting out the vacuum expectation values leads to the uniqueness of the Green's function, namely the *causal*, to use for solving the inhomogeneous equations of motion:

$$\varphi(x) = -\int \mathrm{d}^4 x' D(x - x')J(x'), \tag{4.184}$$

where D is the Green's function with the causal boundary conditions, the *free Feynman propagator*. It can be calculated by taking the Fourier transform of the defining equation

$$-(\Box_x + m^2 - \mathrm{i}\epsilon)D(x - x') = \delta^4(x - x') \Rightarrow D(x - x') = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp[-\mathrm{i}p(x - x')]}{p^2 - m^2 + \mathrm{i}\epsilon}. \tag{4.185}$$

This is, of course, the *free Feynman propagator* we know from chapter 3, where we have obtained it by calculating the vacuum expectation value of the time-ordered product of two field operators (i.e. as the contraction in the sense of Wick's theorem).

Inserting the solution (4.184) into the action we find for the path integral (again absorbing a J-independent constant in the normalisation)

$$Z_0[J] = \exp\left[\frac{\mathrm{i}}{2}\langle \varphi J\rangle\right] = \exp\left[-\frac{\mathrm{i}}{2}\langle J_1 D_{12} J_2\rangle_{12}\right]. \tag{4.186}$$

Herein we have used the abbreviation $D_{12} = D(x_1 - x_2) = D(x_2 - x_1)$ where the symmetry of the Feynman propagator can be seen from (4.185) by substituting $p \to -p$. The same time we have fixed the normalisation constant such that $Z_0[0] = \langle 0\,|\,0\rangle = 1$.

4.5.2 The Feynman Rules for ϕ^4 revisited

Now we are ready to rederive the Feynman rules with help of our functional technique. Here we like to find the Feynman rules for the disconnected Green's functions.

Thus we write down again our path integral in the Lagrangian form:

$$Z[J] = N\int \mathrm{D}\phi \exp(\mathrm{i}S_0[\phi] + \mathrm{i}S_I[\phi] + \mathrm{i}\langle J_1\phi_1\rangle_1). \tag{4.187}$$

Here we have split the action functional in a free part S_0, which is a quadratic functional in ϕ and an interaction part S_I.

Now we factorise the exponential in the free (including the source term) part and the interaction part. Then we expand the interaction part in powers of S_I. Since S_I is (for the most fundamental cases) a polynomial in ϕ and eventually its derivatives each field can be substituted by $1/\mathrm{i}(\delta/\delta J)$. Formally we may write this in the form

$$Z[J] = N\exp\left\{\mathrm{i}S_I\left[\frac{\delta}{\mathrm{i}\delta J}\right]\right\}\int \mathrm{D}\phi\exp(\mathrm{i}S_0[\phi] + \mathrm{i}\langle J_1\phi_1\rangle) = N\exp\left\{\mathrm{i}S_I\left[\frac{\delta}{\mathrm{i}\delta J}\right]\right\} Z_0[J]. \tag{4.188}$$

Again we have absorbed all J-independent factors into N. Now we have found $Z_0[J]$ above and are ready to obtain the Feynman rules for the disconnected n-point Green's functions in the form

$$\mathrm{i}G^{(n)}(x_1, x_2, \ldots x_n) = \left(\frac{\delta^n Z[J]}{\mathrm{i}\delta J(x_1)\mathrm{i}\delta J(x_2)\cdots\mathrm{i}\delta J(x_n)}\right)_{J=0}. \tag{4.189}$$

For this we use the explicit form for Z_0 to calculate the two-point function up to first order in λ as an example to find again the Feynman rules we have already derived in chapter 2 with help of the operator method. To that end we have to perform all the functional derivatives. This could be done completely analytically, but it is much more convenient to use from the very beginning a diagrammatical picture for the calculation.

To do so at first we introduce the basic diagrammatical elements, which are in our case dots (symbolising space-time points), lines (representing propagators), and full blobs with a dot (representing external currents with its space-time argument): Now we can write down the generating functional for the free particles (4.186) as the series of the exponential function (see fig. 4.2).

We have shown the factors 1/2 explicitly.

Now it is very simple to find the rules for calculating the Dyson-Wick series with help of (4.188). As a very simple example let us calculate the first-order correction to the two-point Green's function. At first we have to calculate $\int \mathrm{d}^4x\delta^4 Z_0[J]/\delta J^4(x)$ as the first-order correction of the generating functional. After that we have to differentiate the result again with respect to $J(x_1)$

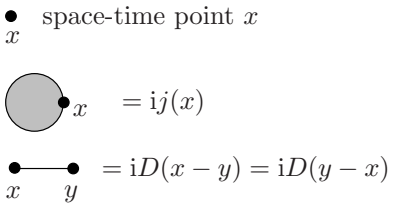


Figure 4.1: Graphical elements for deriving the Feynman rules

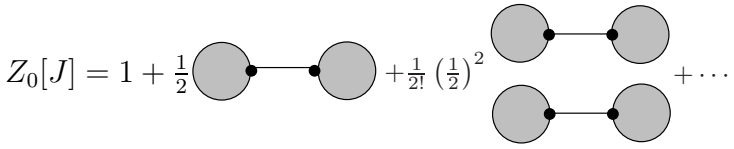


Figure 4.2: Diagrammatic representation for Z_0. All internal space-time points have to be integrated out $\int \mathrm{d}^4x$

and $J(x_2)$. After all we have to set $J = 0$ and renormalise the result by dividing through $Z[0]$ up to the given order (here it is the first-order) in λ.

Now the differentiation with respect to J means truncating a "current-blob" from the diagrams in fig. 4.2 and to multiply with the correct factor (coming from the rule of differentiating powers of J).

From this we learn that we have to keep only the terms in the power expansion in fig. 4.2 which contain the same number of external currents as we have to differentiate the expansion to obtain the searched expression. Indeed, a term which contains less currents than the number of differentiations is cancelled by the differentiations, one with more currents will have at least one current left after performing the differentiations and cancels when setting $J = 0$ at the very end of the calculation.

In our case we have to keep only the term with six external currents. Then one has to build $\delta^4/\delta J(x)^4 Z_0[J]$ with help of the product rule for functional differentiation. The final result is given by fig. 4.3.

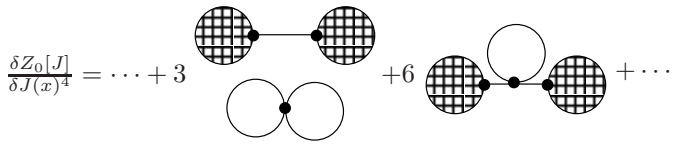


Figure 4.3: Contribution to Z in first order perturbation theory with two external currents. The factor for each diagram is given explicitly and is *not* contained in the diagram!

The same has to be done to calculate the first-order correction to Z[0] for renormalising the propagator up to the given order. The result is shown in fig. 4.4.

At the end we find that the vacuum to vacuum amplitude cancels with the same part in the two-point result shown in fig. 4.3.

The reader should do this graphical calculations on his own. One realizes immediately from this very simple example that we get the same Feynman rules as in the canonical technique but now without using Wick's theorem from the operator technique. Here it is simply the result

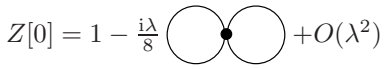


Figure 4.4: Contribution to the vacuum-to-vacuum amplitude in first order perturbation theory

of functional differentiation rules. The only difference to the canonical formalism is, that *we have lost the normal ordering* of the operators which gave the additional rule, that all tadpole contributions can be dropped. Now normal ordering was used to fix the vacuum energy for the free fields to the value 0.

But on the other hand this is not as worse as it looks like, because we have to renormalise the perturbative results order by order in λ anyway. In general, many diagrams containing at least one loop are not well defined as we have already seen at the end of chapter 2. We shall solve this problem of infinities in chapter 6 and appendix C.

4.6 Generating Functionals

Now we shall derive some other formal features for our generating functionals which are important to obtain the *LSZ reduction formula* and some important subclasses of diagrams which can be used as building blocks for higher order calculations. One of such subclasses are the connected diagrams, which can be used to prove that all vacuum to vacuum amplitudes cancel exactly when calculating the n-point Green's functions.

The main advantage is that the number of diagrams to calculate, which are contained in a certain subclass is much smaller than the number of all diagrams.

We shall also define the *effective action*, which is shown to be the generating functional for *one-particle-irreducible* or 1PI truncated diagrams. These are important building blocks for the connected Green's functions. As we shall see they are in a certain sense those diagrams we have to calculate (including regularisation and renormalisation to get rid of the infinities) at least to build up all other sorts of diagrams mentioned so far.

4.6.1 LSZ Reduction

As we have already seen in chapter 3 we can get the S-matrix-elements, i.e., the transition amplitude to find a system in the asymptotic free out-state $|f\rangle$ after a reaction which was prepared in a asymptotic free in-state $|i\rangle$, by truncating the (disconnected) n-point functions and multiplication with the appropriate free-particle amplitudes for in- and out-states respectively. This connection is known as *LSZ reduction formula* and was proved in section 3.6 with help of the operator formalism.

Now we shall again prove this important theorem with help of our path integral formalism. We shall again use ϕ^4-theory as an example, which can be extended to more complicated theories immediately.

Due to eq. (4.150) the only difference between the path integral formula for the generating functional for n-point functions Z and the generating functional S for S-matrix elements are the boundary conditions for the fields. In the former case we have $\phi \to 0$ for $t \to \pm\infty$ in the latter we have

$$\phi(x) \underset{t\to\pm\infty}{\cong} \varphi_0(x) \tag{4.190}$$

where φ_0 is a given free state. This can be written as

$$\varphi_0(x) = \int \frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p}) \exp(-\mathrm{i}px) + a(\vec{p})^* \exp(\mathrm{i}px)\right]_{p_0=\omega(\vec{p})-\mathrm{i}\epsilon}. \tag{4.191}$$

Here the $\mathrm{i}\epsilon$-description is to project out the in- and out-states with definite momentum for $t \to \pm\infty$.

The generating functional for S-matrix elements is given by

$$\Sigma[J, \varphi_0] = \int_{\varphi_0}^{\varphi_0} \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \langle \phi_1 J_1 \rangle_1\right]. \tag{4.192}$$

It is clear that we obtain a certain S-matrix element by inserting (4.191) in (4.192) and taking the functional derivative with respect to $a(\vec{p})$ for a in-particle with momentum $\vec{p}$ and $a^*(\vec{p})$ for a out-particle with momentum $\vec{p}$. It is also clear that one can do the same directly in space-time by differentiating $S[J, \varphi_0]$ with respect to φ_0 for each in- resp. out-field, multiply the result with the one-particle free in- resp. out-state and integrate over the space-time points.

All this is done in presence of the external source J which has to be set to zero at the very end of the calculation.

But now we can use the external source to write down the Dyson series for the generating functional S in the same way as we did for the generating functional Z in (4.188):

$$\Sigma[J, \varphi_0] = \exp\left\{\mathrm{i}S_I\left[\frac{\delta}{\mathrm{i}\delta J}\right]\right\} \Sigma_0[J, \varphi_0], \tag{4.193}$$

where $S_0[J, \varphi_0]$ is the generating functional for the free particles under influence of an external source J and $S_I[\phi]$ is the interaction part of the action functional.

Now we substitute

$$\phi(x) = \tilde{\phi}(x) + \varphi_0(x). \tag{4.194}$$

Together with the translational invariance of the path integral measure we obtain

$$\Sigma_0[J, \varphi_0] = \int \mathrm{D}\tilde{\phi} \exp\left\{\mathrm{i}S_0[\tilde{\phi} + \varphi_0 + \mathrm{i}\left\langle J_1(\tilde{\phi}_1 + \varphi_{01})\right\rangle_1\right\}, \tag{4.195}$$

where $\tilde{\phi}$ fulfils the homogeneous boundary conditions $\tilde{\phi}(x) \to 0$ for $x_0 \to \pm\infty$. Since for φ_0 the free Klein-Gordon equation holds, integrating the action functional by parts we have:

$$S_0[\varphi_0 + \tilde{\phi}] = S_0[\tilde{\phi}]. \tag{4.196}$$

This leads to

$$\Sigma_0[J, \varphi_0] = Z_0[J] \exp[\mathrm{i}\langle J_1 \varphi_{01} \rangle_1]. \tag{4.197}$$

Inserting this result into (4.193) we find using

$$[\Box_x + m^2]\frac{\delta Z_0}{\delta J(x)} = \mathrm{i}J(x)Z_0[J], \tag{4.198}$$

which is obtained from (4.186) and the fact, that the Feynman propagator is the resolvent of the Klein-Gordon operator (4.185) on the left hand side in eq. (4.198),

$$\Sigma[J, \varphi_0] = \exp\left[\left\langle \varphi_0(x)(-\Box_x - m^2)\frac{\delta}{\mathrm{i}\delta J(x)}\right\rangle_x\right] Z[J]. \tag{4.199}$$

From this we obtain the reduction formula by functional differentiating with respect to $a(\vec{p})$ according to the in- and with respect to $a^*(\vec{p})$ out-states one likes to have in the S-matrix element.

Performing the above explained calculation for an arbitrarily given S-matrix element S_{fi} we find (up to a phase factor for the wave functions, which is physically irrelevant) again the *LSZ reduction formula* (3.110). The difference is that the renormalisation factors Z are now absorbed in the overall renormalisation of the generating functional Z which is such that $Z[J=0] = \langle 0, \text{out} \,|\mathbf{S}|\, 0, \text{in}\rangle = 1$.

4.6.2 The equivalence theorem

Now we shall prove a theorem which will become important for the treatment of non-abelian gauge theories in chapter 7.

Let $F_1[\phi]$ be an arbitrary local functional of ϕ. Suppose we define a modified generating functional $Z'[J]$ by

$$Z'[J] = N \int \mathrm{D}\phi \exp[\mathrm{i}S[\phi] + \mathrm{i}\,\langle J_1(\phi_1 + F_1[\phi])\rangle_1]. \tag{4.200}$$

It is clear that with this definition we create new n-point functions compared to those arising from (4.187) by taking derivatives with respect to J. Nevertheless, as we shall show now, the S-matrix elements derived from

$$\Sigma'[J,\varphi_0] = \exp\left[\left\langle \varphi_{01}(\Box_1 + m^2)\frac{\delta}{\delta J_1}\right\rangle_1\right] Z'[J] \tag{4.201}$$

are the same as those derived from Σ. It is clear that we can write

$$Z'[J] = \exp\left(\mathrm{i}\left\langle J_1 F_1\left[\frac{\delta}{\delta J}\right]\right\rangle_1\right) Z[J] \tag{4.202}$$

Since the functional derivatives in (4.201) and (4.202) are commuting we find immediately

$$\Sigma'[J,\varphi_0] = \exp\left(\mathrm{i}\left\langle J_1 F_1\left[\frac{\delta}{\delta J}\right]\right\rangle_1\right) \Sigma[J,\varphi_0]. \tag{4.203}$$

Since the S-matrix elements are generated by $\Sigma[J=0,\varphi_0]$ it is clear that also $\Sigma'[J=0,\varphi_0]$ generates the same S-matrix-elements.

4.6.3 Generating Functional for Connected Green's Functions

From a diagrammatical point of view it is immediately clear that one has to calculate only the connected Green's functions, because if the diagram is composed out of two or more connected diagrams which are not linked together, then the whole expression is given as the product of its connected parts.

At first we introduce the diagrammatics for the disconnected n-point functions.

Now we call a diagram *connected*, if it does not split into two or more subdiagrams which are not connected together. Now we prove the following theorem. For first reading the reader may omit the proof, because it is rather technical.

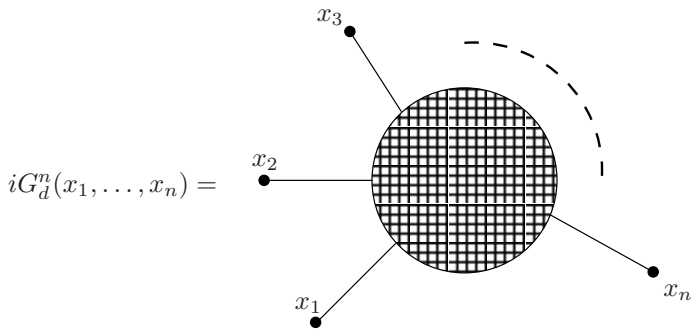


Figure 4.5: Diagrammatic representation for the exact disconnected Green's functions

Theorem 2 (Connected Green's Functions)**.** *The functional*

$$\mathrm{i}W[J] = \ln\{Z[J]\} \Leftrightarrow Z[J] = \exp\{\mathrm{i}W[J]\} \tag{4.204}$$

is the generating functional for connected n-point functions.

$$G_c^{(n)}(x_1,\ldots,x_n) = \left(\frac{1}{\mathrm{i}}\right)^n \left(\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right)_{J=0} \tag{4.205}$$

Proof. We represent the connected Green's functions diagrammatically as shown in fig. 4.6.

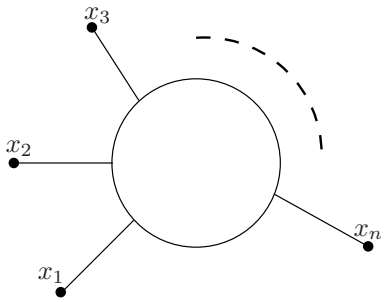


Figure 4.6: Diagrammatic representation for the exact connected Green's functions

The main difficulty in proving this theorem is to formulate the diagrammatical statement in analytical terms. The mathematics is rather simple. Only the use of chain and product rule of functional differentiating is needed.

At first we have to solve the combinatorial problem to express the fact that the disconnected n-point Green's function is given by the sum of all products of connected ones with together n space-time arguments $x_1,\ldots x_n$.

In mathematical terms we have to label all disjoint partitions of space-time coordinates. Any summand entering the sum of products of connected functions can be classified after the number of contained factors and the space-time points attached to a connected subdiagrams the diagram is built off.

Thus we define the following quantities

- P_k^n: The set of all disjoint partitions of the index set $\mathbb{N}_n = \{1, 2, \ldots n\}$ in k subsets.

- r_{nkjl}: The number of elements in the lth set of the jth partition out of P_k^n.
- $m_{nkjl1}, \ldots, m_{nkjlr_{nkjl}}$ labels the elements contained in the jth partition out of P_k^n.

To prove the theorem we define at first the functions with external source J:

$$\mathrm{i}G_J^{(n)} = \frac{1}{\mathrm{i}^n}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)};\; G_{cJ}^{(n)} = \frac{1}{\mathrm{i}^n}\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}. \tag{4.206}$$

The arguments of this functions are always $x_1, \ldots, x_n$. Now we state that the following equation holds:

$$G_J^{(n)} = \frac{Z[J]}{\mathrm{i}^n}\sum_{k=1}^{n}\mathrm{i}^k\sum_{P_k^n}\sum_j\prod_{l=1}^{k}\mathrm{i}^{r_{nkjl}-1}G_{cJ}^{(r_{nkjl})}(x_{m_{1nkjl}}, \ldots, x_{m_{njklr_{nkjl}}}). \tag{4.207}$$

The first is to understand that this formula is identical to the statement of the theorem. Of course the n-point Green's function is given by the sum over all possible products of connected Green's functions with together n external points and with each argument $x_1, \ldots, x_n$ one and only one time appearing as these external points. This is the reason why we had to take the *disjoint* partitions of the external points. It is also clear that $k = 1, \ldots, n$, because there is one disjoint partition with one element, namely the set N_n of the indices itself, and one disjoint partition with n elements containing all sets with one element. Thus (4.207) is the statement of the theorem written in analytical form.

Finally we see that the i-factors in (4.207) cancel. Since the partitions have to be defined as disjoint we have indeed

$$\sum_{l=1}^{k} r_{njkl} = n. \tag{4.208}$$

The proof is now given easily by induction:

The statement is true for $n = 1$, because $G_J^{(1)}(x_1) = Z[J]G_{cJ}^{(1)}(x_1)$ follows directly by differentiating the definition of W (4.204).

Now we suppose the statement is true for n. To perform the induction, we have to do is to differentiate (4.207) after $J(x_{n+1})$. Using the product rule for functional derivatives we find:

$$\begin{aligned} G_J^{(n+1)} &= \frac{1}{\mathrm{i}}\frac{\delta G_J^{(n)}}{\delta J(x_{n+1})} = \\ &= \frac{1}{\mathrm{i}^{n+1}}\frac{\delta Z[J]}{\delta J(x_{n+1})}\sum_{k=1}^{n}\mathrm{i}^k\sum_{P_k^n}\sum_j\prod_{l=1}^{k}\mathrm{i}^{r_{nkjl}-1}G_{cJ}^{(r_{nkjl})}(x_{m_{nkjl1}}, \ldots, x_{m_{nkjlr_{nkjl}}}) + \\ &+ \frac{Z[J]}{\mathrm{i}^{n+1}}\sum_{k=1}^{n}\mathrm{i}^k\sum_{P_k^n}\sum_j\frac{\delta}{\delta J(x_{n+1})}\prod_{l=1}^{k}\mathrm{i}^{r_{nkjl}-1}G_{cJ}^{(r_{nkjl})}(x_{m_{nkjl1}}, \ldots, x_{m_{nkjlr_{nkjl}}}) \end{aligned} \tag{4.209}$$

The last step is to understand that with this expression we have taken account of all disjoint partitions of the set $\mathbb{N}_{n+1}$ of indices.

Since

$$\frac{\delta Z[J]}{\delta J(x_{n+1})} = \mathrm{i}Z[J]G_{cJ}^{(1)}(x_{n+1}) \tag{4.210}$$

the first term in (4.209) is the sum over all disjoint partitions of $\mathbb{N}_{n+1}$, which are partitions of $\mathbb{N}_n$ with the one-element set $\{x_{n+1}\}$ added to the partition. Also the factor i is correct for the one-point Green's function. Indeed by adding the one-element set to a partition P_k^n of the index set $\mathbb{N}_n$ makes a partition of the class P_{k+1}^{n+1} of partitions of the index set $\mathbb{N}_{n+1}$.

Further holds

$$\frac{\delta}{\delta J(x_{n+1})} G_{cJ}^{r_{nkjl}}(x_{m_{nkjl1}}, \ldots, x_{m_{nkjlr_{nkjl}}}) = \mathrm{i} G_{cJ}^{r_{nkjl}}(x_{m_{nkjl1}}, \ldots, x_{m_{nkjlr_{nkjl}}}, x_{n+1}). \tag{4.211}$$

In the 2nd part of (4.209) we can use the product rule again. Then we that this procedure corresponds to making a partition of class P_{n+1}^k of $\mathbb{N}_{n+1}$ out of a partition of class P_n^k of $\mathbb{N}_n$ by adding the element $(n+1)$ to one of the subsets of the partitions in P_n^k. The additional factor i has to be counted with $\mathrm{i}^{r_{njkl}}$.

Now we are finished with our proof because all disjoint partitions of $\mathbb{N}_{n+1}$ are obtained from those of $\mathbb{N}_n$ by adding the element $(n+1)$ as a one-element set to a partition of $\mathbb{N}_n$ or to add this element to one of the sets already existing in a partition of $\mathbb{N}_n$.

Q.E.D.

Now the fact, that the vacuum-to-vacuum subdiagrams cancel, is clear too: Since $Z = \exp(\mathrm{i}W)$ after performing all the functional derivatives to obtain the n-point Green's functions in terms of connected ones, setting $J = 0$ one has to divide by $Z[0]$ to get the proper normalisation of the vacuum-to-vacuum amplitude to 1. This cancels $Z[0] = \exp(\mathrm{i}W[0])$.

The same holds order by order in perturbation series, because then we apply our theorem to Z_0 and we have already shown that one obtains the Dyson-Wick expansion of the exact Green's functions just by calculating functional derivatives of the free generating functional $Z_0 = \exp(\mathrm{i}W[0])$.

4.6.4 Effective Action and Vertex Functions

The next subset of diagrams is given by the *exact n-point vertex functions*. These are defined as the *one-particle irreducible truncated* diagrams with n external space-time points. Here a diagram is called *one-particle irreducible* or 1PI if it is not possible to disconnect the diagram by cutting only a single exact propagator line. The diagram is called *truncated* if all propagators connecting an internal space-time point with an external one are left out. Thus the external points are connected with the diagram by at least two propagator lines.

Before we go into the analytical details we should explain why this is a useful subset of diagrams. The most practical reason is given by the momentum space representation of the Feynman rules. Due to momentum conservation a diagram which consists of two parts connected only through one propagator line splits in the product of the two parts times the propagator. This is a direct consequence of the translation invariance of the theory!

In addition we shall see in the next section that we have a description of the quantum field theory in form of a variational principle which is close to classical field theory in this sense. For instance we shall derive Noether's theorem for the quantum case with help of the *effective action* which is introduced in the following theorem. Its proof is also quite involved (but not really difficult) and may be omitted at the first reading.

Theorem 3 (The Effective Action)**.** *By performing a functional Legendre transformation of the*

generating functional W,

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x \varphi_J(x) J(x) \text{ with } \varphi_J(x) = \frac{\delta W[J]}{\delta J[x]}, \tag{4.212}$$

one obtains a generating functional for exact vertex functions.

Especially the functional inverse of the exact connected two-point Green's function (propagator) is given by

$$\Gamma_J^{(2)}(x_1, x_2) = -\frac{\delta^2 \Gamma[\varphi_J]}{\delta\varphi_J(x_1)\delta\varphi_J(x_2)}. \tag{4.213}$$

In general the ***proper vertex functions*** *are defined without special sign conventions by*

$$\Gamma_J^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n \Gamma[\varphi_J]}{\delta\varphi_J(x_1)\cdots\delta\varphi_J(x_n)}. \tag{4.214}$$

Proof. The vertex functions are represented diagrammatically as shown in fig. 4.7.

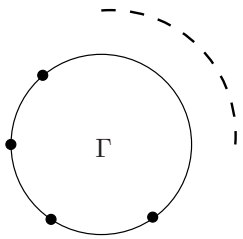


Figure 4.7: Diagrammatical representation for the vertex functions

The first step is to show, that $\Gamma^{(2)}$ is the negative functional inverse for the two-point connected Green's function. This follows immediately from (4.212),

$$\delta\Gamma = \frac{\delta W}{\delta J}\delta J - \varphi_J \delta J - J\delta\varphi_J = -J\delta\varphi_J. \tag{4.215}$$

Thus Γ depends only on φ_J and only implicitly on J. From the last equation it follows that

$$\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(x)} = -J(x). \tag{4.216}$$

Because of the definition (4.213) we have

$$\int \mathrm{d}^4y_2 \Gamma_J^{(2)}(x_1, y_2) G_{cJ}^{(2)}(y_2, x_2) = -\int \mathrm{d}^4y_2 \frac{\delta J(x_1)}{\delta\varphi_J(y_2)} \frac{\delta\varphi_J(y_2)}{\delta J(x_2)} = -\delta^{(4)}(x_1 - x_2), \tag{4.217}$$

which proves our statement that the two-point vertex function is the negative functional inverse of the two-point connected Green's function.

With help of this equation we find

$$G_{cJ}^{(2)}(x_1, x_2) = \int \mathrm{d}^4y_1 d^4y_2 \Gamma_J^{(2)}(y_1, y_2) \mathrm{i} G_{cJ}^{(2)}(x_1, y_1) \mathrm{i} G_{cJ}^{(2)}(x_2, y_2) \tag{4.218}$$

which is equivalent to the graphical statement that the two-point vertex function is identical with the truncated two-point connected Green's function (keep in mind, that a line with a open

circle means the full two-point propagator, $\mathrm{i}G^{(2)}_{cJ}$). Of course there is no possibility for the exact two-point connected Green's function to contain a self-energy insertion, because it is already fully dressed. The two-point vertex function thus has just to be dressed again with two exact propagators, as stated by (4.218).

The systematics for an inductive proof starts with the three-point functions. Diagrammatically it is clear, that also the three-point connected Green's function cannot be one-particle reducible. Thus the three-point function should be expressed just by dressing each external point of the three-point vertex with an exact propagator. This is shown to be true for the analytical statement by differentiation of (4.218) with respect to $J(x_3)$. Using (4.214) we find with help of the chain rule of functional differentiation

$$\frac{1}{\mathrm{i}}\frac{\delta\Gamma^{(n)}_J[y_1,\ldots,y_n]}{\delta J(x)} = \int \Gamma^{(n+1)}_J(y_1,\ldots,y_n,y_{n+1})\mathrm{i}G^{(2)}_{cJ}(y_{n+1},x). \tag{4.219}$$

With help of this relation we obtain by differentiation of (4.218) and (4.217) bringing all $G^{(3)}_{cJ}$ to the left hand side of the equation

$$G^{(3)}_{cJ}(x_1,x_2,x_3) = \int \mathrm{d}^4y_1 d^4y_2 d^4y_3 \Gamma^{(3)}_J(y_1,y_2,y_3)\prod_{k=1}^{3}\mathrm{i}G^{(2)}_{cJ}(x_k,y_k). \tag{4.220}$$

But this is the statement: To obtain the three-point connected Green's function one has to dress the three-point vertex function with exact propagators.

Now the rest is done by induction. Let us first think about the diagrammatical statement of the theorem. It says that we can calculate the n-point connected Green's function by summing all possible diagrams built with help of the vertex functions linked together with the exact connected two-point Green's functions (shortly called the exact propagator) with n external points, which have to be dressed with exact propagators.

In another way we can express this as follows: One has to build the exact Green's functions with help of the exact vertices and propagators as in the case of perturbation series diagrams but taking only diagrams without loops. Now a diagram without loop is called a *tree diagram* and in perturbation theory the sum over all tree diagrams contributing to a n-point Green's function (connected or disconnected) is called the *tree approximation* of this function. Thus the short form of the theorem is: *The exact n-point Green's function is given by the tree-level approximation using exact vertex functions and exact the Green's function as building blocks.*

As we shall see in the next section, the perturbation series can also be seen as an expansion in powers of $\hbar$. A diagram with L loops is of order $\hbar^{L-1}$. The analogue in ordinary quantum mechanics is known as *WKB approximation*[1].

So let us come back to our proof. Similarly to the proof of the theorem in the last section we have to classify now our diagrams as built by tree level diagrams of exact vertex functions and propagators.

Again we can classify the diagrams by the disjoint partitions of the set $\mathbb{N}_n$ in k non-empty subsets. These sets contain the external points which are linked to the 1PI subdiagrams of the tree diagram. The topology of the diagram is fixed by the number of elements contained in the k subsets of $\mathbb{N}_n$. A one-element set in a partition simply tells us to dress connect the corresponding external point with a point of a vertex function. Although this does not classify the topology of

[1] named after the inventors of the method, Wentzel, Kramers and Brillouin

the diagram this special case is important as it will become clear when doing the induction from n' to $n'+1$ below.

The topological type of a diagram is given by a tuple of natural numbers $(n_1, n_2, \ldots, n_k)$, where $n_j \geq 2$ is the number of external points connected with the jth ($j = 1, \ldots, k$) 1PI subdiagram of the tree diagram.

Let us take the four-point connected Green's function as an example. One has partitions of type (4) and of type $(2,2)$. The first one corresponds to the dressed four-point vertex, the second one to two three-point vertices connected with one internal propagator line and with dressed external points. This is shown in fig. 4.8.

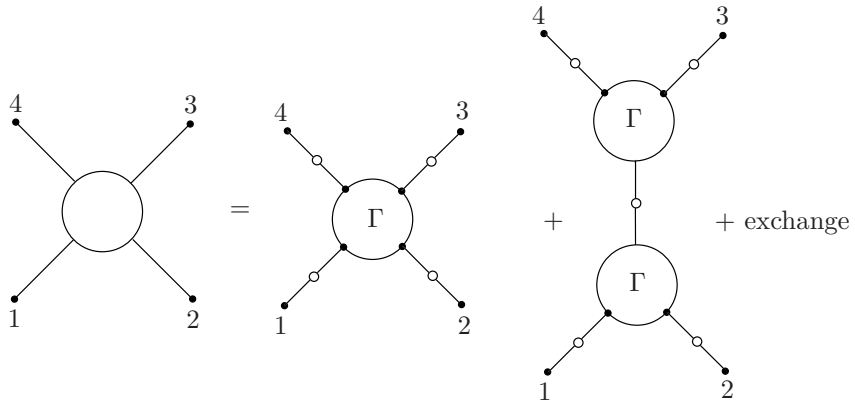


Figure 4.8: The connected four-point function as a tree diagram built out of exact vertex functions and propagators. By "exchange diagrams" we mean the diagrams with the three point vertices of the same topology but with interchanged external legs (there are three tree diagrams of this kind).

Now we have to show that our analytical statement coincides with these diagrammatical rules. We have seen that it holds for $n = 3$ and we can show the rest by induction. So we suppose, that the theorem is correct for all connected n-point functions with $n \leq n'$. We have to show that the theorem then also holds for $(n'+1)$-point functions.

But this can be done diagrammatically, if we find the diagrammatical manipulations corresponding to differentiating the sum of n'-point tree diagram with respect to the external current. The result are $n'+1$-point tree diagrams and, because the theorem is claimed to hold for $n \neq n'$, the result is the complete $(n'+1)$-point functional.

To differentiate an n'-point function one has to use the product rule under the integral and in the case of differentiating a vertex function the chain rule already given in (4.219). The diagrammatical meaning of this calculations is as follows:

- If the derivative $1/\mathrm{i}\delta/\delta J$ is applied to a vertex function, according to (4.219) this generates a vertex function with one more external point which is dressed by a propagator $\mathrm{i}G_{cJ}$. So we have again a diagram consistent with our rules. In the description by disjoint partitions of $\mathbb{N}_{n'+1}$ this corresponds to adding the new point $x_{n'+1}$ to one of the sets in $\mathbb{N}_{n'}$, which contain at least two points.

- The derivative is applied to a propagator which connects a vertex point with an external point. This propagator gives a three-point connected Green's function which is given as

the dressed three-point vertex function according to (4.220). Thus the resulting diagram has one internal propagator line more than the original one. This means making a disjoint partition of $\mathbb{N}_{n'+1}$ by adding the element $x_{n'+1}$ to a one-element set in the partition of $\mathbb{N}_{n'}$ describing the original diagram.

- The derivative is applied to a propagator, which connects two vertex functions within the diagram. The result is again a three-point vertex function with one point connected to the external point $x_{n'+1}$ and with its two remaining points connected to the rest of the diagram by propagator lines. This means adding the one-element set $\{x_{n'+1}\}$ to the partition of $\mathbb{N}_{n'}$ corresponding to the original diagram.

Now it is clear, that from claiming the theorem to hold for $n = n'$ follows, that it also holds for $n = n' + 1$ since one obtains all disjoint partitions of $\mathbb{N}_{n'+1}$ out of the partitions of $\mathbb{N}_{n'}$ by adding the $n' + 1$th point either to a set of the partition of $\mathbb{N}_{n'}$ or adding the one-element set $\{x_{n'+1}\}$ to a partition of $\mathbb{N}_{n'}$ and these possibilities correspond to product and chain rule of functional derivative of $G_{cJ}^{(n')}$ written as the sum of tree diagrams according to the theorem.

Particularly this shows, that for $n = n' + 1$ there is no propagator which connects two arguments of the same vertex function if this is true for $n = n'$, but this has been the assumption of our induction proof. Of course if there was such a line connecting two arguments of one and the same vertex, we could cut this internal line without disconnecting the diagram. This shows that the n-point vertex function is the complete truncated 1PI n-point Green's function. Q.E.D.

Finally we should say that we have to set J to zero after all derivatives are taken. According to (4.216) for the vertex functions, seen as functionals of φ_J, this means, that we have to insert the field φ for φ_J, which solves the equation of motion

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = 0. \tag{4.221}$$

Our proof also shows, why Γ as a functional of φ is called the "effective action": The full connected n-point Green's functions are given as the tree-level approximation of the field theory defined by the effective action as action functional. According to the $\hbar$-expansion in the next section there is some analogy of the tree-level approximation of the quantum field theory with the corresponding classical field theory.

It should also be mentioned that in momentum space it is convenient to define the self-energy Σ by

$$\Sigma(p) = G_0^{-1} - G^{-1} = p^2 - m^2 + \Gamma^{(2)}(p), \tag{4.222}$$

fulfilling *Dyson's equation*:

$$G = G_0 + G\Sigma G_0. \tag{4.223}$$

This shows that Σ is perturbatively given as the sum over all amputated 1PI diagrams with at least one loop[2].

[2]Since in the perturbative diagrams a line stands for $\mathrm{i}G_0$ any diagram with these properties contributes to $-\mathrm{i}\Sigma$.

4.6.5 Noether's Theorem (Quantum Part)

In section 3.3 we have seen the important connection between symmetries and conservation laws, known as *Noether's theorem*, for the classical field theory.

Now we can describe the quantum field theory also by a variational principle, but with the quantum effective action $\Gamma[\varphi]$ introduced in the last section, rather than with the classical action $S[\varphi]$.

We start from the invariance of the classical action and ask for the consequences for the effective action of the corresponding quantised theory. For this we *suppose, that the path integral measure of the fields* $\mathrm{D}\phi$ *is invariant under the symmetry transformation and that it can be written in the Lagrangian form.*

Now we make the substitution $\phi = \phi' - \delta\phi$ in the generating functional (4.152). Since $S[\phi] = S[\phi']$ and $\mathrm{D}\phi = \mathrm{D}\phi'$ by assumption we have

$$Z[J] = N \int \mathrm{D}\phi' \exp\{\mathrm{i}S[\phi'] + \mathrm{i}\left\langle J_1\phi_1'\right\rangle_1\} \exp\{\mathrm{i}\delta\left\langle J_1\phi_1\right\rangle\}. \tag{4.224}$$

Now J is an auxiliary external current, which we can give arbitrary behaviour under the symmetry transformation. We shall assume that it transforms as a set of scalar fields under Poincaré transformations and as a scalar under intrinsic transformations of the fields ϕ.

Now we subtract the original form (4.152) of the functional and expand the expression to linear order in the infinitesimal transformation:

$$\int \mathrm{D}\phi\delta\left\langle J_1\phi_1\right\rangle_1 \exp[\mathrm{i}S[\phi] + \mathrm{i}\left\langle J_2\phi_2\right\rangle] = 0. \tag{4.225}$$

Now we have defined J to be a set of scalar fields, leading to

$$\delta\left\langle J_1\phi_1\right\rangle_1 = \int \mathrm{d}^4x J(x)[\delta\phi(x) - \delta x^\mu\partial_\mu\phi], \tag{4.226}$$

as it was shown when proving Noether's theorem for classical fields in section 3.3.

To understand the meaning of (4.225) better, we write the most general infinitesimal transformation we want to take into account in the following form (also used in section 3.3):

$$\delta\phi(x) = \tau_a(x,\phi)\delta\eta^a,\ \delta x^\mu = -T_a^\mu(x)\delta\eta^a, \tag{4.227}$$

where the $\delta\eta^a$ are the infinitesimal independent parameters of the Lie group.

According to the general Feynman-Kac formula (4.151) and using the effective action formalism we obtain from (4.216)

$$\int \mathrm{d}^4x \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(x)}\delta\eta^a \left\langle 0\right| T_c[\tau_a(x,\boldsymbol{\phi}) - T_a^\mu\partial_\mu\boldsymbol{\phi}]\left|0\right\rangle = 0, \tag{4.228}$$

which is exactly of the same form, we deduced for the classical action to prove Noether's theorem in the classical case, *if* τ *is linear in* ϕ, because then we have $\langle\tau(\boldsymbol{\phi},x)\rangle = \tau\left(\langle\boldsymbol{\phi}\rangle, x\right)$ and $\langle\boldsymbol{\phi}\rangle = \varphi_J$, where $\langle\cdots\rangle$ is the vacuum expectation value for the operators under influence of the external current J.

Together we have shown, that, if the classical action functional is invariant under a infinitesimal transformation, which is linearly realized on the fields and leaves the path integral measure

invariant, then the effective action of the corresponding quantum field theory also is invariant under the same transformation. We can immediately conclude, that then Noether's theorem holds for the quantum case, i.e., there exists a *conserved Noether current* for each independent symmetry transformation, which operates linearly on the fields.

Warning! All this is purely formal, because we have not taken into account that one has to regularise and renormalise the field theory. All this formal manipulations with path integrals have only sense if we are able to regularise the theory in some way. This regularisation has to be done such that the symmetries hold for the regularised theory too. One nice regularisation technique is the *dimensional regularisation* invented by Veltman and van 't Hooft especially for gauge theories (explained in chapter 6 of this notes). Now it may happen that there is a transformation which leaves the classical action invariant in the physical $1 + 3$-dimensional space-time, but that there is no symmetry in other space-time dimensions. Then the regularised theory has no symmetry and thus it happens that the effective action fails also to be symmetric under the symmetry transformation, although the formal assumptions may be fulfilled.

It can also be that the path integral measure is not invariant under the symmetry transformation, which again breaks the symmetry for the effective action. All these cases, when a symmetry of the classical action is not a symmetry of the effective action, are called an *anomaly*. Thus we should keep in mind that we have to check carefully if the theory is free of anomalies before we use any symmetry argument in the quantum case.

In the next section we show in which sense Noether's theorem holds for the Dyson-Wick expansion of the quantum field theory, if all the conditions for the transformation to be a symmetry of the exact effective action is fulfilled.

4.6.6 $\hbar$-Expansion

In this section we reintroduce $\hbar$ explicitly in the path integral formulation of quantum field theory. We expect that it should be possible to expand around the *classical solutions for the fields* which we denote by φ_0. In order to discuss the generating functionals for disconnected and connected Green's functions and the effective action which generates proper vertex functions we include an external source J:

$$\left.\frac{\delta S}{\delta \phi_1}\right|_{\phi=\varphi_0} + J_1 = 0. \tag{4.229}$$

The generating functional for disconnected Green's functions is defined according to 4.187. As explained above we reintroduce $\hbar$ explicitely:

$$Z[J] = N \int \mathrm{D}\phi \exp\left[\frac{\mathrm{i}}{\hbar}\left(S[\phi] + \{J_1\phi_1\}_1\right)\right] := \exp\left(\frac{\mathrm{i}}{\hbar} W[J]\right). \tag{4.230}$$

Now we substitute $\sqrt{\hbar}\phi' = \phi - \varphi_0$ where φ_0 is the classical field subject to the equation of motion (4.229). The classical field is of course of order $O(\hbar^0)$ since it is completely classical while the quantum part should be scaled with $\sqrt{\hbar}$ according to its dimension. With this substitutions we find

$$Z[J] = \exp\left[\frac{\mathrm{i}}{\hbar}\left(S[\varphi_0] + \{J_1\varphi_{01}\}_1\right)\right] Z_1(J) \tag{4.231}$$

with

$$Z_1[J] = N \int \mathrm{D}\phi' \exp\left[\left(\frac{\mathrm{i}}{2}\left\{\mathscr{D}_{12}^{-1}\phi_1'\phi_2'\right\}_{12} - \frac{\mathrm{i}\lambda\sqrt{\hbar}}{3!}\left\{\varphi_{01}\phi_1'^{\,3}\right\}_1 - \frac{\mathrm{i}\lambda}{4!}\hbar\left\{\phi_1'^{\,4}\right\}_1\right)\right], \qquad \mathscr{D}_{12}^{-1} = \left(-\Box_1 - m^2 - \frac{\lambda}{2}\varphi_{01}^2\right)\delta^{(4)}(x_1 - x_2). \tag{4.232}$$

This shows that we can evaluate an $\hbar$-expansion for Z_1 if we apply the perturbative expansion given for Z itself by (4.188). There are only two differences: The first is that the propagator is given by $\mathscr{D}_{12}$ which depends on the classical field φ_0 and thus implicitly on the external source J. The second is that we are not allowed to cancel the additional factor for $Z_{10}[J,K]$ (where K denotes a new independent external source), because it depends on φ_0 and thus also on J.

Thus we obtain

$$Z_1[J] = N\left\{\exp\left[-\mathrm{i}\sqrt{\hbar}\tilde{V}^{(3)}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta K_1}\right] - \mathrm{i}\hbar\tilde{V}^{(4)}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta K_1}\right]\right] Z_1^{(0)}[J,K]\Big|_{K=0}\right\}_1. \tag{4.233}$$

Here we have abbreviated the interaction parts of the Lagrangian in (4.232) by

$$\tilde{V}^{(3)}(\varphi) = \frac{\lambda}{6}\varphi_0\phi'^3,\ \tilde{V}^{(4)}(\varphi) = \frac{\lambda}{24}\phi'^4, \tag{4.234}$$

while $Z_1^{(0)}(J,K)$ is given by the quadratic part of the Lagrangian in (4.233). We apply to it the same trick as for obtaining (4.186) but here we have to keep the overall factor, because it depends in J via its dependence on φ_0:

$$\begin{aligned} Z_1^{(0)}[J,K] &= \int \mathrm{D}\phi' \exp\left[\frac{\mathrm{i}}{2}\left\{\mathscr{D}_{12}^{-1}\phi_1'\phi_2'\right\}_{12} + \mathrm{i}\left\{K_1\phi_1'\right\}_1\right] \\ &= \exp\left[-\frac{\mathrm{i}}{2}\left\{\mathscr{D}_{12}K_1K_2\right\}_{12}\right]\int \mathrm{D}\phi' \exp\left[\frac{\mathrm{i}}{2}\left\{\mathscr{D}_{12}^{-1}\phi_1'\phi_2'\right\}_{12}\right]. \end{aligned} \tag{4.235}$$

To calculate the path integral factor we think of the path integral in terms of its discretised form. Then after a Wick rotation of the integrand we can use (1.77) to obtain immediately

$$\begin{aligned} \int \mathrm{D}\phi' \exp\left[\frac{\mathrm{i}}{2}\left\{\mathscr{D}_{12}^{-1}\phi_1'\phi_2'\right\}_{12}\right] &= \frac{N'}{\sqrt{\mathrm{Det}(\mathscr{D}_{12}^{-1})}} = \\ &= N'' \exp\left[-\frac{1}{2}\,\mathrm{Tr}\ln(\mathscr{D}^{-1}D)\right], \end{aligned} \tag{4.236}$$

where we have introduced other indefinite factors N' and N'' which are independent of both J and K. In the second term we have used the free propagator, defined by

$$D^{-1} = (-\Box_1 - m^2)\delta^{(4)}(x_1 - x_2), \tag{4.237}$$

to normalise the whole expression and to avoid dimensionful quantities within logarithms. As we shall see below in the Tr ln-form we can calculate the functional determinant with help of a perturbation series at any given order of $\hbar$.

Now it is our goal to calculate the effective action, defined by (4.212), up to order $\hbar^2$. According to (4.230) for that purpose we have to expand the functional Z_1 (4.232) up to order $\hbar$. Thus we expand the functional differential operator (4.233) up to order $\hbar$ leading to

$$\begin{aligned} Z_1^{(2)}[J] = N\Bigg[1 + \left\{\left(\frac{\lambda}{3!}\sqrt{\hbar}\frac{\delta^3}{\delta K_1^3} - \mathrm{i}\hbar\frac{\delta^4}{\delta K_1^4}\right) Z_1^{(0)}[J,K]\right\}_1 + \\ + \frac{\hbar}{2}\frac{\lambda^2}{(3!)^2}\left\{\varphi_{01}\varphi_{02}\frac{\delta^3}{\delta K_1^3}\frac{\delta^3}{\delta K_2^3} Z_1^{(0)}[J,K]\right\}_{12}\Bigg]\Bigg|_{K=0}. \end{aligned} \tag{4.238}$$

Straight-forward calculation leads after some algebra[3] to the result

$$\begin{aligned} W[J] = -\mathrm{i}\hbar \ln Z[J] = S[\varphi_0] + \{J_1\varphi_{01}\}_1 + \frac{\mathrm{i}}{2}\hbar \ln\left(\mathscr{D}^{-1}D\right) + \mathrm{i}\hbar^2\frac{\lambda}{8}\left\{\mathscr{D}_{11}^2\right\}_1 + \\ + \hbar^2\lambda^2\left\{\frac{1}{8}\varphi_{01}\mathscr{D}_{11}\mathscr{D}_{12}\mathscr{D}_{22}\varphi_{02} + \frac{1}{12}\varphi_{01}\mathscr{D}_{12}^3\varphi_{02}\right\}_{12}. \end{aligned} \tag{4.239}$$

The next task is to invert the definition

$$\varphi_1 = \frac{\delta W[J]}{\delta J_1} \tag{4.240}$$

and substitute φ instead of J in order to perform the Legendre transformation according to (4.212). This is of course impossible to obtain exactly. But it is not too difficult to provide the systematic $\hbar$-expansion. Here it is important to note that by definition (4.229) φ_0 is the stationary point of the functional $S[\varphi] + \{J_1\varphi_1\}_1$. Thus we have

$$S[\varphi] + \{J_1\varphi_1\}_1 = S[\varphi_0] + \{J_1\varphi_{01}\}_1 + \frac{1}{2}\left\{\mathscr{D}_{12}\varphi_1'\varphi_2'\right\}_{12} + O(\hbar^3) \text{ with } \varphi' = \varphi - \varphi_0. \tag{4.241}$$

Here we have used the fact that $\varphi' = O(\hbar)$ which we have to prove now. To that end we derive the expansion (4.239) with respect to J. Since (4.239) depends only implicitly on J we do first the derivative with respect to φ_0. Using (4.229) we find

$$\frac{\delta W}{\delta\varphi_{01}} = -\frac{\mathrm{i}\lambda}{2}\hbar\varphi_{01}\mathscr{D}_{11} + O(\hbar^2). \tag{4.242}$$

Again using (4.239) we get

$$\frac{\delta\varphi_{01'}}{\delta J_1} = \left(\frac{\delta J_1}{\delta\varphi_{01'}}\right)^{-1} = -\mathscr{D}_{11'}. \tag{4.243}$$

Combining (4.242) and (4.243) leads to

$$\varphi_1 = \frac{\delta W}{\delta J_1} = \varphi_0 + \frac{\mathrm{i}\lambda}{2}\hbar\left\{\varphi_{01'}\mathscr{D}_{1'1'}\mathscr{D}_{1'1}\right\}_{1'} + O(\hbar^2). \tag{4.244}$$

This proves that $\varphi' = \varphi - \varphi_0 = O(\hbar)$. Now we have also

$$\mathscr{D}_{12}^{-1} = D_{12}^{-1} - \frac{\lambda}{2}\varphi_{01}\varphi_{02}\delta_{12} = D_{12}^{-1} - \frac{\lambda}{2}\varphi_1\varphi_2\delta_{12} + O(\hbar) := \mathscr{D}_{12}^{-1}(\varphi) + O(\hbar). \tag{4.245}$$

[3] A simple computer algebra system like `FORM` is recommended for such cases!

This means that up to corrections of order $O(\hbar)$ we can substitute φ instead of φ_0 in $\mathscr{D}_{12}$. Now we are ready to expand (4.239) systematically up to order $O(\hbar^2)$.

Making use of (4.239), (4.244) and (4.245) we find

$$S[\varphi_0] + \{J_1\varphi_{01}\}_1 = S[\varphi] + \{J_1\varphi_1\}_1 + \frac{\lambda^2}{8}\hbar^2 \{\varphi_1\mathscr{D}_{11}(\varphi)\mathscr{D}_{12}(\varphi)\mathscr{D}_{22}(\varphi)\varphi_2\}_{12} + O(\hbar^3). \quad (4.246)$$

For the logarithm we have

$$\begin{aligned}\frac{\mathrm{i}}{2}\hbar \operatorname{Tr}\ln(\mathscr{D}_{12}^{-1}D) &= \frac{\mathrm{i}}{2}\hbar \operatorname{Tr}\ln\left(\delta_{13} - \frac{\lambda}{2}\varphi_1^2 D_{13}\right) = \\ &= \frac{\mathrm{i}}{2}\hbar \operatorname{Tr}\ln[\mathscr{D}^{-1}(\varphi)D] - \frac{\lambda^2}{4}\hbar^2 \{\varphi_1\mathscr{D}_{11}(\varphi)\mathscr{D}_{12}(\varphi)\mathscr{D}_{22}(\varphi)\varphi_2\}_{12} + O(\hbar^3).\end{aligned} \quad (4.247)$$

In the remaining terms of (4.239) which are explicitely of order $O(\hbar^2)$ we can simply set φ instead of φ_0 which gives only differences of order $O(\hbar^3)$.

The final result for the action up to order $\hbar^2$ is thus

$$\Gamma[\varphi] = S[\varphi] + \frac{\mathrm{i}}{2}\hbar \operatorname{Tr}\ln[\mathscr{D}^{-1}(\varphi)D] + \frac{\lambda}{8}\hbar^2 \left\{\mathscr{D}_{11}^2(\varphi)\right\}_1 + \frac{\lambda^2}{12}\hbar^2 \left\{\varphi_1\mathscr{D}_{12}^3(\varphi)\varphi_2\right\}_{12} + O(\hbar^3). \quad (4.248)$$

For sake of completeness we derive the rule how to get derivatives of $\mathscr{D}(\varphi)$ with respect to φ. These are needed if we like to calculate the proper vertex functions using (4.214). We have

$$\begin{aligned}\delta_{12} = \left\{\mathscr{D}_{11'}^{-1}\mathscr{D}_{1'2}\right\}_{1'} \Rightarrow \\ \left\{\frac{\delta\mathscr{D}_{11'}^{-1}}{\delta\varphi_3}\mathscr{D}_{1'2}\right\}_{1'} = -\left\{\mathscr{D}_{11'}^{-1}\frac{\delta\mathscr{D}_{1'2}}{\delta\varphi_3}\right\}_{1'}.\end{aligned} \quad (4.249)$$

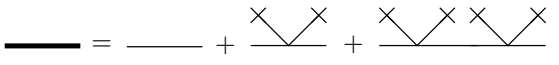
Using the explicit form of $\mathscr{D}^{-1}$ given by (4.245) we find

$$\frac{\delta\mathscr{D}_{12}(\varphi)}{\delta\varphi_3} = \lambda\mathscr{D}_{13}(\varphi)\varphi_3\mathscr{D}_{32}(\varphi). \quad (4.250)$$

Now we shall give a diagrammatical interpretation in terms of the propagator $\mathscr{D}$. To obtain an interpretation in terms of the perturbative propagator $D = \mathscr{D}(\varphi = 0)$ we have to use (4.245) which is inverted with help of a resummation which reads in obvious functional matrix notation:

$$\mathscr{D}(\varphi) = D + D\frac{\lambda}{2}\varphi^2 D + \ldots = D\sum_{k=0}^{\infty}\left(\frac{\lambda}{2}\varphi^2 D\right)^k. \quad (4.251)$$

Diagrammatically this is given by



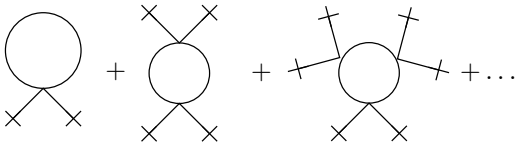
(4.252)

where the bold line stands for $\mathscr{D}$ and the thin one for the perturbative propagator D. The single leg with a cross at the end symbolises the field φ.

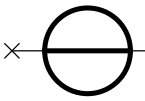
The logarithmic contribution to order $\hbar$ is given by

$$\frac{\mathrm{i}}{2}\hbar\ln(\mathscr{D}^{-1}D) = -\frac{\mathrm{i}}{2}\hbar\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\lambda}{2}\varphi^2\mathscr{D}\right)^k. \quad (4.253)$$

Diagrammatically these are the following "ring diagrams":

$$-\tfrac{\hbar}{2}\operatorname{Tr}\ln(\mathscr{D}^{-1}D) =$$  (4.254)

For the two-loop contribution it is more convenient to give the diagrams in terms of $\mathscr{D}$ rather than of D:

$$\tfrac{\mathrm{i}\lambda^2\hbar^2}{12} =$$  . (4.255)

Here it becomes clear that the number of loops in the diagrams for the proper vertices, which are obtained by taking the appropriate derivative due to (4.214) and setting $\varphi = 0$ at the end give the order or $\hbar$. This can be understood directly from the diagrams as follows: The number of loops is given by the number of independent loop momenta. If I is the number of internal propagator lines and V the number of vertices we have $L = I - (V - 1)$ loops, since on each vertex energy-momentum conservation has to hold but one of this restrictions is fulfilled due to the overall energy-momentum conservation for the external legs. Each vertex multiplies the diagram with a $1/\hbar$ due to the factor $1/\hbar$ in the original path integral (4.230) and each propagator delivers a factor $\hbar$. Then there is the overall factor $\hbar$ from the definition of W as $W = -\mathrm{i}\hbar\ln Z$. So a diagram has $\hbar$-order $I - V + 1$ which was seen to be the number of loops of the diagram.

The calculation of the second order terms also shows how the one particle reducible contributions to Γ cancel order by order in the $\hbar$-expansion as we have already proven in a semi-combinatorial way in section 4.6.3.

4.7 A Simple Interacting Field Theory with Fermions

We end this chapter with deriving the Feynman rules for a very simple quantum field theory including fermionic fields. It will be quite the same line of arguments as given in the scalar case, but we have to be careful with the signs coming from the anti-commuting character of fermionic fields (Grassmann fields in the path integral formalism).

This theory contains massive Dirac fields ψ with mass m and a massive scalar field with mass M. We couple these fields using a pure vector coupling. As we shall see in chapter 6 in order to keep the theory renormalisable we also need a ϕ^4-coupling: Thus the Lagrangian is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{M^2}{2}\phi^2 + \bar\psi(\mathrm{i}\not\partial - m)\psi - \frac{\lambda}{4!}\phi^4 - g\phi\bar\psi\psi. \tag{4.256}$$

This Lagrangian is relativistically invariant because we coupled the scalar field to the scalar bilinear form of the Dirac spinor $\bar\psi\psi$. Since the Lagrangian again contains no derivative couplings we can go immediately over to the Lagrange-an formalism of the path integral. So we define the generating functional

$$Z[J,\bar\eta,\eta] = N\int \mathrm{D}\phi\mathrm{D}\bar\psi\mathrm{D}\psi\exp\left[\mathrm{i}S[\phi,\bar\psi,\psi] + \mathrm{i}\left\langle J\phi\right\rangle + \mathrm{i}\left\langle\bar\eta\psi\right\rangle + \mathrm{i}\left\langle\bar\psi\eta\right\rangle\right], \tag{4.257}$$

where J is a real and, $\bar\eta$ and η are Grassmann external currents.

Now the Dyson-Wick series can be derived along the same lines as done with the scalar ϕ^4-theory above. The first step is to calculate the free generating functional. Fortunately this functional factorises in a scalar and a spinor part. So we can use the scalar free functional from above and have only to calculate the functional for the fermions. So we have to solve for

$$Z_{0F}[\bar{\eta},\eta] = N \int \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[\mathrm{i}S_{0F}[\bar{\psi},\psi] + \mathrm{i}\left\langle\bar{\eta}\psi\right\rangle + \mathrm{i}\left\langle\bar{\psi}\eta\right\rangle\right]. \tag{4.258}$$

The free Dirac spinor action is

$$S_{0F}[\bar{\psi},\psi] = \int \mathrm{d}^4x\bar{\psi}(\mathrm{i}\not{\partial} - m)\psi. \tag{4.259}$$

Now again we have an action quadratic in the fields under the path integral. Due to our calculations above for Grassmann fields we can use the same trick as before: The value of the generating functional is a current-independent constant times the exponential of the action functional at its stationary point. Fortunately it is enough to solve the equation of motion for ψ:

$$(\mathrm{i}\not{\partial} - m)\psi = -\eta. \tag{4.260}$$

This is the inhomogeneous free Dirac equation, which can be solved with help of the causal Green's function with appropriate boundary conditions. We shall do the calculation a little sloppy. The reader is invited to do the exact calculation by first going to Euclidean space-time and then back to Minkowski space by analytic continuation. Thus we set

$$\psi(x) = -\int \mathrm{d}^4x' G_0(x-x')\eta(x'), \tag{4.261}$$

where we have named the free Green's function for Dirac fields G_0 in order to distinguish from the scalar Green's function D. Again the Green's function can be found by solving its equation of motion

$$(\mathrm{i}\not{\partial}_x - m)G_0(x-x') = \delta(x-x') \tag{4.262}$$

with help of the Fourier transformation

$$G_0(x-x') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\exp[-\mathrm{i}p(x-x')]G_0(p) \Rightarrow (\not{p} - m)G_0(p) = \hat{1}. \tag{4.263}$$

Multiplying this algebraic equation from the left side with $/p + m$ and using

$$\not{p}^2 = p_\mu p_\nu \gamma^\mu\gamma^\nu = \frac{1}{2}p_\mu p_\nu \underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2g^{\mu\nu}} = p^2 \tag{4.264}$$

we find

$$G(p) = \frac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon} = (\not{p} + m)D(p). \tag{4.265}$$

Here we have used the $\mathrm{i}\epsilon$-description known from scalar theory in order to define the propagator properly. As said above this result may be obtained by a formal Wick rotation and analytical continuation back to Minkowski space. Physically it is also clear that the causality of this solution is correct in the sense of the Feynman-Stueckelberg interpretation of the "negative energy states".

The insertion of (4.261) gives

$$Z_{0F}(\bar{\eta},\eta) = \exp[-\mathrm{i}\left\langle\bar{\eta}_1 G_{012}\eta_2\right\rangle_{12}]. \tag{4.266}$$

Now to write the action functional for the interacting theory as a series in powers of interactions we use the same trick as above in the scalar case. The only thing we have to be careful with is the anti-commuting nature of the Grassmann fields and functional derivatives with respect to the Grassmann external currents (signs!). With this caveats in mind the Feynman rules can be derived the same way as in the scalar case. The only point is, that the fermionic propagator G_0 has to be represented with an arrow which gives the order of the arguments. Our convention will be such that Feynman diagrams have to be translated to analytic expressions by reading against the direction of the arrows. This will give also the correct order for the matrix valued propagators (and vertices if there is, for instance, an axial scalar coupling involving γ_5).

In the following we shall only use *left functional derivatives* and writing $\delta/\delta\eta$ as abbreviation for $\delta^L/\delta\eta$. With this notation we have, using (4.173):

$$\mathrm{i}G_F^{(m,n)}(x_1,\ldots,x_m;y_1,\ldots,y_n)=\left(\frac{\delta^{m+n}Z[J,\bar{\eta},\eta]}{\delta(\mathrm{i}\bar{\eta}(x_1))\cdots\delta(\mathrm{i}\bar{\eta}(x_m))\delta(-\mathrm{i}\eta(y_1))\cdots\delta(-\mathrm{i}\eta(y_n))}\right)_{J,\eta,\bar{\eta}=0}. \tag{4.267}$$

It is simple to see from the Dyson-Wick series that to all orders perturbation theory this is only non-vanishing for $m=n$. Our diagrammatical convention is thus as given in fig. 4.9.

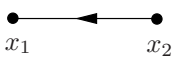
$\mathrm{i}G_0(x_1,x_2) =$ 

Figure 4.9: The diagrammatical representation of the Dirac-fermion propagator

It is also clear that the Feynman rules can be calculated the same way as we did in case of pure scalar ϕ^4-theory. The only change is that the external fermionic lines have an arrow, which gives *the direction of the current $\bar{\psi}\gamma^\mu\psi$ rather than the direction of momentum.* Thus an incoming (outgoing) external fermionic line can mean an incoming (outgoing) particle or an outgoing (incoming) antiparticle. The vertex coming from the $\phi\bar{\psi}\psi$-interaction always contains an incoming and an outgoing fermionic line and a bosonic line (further on drawn as wavy line) which does not necessarily need an arrow. Interchanging two external fermion lines changes the sign of the whole diagram. The overall sign is not important because it adds only an unphysical phase factor to the S-matrix.

There is also another sign rule saying that for each loop which consists of a closed ring of fermionic lines, also gets an additional sign, the so-called *sign rule for fermionic loops.* To prove this we look on the general structure of such a loop drawn in fig. 4.10.

This diagram comes from a term in the Dyson-Wick series of the form

$$Z_{\mathrm{loop}}=(-\mathrm{i}g)^n\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_n\prod_{k=1}^{n}\frac{\delta}{\mathrm{i}\delta J(x_k)}\frac{\delta}{\delta(\mathrm{i}\bar{\eta}(x_k))}\frac{\delta}{\mathrm{i}\delta\eta(x_k)}. \tag{4.268}$$

In order to get the n free Green's functions as drawn in the diagram, one has to interchange the order of the derivatives with respect to $\bar{\eta}_n$, which has to be written to the very left of the whole expression. Since this is done by interchanging this derivative operator an odd number of times with another Grassmann derivative operator this gives the sign change due to the *fermionic loop theorem.*

It is clear further that there is no other formal change in the Feynman rules for S-matrix elements. It is also the same Fourier transformation technique as used in chapter 3 to get the momentum space representation for the Feynman rules.

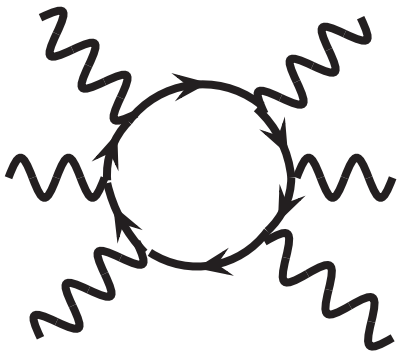


Figure 4.10: A fermionic loop with additional sign

1. Draw all topological different diagrams for the process due to initial and final state in S_{fi}. A diagram with n vertices is to multiply with $1/n!$ (from the expansion of the exponential of the interaction part).[4] Leave all diagrams out which contain vacuum-to-vacuum parts (closed diagrams).

2. For any ϕ^4-vertex there is a factor $-\mathrm{i}\lambda/4!$, for any $\bar{\psi}\psi\phi$ vertex a factor $-\mathrm{i}g$. Each of the latter vertices is attached to one incoming and one outgoing fermionic current.

3. There is to obey energy-momentum conservation on each vertex and for the overall diagram (with an overall factor $(2\pi)\delta^4(P_f - P_i)$ cancelled).

4. For any fermionic line write down a factor $\mathrm{i}G_0$, for any bosonic line $\mathrm{i}D$.

5. Integrate over all independent momenta (loops of the diagram).

6. Any diagram is to be multiplied by the number of ways one can draw the given diagram out of the vertices and external points, which is called *symmetry factor of the diagram.*

7. Multiply the diagram with a factor (-1) for each fermionic loop contained. The overall sign of the diagrams to a given order is arbitrary but there is a relative sign for each interchange of external fermionic legs compared to an arbitrary diagram with a certain reference of order of the external points.

8. For any external fermionic line multiply with the proper normalised free fermion amplitudes. For a fermionic line with outgoing arrow this can be either a $\bar{u}_+(p,\sigma)$ (outgoing particle in the final state) or a $\bar{u}_-(-p,\sigma)$ (incoming antiparticle in the initial state). For a fermionic line with incoming arrow this can be either a $u_+(p,\sigma)$ (incoming particle in the initial state) or a $u_-(-p,\sigma)$ (outgoing antiparticle in the final state).

9. All the external momenta are to be taken on the mass shell.

10. If one measures the unpolarised cross section one has to average over the spins in the initial state and to sum over the spins in the final state.

[4] This factorial is not cancelled completely since there are two different vertices. Only the interchange of the different vertices of the same sort gives a factor. Thus it is more save to count all contractions (or possibilities to connect external points and vertex points to give the partial diagram). It is also important to keep in mind that the external points have fixed labels!

The changes in rule 8. for calculating connected or disconnected Green's functions is to substitute a Green's function for the external amplitudes and just to cancel this factors completely when calculating truncated diagrams. If one likes to calculate vertex functions perturbatively one has to keep 1PI diagrams only.

Chapter 5

Renormalisation

5.1 Infinities and how to cure them

We have already seen in chapter 3 that Feynman diagrams contributing to the perturbation series for the Green's functions are divergent integrals over momentum space if we go beyond the lowest order tree-level diagrams.

The physicists have found such divergences in quantum field theory quite from the very beginning of quantum field theory in the late twenties and early thirties when Heisenberg and Pauli invented Quantum Electrodynamics (QED). The problem of infinities was finally solved in 1948 for the case of QED by Feynman, Schwinger and a little bit earlier during the war by Tomonaga in Japan.

Nevertheless this solution was only formal and the physicists stayed to feel uneasy with this solution. It looked like as "sweeping the garbage under the rug" (Feynman). It took until the early seventies when *Wilson* discovered its full physical meaning by looking at quantum field theory from the point of view of a condensed-matter physicist. It is this Wilsonian point of view we are after in this chapter.

Because the mathematics of renormalisation is rather involved we shall take a little time in this introducing section to get a qualitative idea, what has to be done and what is the physical meaning of this complicated procedure.

For this purpose let us look back on what has been done in the chapters before. We started with ordinary non-relativistic quantum mechanics for a point particle. Using this simple example for a quantum system we repeated how the general structure of quantum theory looks like: As an ingredient from classical physics quantum theory inherits the structure of space and time. Almost all of the physical content of space and time can be summarised in the symmetries the spacetime manifold respects and the principle of causality, i.e., the direction of time. Further the quantum system is defined by an algebra of observables which is realized as a representation by self-adjoint operators on a Hilbert space. These observables have to be generators of the symmetry group of spacetime, which must be a symmetry of the quantum formalism, because otherwise the theory would not be consistent with the structure of space and time.

We have also seen that any symmetry has to be realized as a unitary (or anti-unitary) transformation in the Hilbert space, i.e., any symmetry group is realized as a unitary or antiunitary ray representation on Hilbert space.

Further to complete the definition of a quantum system, we have to define the Hamiltonian (contained in the algebra of observables). The Hamiltonian describes the causal evolution of the system in time. Since the physical laws have to be the same at any instant of time this time evolution has to be also a symmetry transformation, which means that it is necessarily also described by a unitary transformation.

The most important special application of quantum mechanics for elementary particle physics is *scattering theory*. Our picture of such processes is motivated by the typical experimental setup of an elementary-particle collision: At first one prepares the particles in a given asymptotically free state (in most experiments as particles with a well determined momentum). These particles collide, and the experimenter observes the asymptotically free state of the scattered and the particles which might have been produced during the collision. The transformation from the asymptotically free initial to the asymptotically free final state is given by time evolution and is thus a unitary transformation in Hilbert space, which we have called the S-operator (or S-matrix).

This S-matrix contains the main physical outcome of the theory, namely the transition probabilities for the scattering events. In order to be consistent with the structure of space and time it has to respect this symmetry. Note that we neglect by definition the question "What happens at the collision time?"

Coming now back to our problem of the divergences we have to take care of this fundamental structure in order to keep the theory physically consistent. This means that we can summarise the main task of renormalisation theory that we have to get rid of the infinities, arising often when we try to calculate loop diagrams in perturbation theory, in such a way that the fundamental features of the S-matrix (namely to be unitary and to respect the symmetries of space and time and other symmetries which we shall use in the next chapter when we come to the description of the standard model of elementary particles) are settled and thus the theory keeps its fundamental physical meaning with which we started when we built this theory as a model for real processes.

Since this section is introductory for the entire topic and because the physical meaning of renormalisation theory is hidden under a carpet of tremendously complicated mathematical formalisms it might be allowed to look from the solution of the problem at the physical results. This might be done from the point of view of the quantum effective action we have introduced as a generating functional for truncated 1PI diagrams, which represent the exact proper vertex functions of the theory. Nevertheless the first way to approximate this effective action is perturbation theory, and we have also seen using formal manipulations of path integrals that the lowest order, corresponding to tree-level diagrams (which means diagrams without loops), corresponds to the classical approximation while the number of loops counts the $\hbar$-power of the diagram. This means that the loop diagrams contribute to the interaction vertices already existent in the classical limit as well as to interactions which are clearly induced as quantum corrections and which are not contained in the classical limit.

Now this picture of a $\hbar$-expansion or an expansion around the classical limit gives us a first intuitive picture about the physical meaning of the radiative corrections in quantum field theory. We want to take QED as an example because we know very well the classical limit, which is nothing else than Maxwell's theory of electromagnetism given in a modern relativistic prescription. The particle content is given by the classical Lagrangian containing a massless spin-1-field, describing photons. This field is necessarily a gauge field as we have seen in chapter 4. This in turn means that this field is necessarily coupled to a conserved current and the most easy way to do so is minimal coupling (the geometric meaning of this formalism will become much clearer in the next

chapter about more general gauge theories). Let us concentrate on QED in the narrow sense where we have an electron-positron field (a massive Dirac-spinor field) minimally coupled to the photon-field.

The point we are interested in here is now the meaning of the constants in this classical Lagrangian from the point of view of quantum theory of fields. There is a mass of the electron and a coupling constant which is given by the elementary charge e of the positron (particle/anti-particle symmetry dictates then that the electron has the same charge with the opposite sign, namely $-e$). The only classical coupling is given by the 3-point vertex describing the Coulomb interaction of positrons and/or electrons by exchange of a virtual photon. Especially note that there are no interactions of photons in the classical limit because these are uncharged particles which do not interact. Now there are radiative corrections to the 3-point vertex and the propagators of the particles. This means there is not only the classical Coulomb interaction between electrons and/or positrons but also interactions with virtual photons and electron-positron pairs created spontaneously due to quantum fluctuations and also absorbed due to these interactions. This means that the real (not perturbative) photons, electrons and positrons are surrounded with a cloud of virtual particles.

The main effect of this cloud to the n-point-functions is a contribution to the mass and charge of the particles. This means that there are fluctuations of the energy contributing to the measured mass of the electrons (and of course also to that of the photons!). The same is true for the correction to the three-point vertex, which means that the main effect of the virtual particles is a contribution to the measured charge of the electrons and positrons. But it hints us to what may be the cure for the infinities arising by the naive use of the perturbative Feynman rules: The couplings and masses written down in the Lagrangian cannot be the physically measured parameters since these correspond to the tree-level diagrams which are shown to leave out the quantum fluctuations or neglect in our intuitive picture the clouds of virtual particles around the physical measured entities. This in turn means that these parameters have no physical meaning because the observed physical particles contain these quantum fluctuations. For this reason we call the parameters given in the Lagrangian *the bare parameters*. These parameters can have any value because they are not observable. Thus we can hope that we shall be able to push the infinities of the naive Feynman rules to these unobservable parameters and writing down the results in terms of the *physical* or *dressed parameters* which have the measured finite values listed in the particle data booklet.

But now as the quantum action functional shows that there will be contributions of the perturbation theory which were not given in the original classical Lagrangian. For instance in QED there is the famous box diagram which gives a contribution to the four-photon vertex which means physically the scattering of "light by light" (also called *Delbrück scattering*), an effect which is not known from classical electromagnetic theory. Now if this diagram was infinite we would need such a coupling in the classical Lagrangian to shuffle the infinity of the box diagram into the associated coupling bare constant, which shows that it might be that we need an infinite number of terms in the classical Lagrangian because all vertices generated by radiative corrections that are infinite force us to introduce a bare parameter where we can push the infinity to. This is indeed the case for almost all quantum field theories.

Nevertheless fortunately there are some quantum field theories which need only a finite number of bare parameters, and these are called *renormalisable*. It is the main task of this chapter to give the strict mathematical condition to prove that a field theory is renormalisable and how it is done in practice to push the infinities to the bare parameters in order to obtain a perfectly

finite result in terms of the physical parameters.

To make our qualitative story about QED complete we have to mention some specialities which are caused by the gauge symmetry of the underlying Lagrangian. The first physical content is that the photon is massless. This is the reason that the theory *must* be gauge invariant and thus the physical mass of the photon should be 0. As we shall see in the next chapter this is indeed the case and the reason for that is, the reader might have guessed it, the gauge invariance of the classical Lagrangian, which cancels the infinity which could force us to introduce a bare-mass term to the Lagrangian. There is also a nice cancellation of an infinity in the four-photon diagram mentioned above, so that we are prevented from introducing a bare four-photon coupling in the classical Lagrangian. As we shall see later, quantum electrodynamics is a renormalisable theory, which means that we can hide all infinities into the bare parameters given in the QED-Lagrangian.

Thus a renormalisable quantum field theory contains only a finite number of parameters, of which the bare ones are given in the classical Lagrangian. These bare parameters absorb the infinities and the physical (dressed) parameters are the finite coupling constants, masses and so on which are measured in experiments.

As one can guess the procedure of renormalisation is not unique since one always may add arbitrary finite renormalisations to the bare parameters of the theory. A certain choice of this finite renormalisation is called a *renormalisation scheme*. This shows that the numerical values of the physical parameters change when we go to a different renormalisation scheme. On the other hand since the theory is defined uniquely by the particle content (i.e. the sorts of fields contained) and the particular form of the Lagrangian this dependence should change nothing with respect to S-matrix elements, i.e., the measurable particle properties. As we shall see this is indeed the case for the fully resummed theory. The dependence of the physical parameters on the renormalisation scheme is governed by the *renormalisation group equations* which describes the independence of the S-matrix elements from the chosen renormalization scheme. As we shall see, for perturbative S-matrix elements these equations provide a resummation of "large logarithms" based on the renormalization-scheme independence of the (or course still approximate) S-matrix elements.

This leads directly to the Wilsonian interpretation of the renormalisation process: The physical parameters like coupling constants and masses have to be fixed at a certain scale of energies involved in the processes. The choice of the scale is in principle arbitrary for a renormalisable theory. But in practice if we use perturbation theory we can only evaluate the Green's functions when the coupling constants are small. This might be the case on certain scales and not on others. For instance in QCD (Quantum Chromodynamics as the most promising candidate for describing the strong interactions of quarks) the couplings are small at high energy scales and perturbation theory can be justified, but this is not the case at lower energies where other non-perturbative techniques (especially lattice QCD calculations) are at place.

This Wilsonian point of view also shows that in the quantum field theory always enters an energy-momentum scale, the *renormalisation scale*. This widens also the applicability of quantum field theory to so called *effective theories* which are not renormalisable in the sense explained above. The Lagrangian we build for such theories may contain an arbitrary number of terms consistent with a given symmetry which is underlying the theory. Then we have to introduce a cutoff-parameter which is higher than the energies of interest and at which the theory is estimated to be valid. As we shall see, the most relevant part of such a theory will be its renormalisable part and this explains why up to the nowadays applicable energies the renormalisable standard model of elementary particle is so successful: There might exist an underlying "first-principle theory" of

which the standard model is an effective theory valid at least up to energies which are available with nowadays particle accelerators.

To clarify this point of view on effective theories we can give one example from solid state physics which is quite enlightening. If one looks at the phenomenon of super-conductivity, one can describe it as quantum electrodynamics where the gauge symmetry is spontaneously broken (in the sense of the *Higgs mechanism*, which we shall describe later). This leads to "phenomenological" theories like the London or the Ginzburg-Landau theory of super-conductivity and explains almost all measured phenomena of super-conducting materials with help of a few parameters which have to be fitted to the data. On the other hand we know that the conducting electrons in a metal are described very well as a free Fermi gas with weak interactions of the electrons and the phonons (the quanta of lattice vibrations). Now the effective interaction of the electrons near the Fermi surface[1] is attractive due to the electron-phonon interaction. Because of this attraction the naive picture of the scattering of electrons fails for small momentum exchange. The reason is that the electrons near the Fermi surface can lose energy by pairing to a state of total spin 0. This is the main ingredient of the famous *BCS-theory*[2]. Due to this Cooper pairing there is an energy gap (which in particular prevents the Cooper pairs to break by weak perturbations) to the next excited state super conductivity can take place, because below a certain energy threshold there are no excitations and thus no dissipation. This means that the electric resistence vanishes. This shows that there are indeed two levels of describing the system: The first one uses only symmetry principles and explains super-conductivity as a phenomenon of spontaneous U(1)-gauge symmetry breaking, while the other explains what happens on a microscopic level and how this symmetry breaking comes about.

It might be that the standard model of elementary particles is also an effective theory of which the underlying "microscopic theory" is not known yet. Of course we do not know up to now the mechanism which explains the spontaneous symmetry breaking. Finding such a more comprehensive theory might answer the question about how the observed masses of the elementary particles come about.

We close this qualitative introduction with a short overview over the rest of this chapter since the mathematical task is difficult and the whole chapter will be lengthy. The reader should keep in mind this overview when going through the rest of the chapter.

5.1.1 Overview over the renormalisation procedure

In the next section we start the renormalisation procedure at hand of the simple example of ϕ^4-theory. The first task is to give the infinite integrals a meaning which is called *regularisation*. Regularisation means to render the infinite integrals finite by parameterising the divergence of the integral. The original ill-defined integral is then given as a diverging limit of the regularisation parameters.

On hand of the most simple example of an infinite integral, namely the tadpole diagram, which is 0 in the operator formalism due to normal ordering but becomes a divergent expressen in the path integral formalism, where normal ordering is lost, we show how to regularise this divergence

[1] Remember that a Fermi gas at zero temperature fills the electrons up to a certain energy level due to Pauli's exclusion principle. This is called the Fermi surface in momentum space. The quantum theory at finite temperatures describes the system with help of particle-like excitations from this ground state, which are called *quasi-particles*.

[2] named after its inventors Bardeen, Cooper and Shriver

with help of a cutoff. We shall also show that the infinity can be absorbed completely into the bare mass rendering the contribution of the tadpole diagram to 0 in terms of the observable mass (in one particular renormalisation scheme, the so called *on-shell scheme*).

This example shows that it is most convenient to calculate the Feynman integrals in Euclidean space time with help of a *Wick rotation*. To go back to Minkowski space is then a task of analytic continuation, as will be worked out together with the regularisation scheme used in these notes.

There are a lot of regularisation schemes on the market, but we shall use only the so-called *dimensional regularisation* scheme. This scheme uses the observation that the Feynman integrals would be finite if the spacetime dimension was less than 4 and that the results of these finite integrals are analytic functions of the dimension of spacetime. The infinities are manifest in this formalism as the Feynman integrals have poles in the complex spacetime-dimension plane at $d = 4$. For dimensional regularisation we give a self contained study of the Γ-function which is needed to calculate some standard integrals which can then be used as a toolbox for calculating Feynman integrals. We evaluate these standard integrals and continue back to Minkowski space. Then we can do all our calculations completely in Minkowski space.

In order to get some experience what to do about the poles for $d = 4$ we go further with another one-loop example of infinite integrals in ϕ^4 theory and calculate it within the dimensional regularisation scheme. We shall see how to renormalise the bare parameters of the theory in order to make the results of the calculation finite.

This experience will help us to formulate the convergence criterion for Feynman integrals, which is known as *Dyson's power-counting theorem*. This theorem was proven by Weinberg and is thus also known as *Weinberg's theorem*. This theorem leads to a *necessary condition* for renormalisability of a quantum field theory.

The next task will be to show that ϕ^4-theory (which fulfils the necessary condition) is indeed renormalisable. As we shall see, this is a rather complicated issue, and we shall also present a self contained complete proof of this fact in terms of the so called BPHZ-renormalisation (also in the version given by Zimmermann).

As we shall see, the BPHZ-formulation is applicable to general quantum field theories. We shall use this wide applicability of the BPHZ-scheme to show the complications coming in when the theory is a gauge theory like QED. Here renormalisability not only means to absorb the infinities into the bare constants but also to make sure that the renormalised theory is gauge invariant.

5.2 Wick rotation

With this section we start the prescription of the mathematical formalism of regularisation as the first step towards renormalisation.

We have shown in the third and fourth chapter that we can restrict our investigation to 1PI truncated diagrams, the so called *proper vertex functions*. We shall use ϕ^4-theory as the most simple example of a quantum field theory which has no symmetries (especially no gauge symmetries) to be fulfilled along the procedure of regularisation. We shall also use the path-integral formalism Feynman rules which corresponds to a non-normal-ordered Lagrangian. From the point of view of renormalisation this means only that we have also to regularise the vacuum energy of the theory and this is done by an additional mass renormalisations. The reason for that is that the path integral formalism is more simple in the case of gauge theories.

Now let us take the most simple 1-loop-diagram in ϕ^4-theory, which is the tadpole contribution to the self-energy shown in fig. 5.1.

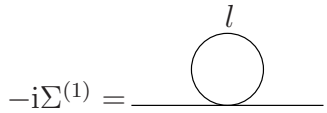


Figure 5.1: *The one-loop tadpole contribution to the self-energy*

Due to the Feynman rules we have found in the third chapter within the canonical operator formalism and in the fourth in the path integral formalism that the analytic expression of this diagram is given by

$$\Sigma^{(1)} = \frac{\mathrm{i}\lambda}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 - m^2 + \mathrm{i}\eta}. \tag{5.1}$$

The first complication we have to get rid of are the poles of the propgator for on-shell loop momenta. Now we find it again important to have used the iη-regulator[3], which came into the game when we calculated the propagator in the operator formalism as well as in the path integral formalism. In the former the reason we had to plug in the iη was the *time-ordering operator* in the definition (3.134) of the propagator, in the latter it was used to project out the vacuum expectation value as shown in section 1.10. From the mathematical point of view it defines the correct causal (i.e. the Feynman-Stückelberg-) weak limit of the propagator in momentum space.

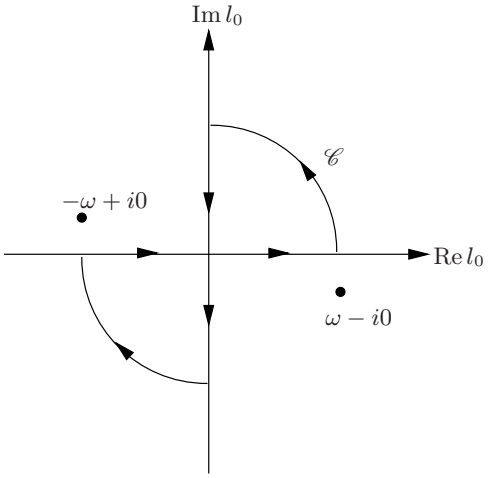


Figure 5.2: *The Wick rotation: The* 0*-component of the loop momentum as a complex variable and the pole structure of the integrand in (5.1)*

It is clear that the iη-prescription helps us to take the correct integral over l_0. Thus we look at figure 5.2, where l_0 is depicted as a complex variable. The two poles $\pm\omega(\vec{p})$ are slightly shifted due to the iη prescription. In (5.1) we are told to integrate over the real l_0-axis. On the other hand the integration over the path $\mathscr{C}$ vanishes due to the theorem of residues, because there are no poles inside this path thanks to the iη-shifts. Since the integrand is $\cong_{l_0 \to \infty} 1/l_0^2$ the two

[3]Beginning with this chapter we change the regulator in the Green's functions from iϵ to iη because in the dimensional-regularisation technique one usually uses $d = 4 - 2\epsilon$ for the spacetime dimension.

quarter circles do not contribute to the integral. Thus we have

$$\int_{\mathscr{C}} \mathrm{d}l_0 f(l_0) = 0 \Rightarrow \int_{-\infty}^{\infty} \mathrm{d}l_0 f(l_0) - \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}l_0 f(l_0) = 0, \tag{5.2}$$

where the limits in the 2nd integral mean the integral along the imaginary l_0-axis from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ (the sign comes in from the fact that this path is run in the opposite direction as part of $\mathscr{C}$). Now substituting in the second integral $l_0 = -\mathrm{i}l_4$ we find

$$\int_{-\infty}^{\infty} \mathrm{d}l_0 f(l_0) = +\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}l_4 f(-\mathrm{i}l_4). \tag{5.3}$$

This rule, which allows one to take the causal pole structure of the propagators into account by integrating over the complex l_0-axis instead along the real axis, is called *Wick rotation* [Wic54], because it can be seen as a rotation of the real path of integration to the complex axis.

Now introducing $\bar{l} = (l_1, \ldots, l_4)$ as a new four vector, we can write (5.1) as

$$\Sigma^{(1)} = \frac{\lambda}{2} \int \frac{\mathrm{d}^4 \bar{l}}{(2\pi)^4} \frac{1}{\bar{l}^2 + m^2}, \tag{5.4}$$

where $\bar{l}^2 = l_1^2 + \cdots l_4^2$ is the *positive definite Euclidean inner scalar product of the four-vector* $\bar{l}$. Thus the Wick-rotation means to go from causal quantum field theory to *Euclidean field theory*. This feature of the iϵ-description we have already seen in section 1.10, where we could either rotate the time integral over the Lagrangian to the imaginary axis (leading to the Euclidean description) or rotate only with a tiny angle (leading to the iϵ-description).

Now we introduce four-dimensional spherical coordinates. Since the integrand depends only on $\bar{l}^2$ we leave out the angular part of the integral, which will be discussed in detail in the next section. It gives only the surface of the three-dimensional sphere Ω_3 in four-dimensional Euclidean space. Now we realise the problem with the integral explicitly, which is divergent, because the volume element reads $L^3 \mathrm{d}L$ (with $L = \sqrt{\bar{l}^2}$) while the integrand decreases only with $1/L^2$ for Large Euclidean loop momenta.

In order to make sense of this most simple example we simply cut off the integral at a Euclidean loop-momentum scale Λ, called the *cut-off scale*. From the power counting above we expect the integral to diverge with Λ^2 for $\Lambda \to \infty$. Indeed we find

$$\Sigma^{(1)}_{\mathrm{reg}} = \frac{\lambda}{32\pi^4} \Omega_3 \int_0^{\Lambda} \mathrm{d}L \frac{L^3}{L^2 + m^2} = \frac{\lambda \Omega_3}{64\pi^4} \left(\Lambda^2 + m^2 \ln \frac{m^2}{\Lambda^2 + m^2} \right). \tag{5.5}$$

This shows that the naive *power counting* was correct for our simple example. The divergence is thus called *quadratic divergence*.

Now we can use the recipe given in the previous section. We try to absorb this divergent contribution of the radiation corrections to the bare parameters of the Lagrangian. This must be done by adding a *counter term* to the *interacting Lagrangian* which is of the same form as a term which is already in the bare Lagrangian (it is not important if the term in the bare Lagrangian is in the "free part" or the "interacting part"). Because this *counter term* should contribute to the self energy, it has to be $\propto \phi^2$.

Now we see that to make the combined contribution (i.e. the sum of (5.5) and the counter term) finite, we can just set

$$\mathscr{L}^{(1)}_{\mathrm{CT}} = \frac{\Sigma^{(1)}_{\mathrm{reg}} + \mathrm{const.}}{2} \phi^2. \tag{5.6}$$

This counter term has to be treated as a vertex in the interaction part of the Lagrangian for later use when evaluating other Feynman diagrams, where the tadpole diagram occurs as a subdiagram, leading to the *counter-term Feynman-rule* in figure 5.3.

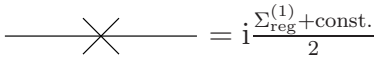


Figure 5.3: *The one-loop counter-term contribution to the bare Lagrangian, which compensates the infinity of the tadpole diagram.*

Due to the Feynman rules the contribution of the counter term to the self energy is given by:

$$-\mathrm{i}\Sigma^{(1)}_{\mathrm{CT}} = \mathrm{i}(\Sigma^{(1)}_{\mathrm{reg}} + \mathrm{const.}). \tag{5.7}$$

Here we have taken into account the factor 2 from connecting the legs to the external points (the legs have of course to be amputated). We find then for the whole contribution

$$\Sigma^{(1)}_{\mathrm{ren}} = \Sigma^{(1)}_{\mathrm{reg}} - \Sigma^{(1)}_{\mathrm{reg}} - \mathrm{const} = -\mathrm{const}. \tag{5.8}$$

This is indeed an arbitrary constant contributing to the bare mass of the particles described by the quantum field ϕ, which is finite for $\Lambda \to \infty$, because it does not depend on Λ at all. We have expected the arbitrariness of the finite part of the counter term, because the only thing we have to fix is the divergent part, which has to be cancelled completely with help of the counter term.

It is also simple to interpret this arbitrariness. From (4.222), Dyson's equation, we know that the approximation for the two-point Green's function to first order in the coupling constant is given by

$$G^{(1)}(p) = \frac{1}{p^2 - m^2 - \Sigma^{(1)}_{\mathrm{ren}} + \mathrm{i}\epsilon}. \tag{5.9}$$

Now it becomes clear that the physical mass squared is given by the pole of the Green's function of the particle, which means that we have

$$m^2_{\mathrm{phys}} = m^2 + \Sigma^{(1)}_{\mathrm{ren}}. \tag{5.10}$$

Now choosing the constant, we define a certain *renormalisation scheme*. This can be done such that $\Sigma^{(1)}_{\mathrm{ren}} = 0$, which is called the *physical renormalisation scheme* or the *on-shell scheme*. In that case we set the mass parameter in the original Lagrangian equal to the physical mass order by order of the Dyson-Wick series. We may also choose another scheme. The only point is that we have to compensate the part of the regularised Feynman integrals which is infinite for $\Lambda \to \infty$ with help of a counter term in the Lagrangian. The counter term should be of the same form as a term which was in the original Lagrangian in order to absorb the infinities into the bare parameters of the theory (which are in this counter-term renormalization approach the sum of the finite parameters and the counter terms, which are divergent for $\Lambda \to \infty$). The numerical values of the finite parameters have to be fitted to scattering data; they are not given from first principles of relativistic quantum field theory. As we shall see later, the only known principles which restrict the choice of parameters are symmetries (most importantly gauge invariance) and renormalisability.

From our simple example the whole idea of renormalisation can be summarised now. Our first step was to handle the pole structure of the Green's function in order to keep the causality of the theory with help of the *Wick-rotation*. Then we have seen that the integral is indeed

divergent and to give it a definite meaning we had to *regularise* this integral. Here we did this by introducing a cut-off Λ for the Euclidean four-momentum. The reader should keep in mind that we have introduced a *momentum scale* into the theory when we keep Λ finite. The next step was to renormalise the integral making the first order Tadpole-contribution to the self energy finite for $\Lambda \to \infty$ by absorbing the infinity of order Λ^2 for $\Lambda \to \infty$ into the *bare mass* of the particle. After this we could take the physical limit $\Lambda \to \infty$. The physical renormalisation scheme was in this case nothing else than enforce the normal ordering description of the path integral which makes the Tadpole contribution vanish from the very beginning within the canonical operator formalism.

We can now give a further outlook of the mathematical solution of the problem of infinite Feynman integrals: For a given theory we have to show that all infinities can be cancelled with help of adding counter-terms to the bare Lagrangian which shuffle the infinities when taking the regularisation parameter (in our example the cut-off) to the physical limit (in our case this was $\Lambda \to \infty$) into a finite set of bare parameters of the theory as there are masses, coupling constants and wave function normalisation constants. Thus a necessary condition for renormalisability is that only a finite set of proper amputated diagrams should be divergent. For ϕ^4-theory only the 2-point and the 4-point function are allowed to be divergent. If another diagram would be divergent, and this divergence had to be compensated by a counter-term which goes with ϕ^6 for example, this would violate the renormalisability of ϕ^4-theory or it would at least force us to introduce a ϕ^6-vertex into the bare Lagrangian from the very beginning. But we shouldn't be forced to introduce infinite many terms into the bare Lagrangian and thus also to use an infinite set of parameters to describe the interaction of the particles involved. Although we might be forced to introduce a finite number of such bare terms, we can define a renormalisable quantum field theory such that it is possible to start with a bare Lagrangian with a finite number of parameters.

A first superficial hint which diagrams are divergent is given by *power counting* of the loop integrals. Beginning with the next section we shall solve the problem of renormalisation. The first step is to introduce a simple regularisation scheme which gives us a recipe to calculate systematically the Feynman integrals and to extract the infinities in order to find the correct counter terms for the bare Lagrangian. Because it is so useful for the renormalisation of gauge theories we shall use *dimensional regularisation* in these notes.

5.3 Dimensional regularisation

In this section we want to show one of the most convenient regularisation prescriptions used in perturbative calculations. The idea is to preserve as many symmetry features as possible in the theory. This is most important in the case of gauge theories, where the local gauge symmetry is not the basis of the model building process to describe interactions which occur in nature. It is also a necessary condition to give the theory a physical meaning at all. It is the only principle known so far to build theories which are at the same time flexible enough to describe all the features of elementary particle up to the highest energies which are available nowadays in accelerator facilities in the world and constraint enough to be renormalisable and give thus a physically consistent theory with a unitary S-matrix to describe scattering processes of interacting elementary particles.

To find a regularisation which keeps especially the gauge invariance of the theory valid at all stages of the calculation (as far as this is possible at all) is not only important for convenience

during the calculations in perturbation theory (although this is indeed also an important point for practical purposes) but it is also an essential key to the proof of renormalisability in the sense that the renormalised S-matrix is unitary as well as gauge invariant, because this proof is only sound if the regularised theory is gauge invariant.

We shall come back to gauge theories (not only QED but also the more general case of non-abelian gauge theories and as a final goal of these notes the standard model for elementary particles) after we have settled the complete renormalisation program. Now as a first step we use again our ϕ^4-toy model theory which we aim to renormalise as the first particular example for a field theory. In that case there is no need for dimensional regularisation. We could do all with the above given regularisation scheme with a cut-off momentum Λ without violating any physical principles. But this has no advantages compared to dimensional regularisation, and since we shall need this technique for all physical relevant theories we like to understand later on, we shall use the dimensional regularisation prescription also in that simple case. This work is also not lost since we shall calculate all the standard integrals, which are useful for practical calculations in perturbation theory, in this section.

Our starting point is the naive power counting approach we have seen to work for the most simple Tadpole-graph in the preceding section. If space-time would not be four-dimensional but only one-dimensional (which would of course be a rather "boring world" so to say), the integral we had to calculate were perfectly finite. Thus we introduce the dimension of space time as the regularisation parameter. It seems just clear that all inner symmetries, i.e. symmetries which have nothing to do with the space-time symmetries, are valid in the arbitrary space time-dimension. We shall see later on that there are very important exceptions of this conjecture, known as *anomalies*, which are not only important for phenomenological reasons (e.g. pion decay to photons) but may also be dangerous for the renormalisability and unitarity of gauge theories. As an anomaly we define the case that a symmetry of the bare Lagrangian, which leads to a conserved current due to Noether's theorem, is not preserved in the quantised theory. But we shall come to that point later. Our ϕ^4-toy theory cannot have an anomaly because there is no symmetry except Lorentz-invariance.

Thus we introduce the space time-dimension d as our regularisation parameter with the physical limit $d \to 4$ which will be taken after renormalising the Feynman-integral under consideration. Now it happens that the Feynman-integrals can be formally seen as *analytic functions* in the complex d-plane. Of course there is no sophisticated geometrical meaning behind that, but it is convenient to expand the integrals around the point $d = 4$ which is a singular point if these integrals are divergent (otherwise they were convergent and we had nothing to regularise). This is perfectly what we like to find, namely a parameterisation of the infinity of the Feynman-integral, which gives us the possibility to push this infinity into the bare parameters of the (hopefully) renormalisable theory. After the subtraction of the infinities by introduction of counter terms into the bare Lagrangian we can take without further problems the physical limit $d \to 4$ to obtain the finite result of the radiation corrections to the tree level order of perturbation theory[4].

We have seen that it is very convenient to get rid of the pole structure of the free propagators by using the Wick-rotation to switch to Euclidean field theory. This will be done here. We should only remark, that we have also to take the Euclidean form of the external momenta, which enter a given diagram, in order to have a $SO(4)$ instead of a $SO(1,3)$ invariant theory. We shall also have to solve the problem of analytic continuation to Minkowskian space time for the external

[4]The next much more involved task will be to show that the renormalised physical result is independent of the regularisation scheme, which will be done beginning with the next section.

momenta.

As mathematical foundation the first step is to remember the properties of the Γ-function which is very useful to calculate the d-dependent regularised Feynman-integrals. What we shall also need is the complete analytic structure of the Γ-function.

5.3.1 The Γ-function

The Γ-function lies at the heart of the dimensional regularisation technique, because its analytic properties allow to manage the problem of continue the dimension d of Euclidean space time from a few integer values where the Feynman integrals are convergent[5] to the whole complex d-plane.

Indeed it was a quite similar task Euler and Gauß solved in the 18th century, namely the continuation of the factorial function, defined on the non-negative integer numbers, to the whole complex plane.

We start with Euler's representation of the Γ-function:

$$\Gamma(z) = \int_0^\infty \mathrm{d}t \exp(-t) t^{z-1}. \tag{5.11}$$

Herein we understand the potential of t as

$$t^{z-1} = \exp[(z-1)\ln t], \tag{5.12}$$

where the logarithm along the positive real axis is defined as real (principal value of the logarithm). Now we show that the integral (5.12) is uniformly convergent in all compact areas in the right z-half-plane, i.e., for all $\mathrm{Re}\, z > 0$. This implies that the Γ-function is an analytic function in the right z-half-plane.

For this purpose we split the integral in the following way:

$$\Gamma(z) = \int_0^1 \mathrm{d}t \exp(-t) t^{z-1} + \int_1^\infty \mathrm{d}t \exp(-t) t^{z-1}. \tag{5.13}$$

At first we look on the second integral:

$$\omega(z) = \int_1^\infty \mathrm{d}t \exp(-t) t^{z-1} : \tag{5.14}$$

For each $t > 1$ the integrand is an analytic function of $z \in \mathbb{C}$. For an arbitrary compact area B of the z-plane there exists $x_0 \in \mathbb{R}$ such that:

$$x_0 = \max_{z \in B} [\mathrm{Re}\, z]. \tag{5.15}$$

Since further for $t \geq 1$ the logarithm is non-negative, we find

$$\forall z \in B : \ |\exp(-t) t^{z-1}| = |\exp[-t + (z-1)\ln t]| \leq \exp(-t) t^{x_0 - 1}. \tag{5.16}$$

Because the integral

$$\int_1^\infty \exp(-t) t^{x_0 - 1} dt \tag{5.17}$$

[5] The whole dimensional regularisation program makes only sense if at least one integer space time dimension exists, where the integral under consideration is convergent.

is converging point-wise for all $z \in B$, due to Weierstraß' convergence criterion this is also the case in the sense of uniform convergence in B and thus ω is a analytic function in B. Thus ω is analytic in the whole complex z-plane.

A little more work is to do for the first integral in (5.13):

$$\phi(z) = \int_0^1 \mathrm{d}t \exp(-t) t^{z-1}. \tag{5.18}$$

The modulus of the integrand is $\exp(-t)t^{x-1}$, and for $x > 1$ the integral converges. Thus (5.18) is an analytic function for $\operatorname{Re} z > 1$. We like to show that this is the case for all compact areas of the right z-half-plane. Because B is supposed to be compact, there exists

$$x_1 = \min_{z \in B} \operatorname{Re} z, \tag{5.19}$$

and it is $x_1 > 0$. For $0 < t \leq 1$ we have $\ln t \leq 0$. Thus also

$$\forall z \in B : \ |\exp(-t)t^{z-1}| \leq \exp(-t)t^{x_1-1} \tag{5.20}$$

holds true.

Since the integral over this real function is converging, again applying Weierstraß' criterion for uniform convergence shows that (5.18) is an analytical function in B.

Since B can be an arbitrary compact area in the right z-half-plane from (5.11) follows the analyticity of Γ in the whole open right z-half-plane.

The next step is to find a maximal analytic continuation of Γ to the left half plane. It is enough to do this for the first integral in (5.13), because the second one has been shown to be an analytic function of z in the whole complex plane.

Now the series

$$\exp(-t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \tag{5.21}$$

is uniformly convergent for fixed $t \in \mathbb{R}$. Plugging this into (5.18) by naive order by order-integration of the series we find for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$:

$$\phi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n+z}. \tag{5.22}$$

Since the series at the right hand side of this equation is uniformly convergent in any compact subset of the right z-half-plane which does *not* contain any of the points $\{0; -1; -2; \dots\}$, the order by order-integration is justified and ϕ is analytically continued to a meromorphic function with simple poles at the non-positive integer numbers. This property is thus also true for the Γ-function itself. From this we read off Weierstraß' expansion of the Γ-function:

$$\Gamma(z) = \int_1^{\infty} \mathrm{d}t \exp(-t) t^{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n+z}. \tag{5.23}$$

In the following we understand this meromorphic function as Γ. Now we like to find some useful properties of the Γ-function.

Fur $n \in \mathbb{N}$ we can calculate the integral (5.11) analytically with the result

$$\Gamma(n+1) = n! \tag{5.24}$$

This is easily shown inductively by integrating (5.11) by parts. For any real positive z this yields also the important functional relation

$$\Gamma(z+1) = z\Gamma(z). \tag{5.25}$$

Since Γ is a meromorphic function this is valid for all $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.
The next relation we like to show is

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{5.26}$$

To prove this we substitute $t = u^2$ in (5.11) and suppose $z \in (0,1) \subset \mathbb{R}$:

$$\Gamma(z) = 2\int_0^\infty \mathrm{d}u \exp(-u^2) u^{2z-1}. \tag{5.27}$$

Now setting $1-z$ for z and renaming the integration variable with v we find

$$\Gamma(1-z) = 2\int_0^\infty \exp(-v^2) v^{1-2z} dv. \tag{5.28}$$

Multiplicating (5.27) with (5.28) yields

$$\Gamma(z)\Gamma(1-z) = 4\int_0^\infty \int_0^\infty \mathrm{d}u\, \mathrm{d}v \exp(-u^2-v^2) \left(\frac{u}{v}\right)^{2z-1}. \tag{5.29}$$

This can be read as an integral over the first quarter of the uv-plane and we transform it into plane polar coordinates:

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= 4\int_0^\infty r\, \mathrm{d}r\ \exp(-r^2) \int_0^{\pi/2} \mathrm{d}\phi\, (\cot\phi)^{2z-1} = \\ &= 2\int_0^{\pi/2} \mathrm{d}\phi\, (\cot\phi)^{2z-1}.\end{aligned}$$

To calculate this integral we substitute $\phi = \operatorname{arccot}(\sqrt{x})$:

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \mathrm{d}x \frac{x^{z-1}}{1+x}. \tag{5.30}$$

The function

$$f(y) = \frac{(-y)^{z-1}}{1+y} \tag{5.31}$$

has an essential singularity in $y = 0$ and we cut the complex y-plane along the positive real axis. Now we go to the sheet of the Riemannian surface for which

$$\lim_{\mathrm{Im} z \to \pm 0} (-y)^{z-1} = |y|^{z-1} \exp[\mp \mathrm{i}\pi(z-1)] \tag{5.32}$$

is valid. Now we integrate over the path shown in figure 5.4.

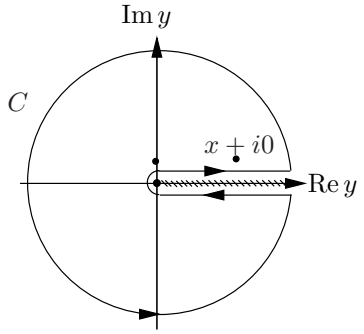


Figure 5.4: *Path for the integral (5.30)*

Letting the radius of the big circle go to infinity and this of the small one to zero these circles do not contribute to the integral, and we find

$$\int_C \mathrm{d}y \frac{(-y)^{z-1}}{1+y} = 2\mathrm{i}\sin(\pi z)\int_0^\infty \mathrm{d}x \frac{x^{z+1}}{1+x}. \tag{5.33}$$

On the other hand using the residuum theorem we see, that

$$\int_C \mathrm{d}y \frac{(-y)^{z-1}}{1+y} = 2\pi\mathrm{i}\operatorname*{Res}_{y=-1} \frac{(-y)^{z-1}}{1+y} = 2\pi\mathrm{i}. \tag{5.34}$$

Both results prove together with (5.29) the conjecture (5.26) for $z \in (0,1)$ and thus for all $z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$, because Γ is a meromorphic function.
Especially plugging in $z = 1/2$ in (5.26) we find

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \mathrm{d}t\sqrt{t}\exp(-t) = \sqrt{\pi}. \tag{5.35}$$

Further we need *Euler's Beta-function*, defined by

$$\mathrm{B}(p;q) = \int_0^1 \mathrm{d}x x^{p-1}(1-x)^{q-1}. \tag{5.36}$$

Substitution of $t = 1 - x$ yields the symmetry of the function under interchange of its two arguments:

$$\mathrm{B}(p;q) = \mathrm{B}(q;p). \tag{5.37}$$

Integration by parts shows that

$$\mathrm{B}(p;q+1) = \frac{q}{p}\mathrm{B}(p+1;q). \tag{5.38}$$

holds. Inspection of (5.27) gives

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 2\int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \exp(-u^2-v^2)u^{2p-1}v^{2q-1} = \\ &= 4\int_0^\infty \mathrm{d}r r^{2(p+q-1)} \int_0^{\pi/2} \mathrm{d}\phi \exp(-r^2)\cos^{2p-1}\phi \sin^{2q-1}\phi,\end{aligned} \tag{5.39}$$

where we have introduced plane polar coordinates in the last step.

Substitution of $t = r^2$ gives

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 2\int_0^\infty \mathrm{d}t \exp(-t) t^{p+q-1} \int_0^{\pi/2} \mathrm{d}\phi \cos^{2p-1}\phi \sin^{2q-1}\phi = \\ &= 2\Gamma(p+q)\int_0^{\pi/2} \mathrm{d}\phi \cos^{2p-1}\phi \sin^{2q-1}\phi. \end{aligned} \tag{5.40}$$

In the remaining integral we substitute $x = \cos^2\phi$ and obtain its value to be $\mathrm{B}(p;q)/2$. Thus we have

$$\mathrm{B}(p;q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{5.41}$$

Now we want to give a proof for Gauß's representation of the Γ-function as an infinite product:

$$\begin{gathered}\frac{1}{\Gamma(z)} = z\exp(\gamma z)\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\exp\left(-\frac{z}{k}\right) \\ \text{with } \gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).\end{gathered} \tag{5.42}$$

γ is the so called *Euler-Mascheroni-constant.*

To prove (5.42) we use the following representation of the exponential function

$$\exp(-t) = \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n \tag{5.43}$$

and define the function series

$$P_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt. \tag{5.44}$$

Naively looking on this definition we see that this series converges to $\Gamma(z)$ in each regular point z. We shall show that this is even the case in the sense of uniform convergence. But at first we show that we then also have proven the product representation (5.42):

Substitution of $t = n\tau$ in (5.44) yields

$$\begin{aligned}P_n(z) &= n^z\int_0^1 \mathrm{d}\tau(1-\tau)^n\tau^{z-1} = n^z\mathrm{B}(z;n+1) = \\ &= \frac{n^z\Gamma(z)\Gamma(n+1)}{\Gamma(z+n+1)} = \frac{n^z n!}{(z+n)(z+n-1)\cdots z}.\end{aligned} \tag{5.45}$$

Here we have used the properties of the B-function given above as well as the functional property (5.25) of the Γ-function.

A little algebra of this result yields:

$$\begin{aligned}\frac{1}{P_n(z)} &= \frac{\exp[z(1+1/2+\cdots+1/n-\ln n)]}{\exp[z(1+1/2+\cdots+1/n)]} z\frac{z+1}{1}\cdots\frac{z+n}{n} = \\ &= z\exp[z(1+1/2+\cdots+1/n-\ln n)]\times \\ &\times\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\exp\left(-\frac{z}{k}\right).\end{aligned}$$

This shows that indeed the uniform convergence of the series P_n to Γ proves Gauß' product representation (5.42).

From the principle of analytic continuation we know that it is sufficient to show this for real positive z. Differentiating with respect to t leads to the following relation:

$$1 - \left(1 - \frac{t}{n}\right) \exp t = \int_0^t \mathrm{d}v \frac{v}{n} \left(1 - \frac{v}{n}\right)^{n-1} \exp v. \tag{5.46}$$

For $0 < t < n$ the integrand is positive. On the other hand we have

$$\int_0^t \mathrm{d}v \frac{v}{n} \left(1 - \frac{v}{n}\right)^{n-1} \exp v < \int_0^n \mathrm{d}v \frac{v}{n} \exp t = \frac{t^2}{2n} \exp t, \tag{5.47}$$

which leads to

$$0 \leq \exp(-t) - \left(1 - \frac{t}{n}\right) < \frac{t^2}{2n}. \tag{5.48}$$

From Euler's definition of the Γ-function (5.11) we know that

$$\Gamma(z) - P_n(z) = \int_0^n \mathrm{d}t \left[\exp(-t) - \left(1 - \frac{t}{n}\right)^n\right] + \int_n^\infty \mathrm{d}t \exp(-t) t^{z-1} \tag{5.49}$$

holds. Within the convergence proof of (5.11) we have shown that the second integral converges uniformly to 0 for $n \to \infty$. From the above inequality we read off

$$\begin{aligned} 0 &\leq \int_0^n \mathrm{d}t \left[\exp(-t) - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1} \leq \\ &\leq \int_0^{n_0} \mathrm{d}t \left[\exp(-t) - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1} + \int_{n_0}^n \mathrm{d}t \exp(-t) t^{z-1} \leq \\ &\leq \int_0^n \mathrm{d}t \frac{t^{z+1}}{2n} + \int_{n_0}^\infty \mathrm{d}t \exp(-t) t^{z-1} \end{aligned} \tag{5.50}$$

for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Because of the uniform convergence of the last integral we may chose n_0 so large that

$$\int_{n_0}^\infty \mathrm{d}t \exp(-t) t^{z-1} < \frac{\epsilon}{2} \tag{5.51}$$

holds. Then we have for $n > n_0$ by using the inequality again:

$$0 \leq \int_0^n \mathrm{d}t \left[\exp(-t) - \left(1 - \frac{t}{n}\right)\right] t^{z-1} < \int_0^{n_0} \mathrm{d}t \frac{t^{z+1}}{2n} + \frac{\epsilon}{2} = \frac{1}{z+2} \frac{n_0^{z+2}}{n} + \frac{\epsilon}{2}. \tag{5.52}$$

From this we can read off immediately that the integral is uniformly convergent in each compact interval of the positive real axis. Thus we finished the proof of Gauß' product representation for the Γ-function.

Taking its logarithm we find

$$-\ln[\Gamma(z)] = \gamma z + \ln z + \sum_{k=1}^\infty \left[-\frac{z}{k} + \ln\left(1 + \frac{z}{n}\right)\right]. \tag{5.53}$$

Deriving of this equation with respect to z leads to

$$\Psi_1(z) := \frac{\mathrm{d}}{\mathrm{d}z}\ln[\Gamma(z)] = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty}\frac{1}{k(z+k)} \tag{5.54}$$

Since the series converges uniformly on each compact subset of $\mathbb{C}$ which does not contain a negative integer number or 0, this is really the logarithmic derivative of the Γ-function.

Within the dimensional regularisation technique we shall also need the Laurent-expansion of the Γ-function around the simple poles at $z \in \mathbb{Z}_{\leq 0}$. It is enough to find the expansion up to the first order:

$$\forall n \in \mathbb{N}:\ \Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \Psi_1(n+1) + O(\epsilon)\right]. \tag{5.55}$$

For proving this equation we state that from (5.54) follows

$$\Psi_1(1) = -\gamma - 1 + \sum_{k=1}^{\infty}\frac{1}{k(k+1)} = -\gamma. \tag{5.56}$$

From (5.25) we obtain

$$\Psi_1(z+1) = \frac{\mathrm{d}}{\mathrm{d}z}\ln[\Gamma(z+1)] = \frac{1}{z} + \Psi_1(z). \tag{5.57}$$

By induction we find

$$\forall n \geq 1:\ \Psi_1(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}. \tag{5.58}$$

Now we look on the Taylor-expansion of the Γ-function around the regular point $z = 1$:

$$\Gamma(1+\epsilon) = 1 + \epsilon\Gamma'(1) + O(\epsilon^2) = 1 + \epsilon\Psi_1(1) + O(\epsilon^2), \tag{5.59}$$

which is valid in the open disc of radius 1 around $\epsilon = 0$, because the next pole of the Γ-function is at $\epsilon = -1$. Dividing this equation through ϵ yields:

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon). \tag{5.60}$$

This is (5.55) for $n = 0$. For all other $n \in N$ the equation can be shown by induction. Suppose it is true for $n = k$. Then we find making use of (5.25) again:

$$\Gamma[-(k+1)+\epsilon] = \frac{\Gamma(-k+\epsilon)}{-(k+1)+\epsilon} = \frac{(-1)^{k+1}}{(k+1)!}\left[\frac{1}{\epsilon} + \Psi_1(k+1) + \frac{1}{k+1} + O(\epsilon)\right]. \tag{5.61}$$

Comparing this with (5.55) we see that this is of course this equation for $n = k+1$, and this was to show.

5.3.2 Spherical coordinates in d dimensions

We have seen in the last section that one can use the iϵ-description of the free Green's functions to make use of the Wick-rotation. This makes it possible to calculate the Feynman integrals in Euclidean space. The final result in Minkowski space is then given by analytic continuation.

The first step to find the standard formulas which we shall use when we calculate Feynman integrals, is to introduce d-dimensional spherical coordinates. For $d = 2$ these are the usual polar coordinates

$$\vec{x} = (r\cos\phi, r\sin\phi) \text{ with } r \in \mathbb{R}_{>0},\ \phi \in (0, 2\pi). \tag{5.62}$$

The d-dimensional spherical coordinates can be defined recursively starting with the 2-dimensional polar coordinates as follows:

$$\vec{x} = r(\vec{n}_{d-1}\sin\theta_{d-2}, \cos\theta_{d-2}). \tag{5.63}$$

Herein $\vec{n}_{d-1}$ is the radial vector of unit length in $(d-1)$-dimensional space. The angles θ_k with $k = 1\ldots(d-2)$ are defined in $(0,\pi)$. The Jacobian of the transformation from Cartesian to spherical coordinates contains a factor r^{d-1}:

$$J_d = \det\frac{\partial(x_1; x_2; \ldots; x_d)}{\partial(r; \phi; \theta_1; \ldots; \theta_{d-2})} = r^{d-1} j_d. \tag{5.64}$$

Using the definition of the Jacobian for (5.63) we obtain the recursion formula by expansion of the determinant with respect to its last row:

$$j_2 = 1;\ j_d = (\sin\theta_{d-2})^{d-2} j_{d-1}, \tag{5.65}$$

from which we get immediately

$$j_d = \prod_{k=1}^{d-2} \sin^k\theta_k \text{ for } d \geq 3. \tag{5.66}$$

Further we remark that the part of $\mathbb{R}^d$ which is not covered by the spherical coordinates is of Lebesgue-measure 0, so that the Euclidean integrals can be calculated by using this one chart.

5.3.3 Standard-integrals for Feynman integrals

In this subsection we shall calculate some integrals which will be useful to calculate Feynman-integrals within the dimensional regularisation scheme. First we consider:

$$I_d(q) = \int \frac{\mathrm{d}^d p}{(m^2 - p^2 - 2pq - \mathrm{i}\epsilon)^\alpha}. \tag{5.67}$$

This integral is written in d-dimensional Minkowski space. If the integral exists in the d-dimensional space we can shift the integration variables by $p' = p + q$:

$$I_d(q) = \int \frac{\mathrm{d}^d p'}{[-(p')^2 + (m^2 + q^2) - \mathrm{i}\epsilon]^\alpha}. \tag{5.68}$$

Now we suppose that q is chosen such that $\mu^2 := m^2 + q^2 > 0$. This integral is of the typical form of Feynman-integrals[6]. It is simply the α-th power of the negative Feynman-Green's function

[6] As α does not need to be an integer along the calculations of multi-loop-integrals it is advantageous to use $[-G(p)]^\alpha$ which enables a Wick rotation without introducing factors $(-1)^\alpha$. Since α in the original Feynman integrals is integer this causes no trouble since one can uniquely introduce an integer number of factors (-1) within the original integral as needed.

for a free scalar particle. We have seen in the previous section that its pole structure allows us to make use of the Wick-rotation and transform the integral to its Euclidean counterpart:

$$I_d(q) = \mathrm{i}\int \mathrm{d}^d\bar{p}\frac{1}{(\bar{p}^2+\mu^2)^\alpha}. \tag{5.69}$$

Herein $\bar{p}^2$ is the positive definite Euclidean quadratic form of the d-dimensional vector $\bar{p}$. Now we introduce the d-dimensional spherical coordinates. From (5.64) and (5.66) we get:

$$\begin{aligned} I_d(q) &= \mathrm{i}\int_0^\infty \mathrm{d}r\, r^{d-1}\frac{1}{(r^2+\mu^2)^\alpha}\times \\ &\quad\times\int_0^{2\pi}\mathrm{d}\phi\int\cdots\int\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{d-2}\prod_{k=1}^{d-2}\sin^k\theta_k, \end{aligned} \tag{5.70}$$

where $r = \|\bar{p}\|$. The integration over the angles can be done with help of Euler's B-function. When we proved (5.41) we found

$$\int_0^{\pi/2}\mathrm{d}\theta\cos^{2p-1}\theta\sin^{2q-1}\theta = \frac{1}{2}\mathrm{B}(p;q) = \frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)}. \tag{5.71}$$

For $p = 1/2$ we have

$$\int_0^{\pi}\mathrm{d}\theta\sin^{2q-1}\theta = \frac{\Gamma(1/2)\Gamma(q)}{\Gamma(p+q)}, \tag{5.72}$$

because the sine is symmetric around $\pi/2$. Setting $k = 2q-1$ yields with help of (5.35):

$$\int_0^{\pi}\mathrm{d}\theta\sin^{k}\theta = \frac{\sqrt{\pi}\Gamma[(k+1)/2]}{\Gamma(k/2+1)}. \tag{5.73}$$

Thus the integration over the angles in (5.70) gives the area of the d-dimensional hyper-sphere of radius 1, which is the d-dimensional analogue of the full solid angle in 3-dimensional space:

$$\Omega_d = 2\pi\prod_{k=1}^{d-2}\frac{\sqrt{\pi}\Gamma[(k+1)/2]}{\Gamma[(k+2)/2]} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{5.74}$$

For $d = 2$ we find the well known length of the circumference of the unit circle 2π, for $d = 3$ the surface area of the unit sphere 4π.

Now the right hand side of this formula is written in the form which is most important for the dimensional regularisation technique, namely it can be interpreted as an analytic function of the dimension d.

Using (5.74) in (5.70) we get

$$I_d(q) = \frac{2\mathrm{i}\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty \mathrm{d}r\frac{r^{d-1}}{(r^2+\mu^2)^\alpha}. \tag{5.75}$$

Substitution of

$$\tau^2 = \frac{1}{t} - 1 \Rightarrow t = \frac{1}{1+\tau^2};\ \tau d\tau = -\frac{1}{2t^2}dt, \tag{5.76}$$

in the definition of the B-function (5.36) yields

$$B(x;y) = 2\int_0^\infty \mathrm{d}\tau\frac{\tau^{2y-1}}{(1+\tau^2)^{x+y}} = 2\int_0^\infty \mathrm{d}\tau\frac{\tau^{2x-1}}{(1+\tau^2)^{x+y}}, \tag{5.77}$$

where we have used the symmetry of the B-function under the interchange of its two arguments. Setting herein

$$x = \frac{\beta+1}{2},\ y = \alpha - \frac{\beta+1}{2},\ \tau = \frac{s}{\mu}, \tag{5.78}$$

we find

$$\int_0^\infty \mathrm{d}s \frac{s^\beta}{(s^2+\mu^2)^\alpha} = \frac{1}{2\mu^{2\alpha-\beta-1}} B\left(\frac{\beta+1}{2}; \alpha - \frac{\beta+1}{2}\right). \tag{5.79}$$

With $\beta = d-1$ in (5.75) we have

$$I_d = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(d/2)} B\left(\frac{d}{2}; \alpha - \frac{d}{2}\right) \mu^{d-2\alpha} = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(\alpha)} \frac{\Gamma(\alpha - d/2)}{(\mu^2)^{\alpha-d/2}}. \tag{5.80}$$

Because of $\mu^2 = m^2 + q^2$ we can write this as:

$$I_d(q) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^\alpha} = \frac{\mathrm{i}}{(4\pi)^{d/2}\Gamma(\alpha)} \frac{\Gamma(\alpha - d/2)}{(q^2+m^2)^{\alpha-d/2}}. \tag{5.81}$$

For $q^2 > -m^2$ the power in the denominator is defined to be real and has to be continued analytically to the whole Minkowski space. For later convenience we introduce $d = 2\omega$:

$$\int \frac{\mathrm{d}^{2\omega} p}{(2\pi)^{2\omega}} \frac{1}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^\alpha} = \frac{\mathrm{i}}{(4\pi)^\omega} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{1}{(q^2+m^2)^{\alpha-\omega}}. \tag{5.82}$$

All other formulas we shall need for the calculation of Feynman-integrals can be obtained by deriving this result with respect to q_μ:

$$\int \frac{\mathrm{d}^{2\omega} p}{(2\pi)^{2\omega}} \frac{p_\mu}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^\alpha} = -\frac{\mathrm{i}}{(4\pi)^\omega} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{q_\mu}{(q^2+m^2)^{\alpha-\omega}}. \tag{5.83}$$

Differentiating this result again with respect to q^ν we obtain:

$$\begin{aligned} \int \frac{\mathrm{d}^{2\omega} p}{(2\pi)^{2\omega}} \frac{p_\mu p_\nu}{(m^2 - p^2 + 2pq - \mathrm{i}\eta)^\alpha} &= \frac{\mathrm{i}}{(4\pi)^\omega \Gamma(\alpha)} \frac{1}{(q^2+m^2)^{\alpha-\omega}} \times \\ &\times \left[q_\mu q_\nu \Gamma(\alpha - \omega) - \frac{1}{2} g_{\mu\nu} (q^2+m^2) \Gamma(\alpha - \omega - 1) \right]. \end{aligned} \tag{5.84}$$

Contracting the indices μ and ν results in:

$$\begin{aligned} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p^2}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^\alpha} =& \frac{\mathrm{i}}{(4\pi)^\omega \Gamma(\alpha)} \frac{1}{(q^2+m^2)^{\alpha-\omega}} \times \\ &\times [q^2 \Gamma(\alpha - \omega) - \omega (q^2+m^2) \Gamma(\alpha - \omega - 1)]. \end{aligned} \tag{5.85}$$

With these formulas and the others listed in appendix C we shall calculate almost all of the Feynman integrals in these lectures. Of course there are some subtleties in those calculations which will be developed in the next section where we give some characteristic examples in ϕ^4-theory.

There is only one general topic we have to mention, concerning the dimensions. In these notes we always set $\hbar = c = 1$ and thus are left with one fundamental unit, namely energy or length (which is in our "natural" units inverse energies). Since we work in momentum space the most time we shall count the dimensions in terms of energies. It is clear that from $\hbar = 1$ the action functionals

have dimension 1 and thus the Lagrangian is of dimension $E^{2\omega}$ where 2ω is the dimension of space-time. In order to keep the coupling constants of the same dimension as in $2\omega = 4$, which is of course the physical case, we have to introduce an energy scale μ. We see that from this an energy scale comes into the game from the regularisation procedure which was introduced in a less formal way also by the crude cut-off regularisation. In the case of renormalisable quantum field theories in the final result the infinities can be absorbed into the bare quantities and the renormalised parameters will depend on the renormalisation scale.

5.4 Renormalization of the ϕ^4 model

In this section we shall look at the divergent truncated 1PI diagrams (depicting proper vertex functions) of the ϕ^4 model up to order λ^2 of the perturbative expansion in a systematic way, using dimensional regularisation. In $d = 2\omega = 4$ space-time dimensions the superficially divergent diagrams are the two-point function or self-energy and the four-point function. At order λ for the self-energy there is only the tadpole diagram and no correction to the four-point function. At order λ^2 we have a one-loop correction of the four-point function and two two-loop corrections for the self-energy. As we shall see shortly, from the point of view of renormalisation one has to systematically calculate first all one-loop diagrams, isolate their divergences, which in terms of dimensional regularization occur in poles $1/(2-\omega)^j = 1/\epsilon^j$, which define counterterms to shift the infinities to the bare parameters. If we only subtract these pole terms, which defines the socalled **minimal subtraction (MS) renormalization scheme** we expect that these counterterms become *independent of all energy scales*, because the divergences are due to large loop momenta. In this "UV region" any finite mass or energy scales should become irrelevant, and thus the MS renormalization scheme should lead to bare parameters (i.e., wave-function normalization, mass and coupling constant) that depend on the renormalisation scale only implicitly via the corresponding renormalized parameters.

5.4.1 The tadpole self-energy diagram

We have evaluated the tadpole self-energy diagram in Fig. 5.1 in Eq. (5.1). Now we use dimensional regularization to regularise the diagram, i.e.,

$$\Sigma^{(1)} = -\frac{\mathrm{i}\lambda}{2}\mu^{4-2\omega}\int \frac{\mathrm{d}^{2\omega}}{(2\pi)^{2\omega}}\frac{1}{m^2-l^2}. \tag{5.86}$$

The factor $\mu^{4-2\omega}$ with the renormalisation scale μ has been introduced to keep the coupling constant λ dimensionless in all spacetime dimensions, i.e., we write $\lambda \to \lambda\mu^{4-2\omega} = \lambda\mu^{2\epsilon}$. Here and in the following we have absorbed the small imaginary part of the free propagator into the mass term, i.e., one should read $m^2 \to m^2 - \mathrm{i}\eta$. Now we can use the standard formulae (C.11), leading to

$$\Sigma^{(1)} = \frac{\lambda m^2}{32\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\Gamma(-1+\epsilon) = \frac{\lambda}{32\pi^2}\left[-\frac{1}{\epsilon} - 1 + \gamma + \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] + \mathcal{O}(\epsilon), \tag{5.87}$$

where in the last step we have performed a Laurent expansion around $\epsilon = 0$, using (C.14).

The infinity for $\epsilon \to 0$, corresponding to the pole $\propto 1/\epsilon$ can be renormalised by a mass counterterm. In the MS scheme we just use it to subtract the pole term, i.e., we set

$$\delta m^2_{1,\mathrm{MS}} = \frac{\lambda m^2}{32\pi^2\epsilon} = m^2\delta Z_{m1,\mathrm{MS}}. \tag{5.88}$$

As we see, the dependence on the renormalisation scale μ is only implicit in the renormalised coupling λ and mass m. The renormalised self-energy the reads

$$\Sigma_{\text{ren, MS}} = \delta m^2_{1,\text{MS}} + \Sigma^{(1)} = \frac{\lambda m^2}{32\pi^2}\left[-1+\gamma+\ln\left(\frac{m^2}{4\pi\mu^2}\right)\right]. \tag{5.89}$$

5.4.2 The dinosaur diagram for the four-point vertex

The one-loop correction to the four-point vertex function is of order λ^2, cf. Fig. 5.5.

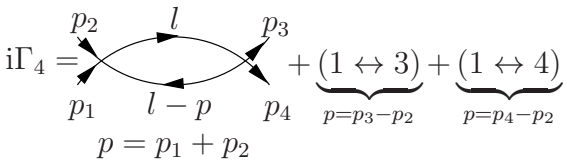


Figure 5.5: *The "dinosaur diagram" as a one-loop contribution to the 4-point function*

Evidently it is sufficient to calculate the first diagram. From momentum conservation and relativistic covariance this diagram is a function of the Mandelstam variable $s = (p_1+p_2)^2$, and the other diagrams lead to the same function of the other Mandelstam variables $t = (p_1-p_3)^2$ and $u = (p_1-p_r)^2$.

The symmetry factor is calculated as follows: There are 8 possibilities to connect the first external point to one of the vertices, remaining 3 possibilities to connect the second one with the same inner vertex, 4 to connect the other vertex with the external point and again 3 to connect this same vertex with the last external point. Then there are 2 possibilities to connect both vertices with the two lines building the loop. There is a factor 1/2 from the Dyson-Wick series while a factor $(-\mathrm{i}\lambda/4!)^2$ comes from the interaction Lagrangian. The Feynman rules in $d = 2\omega = 4-2\epsilon$ spacetime dimensions, including the scale factor $\mu^{2(4-2\omega)} = \mu^{4\epsilon}$ leads to

$$\begin{aligned}\Gamma^{(4)}_{\text{reg}}(s,t,u) =& -\mu^{2\epsilon}[A(s)+A(t)+A(u)] \text{ with}\\ -\mathrm{i}A(p^2) =& \frac{\lambda^2}{2}\mu^{4\epsilon}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{(m^2-l^2-m^2)[m^2-(l-p)^2]}.\end{aligned} \tag{5.90}$$

To apply formula (5.82) we introduce the *Feynman parameterisation* making use of

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x \frac{1}{[ax+b(1-x)]^2}. \tag{5.91}$$

Setting this into (5.90) we find

$$-\mathrm{i}A(p^2) = \int_0^1 \mathrm{d}x \frac{\lambda^2}{2}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{\mu^{2\epsilon}}{\{(1-x)(l^2-m^2)+x[(l-p)^2-m^2]\}^2}. \tag{5.92}$$

Introducing $l' = l - xp$ leads to

$$-\mathrm{i}A(p^2) = \frac{\lambda^2\mu^{2\epsilon}}{2}\int_0^1 \mathrm{d}x \int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{[l'^2+x(1-x)p^2-m^2]^2}, \tag{5.93}$$

and applying (5.82) gives

$$-\mathrm{i}A(p^2) = \frac{\lambda^2(4\pi\mu^2)^\epsilon}{32\pi^2}\int_0^1 \mathrm{d}x \frac{\Gamma(\epsilon)}{[m^2 - x(1-x)p^2]^\epsilon}. \tag{5.94}$$

Now using (5.55) and (5.58) this gives

$$-\mathrm{i}A(p^2) = \frac{\mathrm{i}\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon} - \gamma - \int_0^1 \mathrm{d}x \ln\left[\frac{m^2 - x(1-x)p^2 - \mathrm{i}\eta}{4\pi\mu^2}\right]\right\}, \tag{5.95}$$

where we have reintroduced explicitly the regulator $\mathrm{i}\eta$ from the propagator in order to define the analytic continuation of the integral from space like momenta p where it is analytic and uniquely defined. The remaining Feynman integral is finite and can be evaluated analytically to be

$$\begin{aligned} A(p^2) = -\frac{\lambda^2}{32\pi^2}\Bigg[&\frac{1}{\epsilon} - \gamma + 2 - \ln\left(\frac{m^2}{4\pi\mu^2}\right) \\ &- 2\sqrt{\frac{p^2 - 4m^2 + \mathrm{i}\eta}{p^2 + \mathrm{i}\eta}}\,\operatorname{artanh}\left(\sqrt{\frac{p^2 + \mathrm{i}\eta}{p^2 - 4m^2 + \mathrm{i}\eta}}\right)\Bigg]. \end{aligned} \tag{5.96}$$

For first investigation of the properties of this function we shall apply the so called *minimal subtraction* scheme which just subtracts the pole term at $\epsilon \to 0$ leading to the finite result

$$A_{\mathrm{MS}}(s) = -\frac{\lambda^2}{32\pi^2}\left[2 - \gamma - \ln\left(\frac{m^2}{4\pi\mu^2}\right) - 2\sqrt{\frac{p^2 - 4m^2 + \mathrm{i}\eta}{p^2 + \mathrm{i}\eta}}\,\operatorname{artanh}\left(\sqrt{\frac{p^2 + \mathrm{i}\eta}{p^2 - 4m^2 + \mathrm{i}\eta}}\right)\right] \tag{5.97}$$

for $\epsilon \to 0$. Thereby we have introduced a counterterm for the coupling constant[7]:

$$\delta A_{\mathrm{MS}} = \frac{\lambda^2}{32\pi^2}\frac{1}{\epsilon} \Rightarrow \delta\lambda_{\mathrm{MS}} = \frac{3\lambda^2}{32\pi^2}\frac{1}{\epsilon} \tag{5.98}$$

into the Lagrangian in order to render the 1-loop contribution to Γ^4 finite.

Now we can investigate the analytic structure of $A(s)$ around the real axis. In the physical sheet of the Riemannian surface this function is analytic for $s < 4m^2$ and has a branch cut along the real axis for $s > 4m^2$, i.e., at $s = 4m^2$ it has an essential singularity (branching point). For $s < 4m^2$ the function is real and from Schwarz' reflection principle this means along the cut we have $A_{\mathrm{MS}}(s - \mathrm{i}\eta) = A^*_{\mathrm{MS}}(s + \mathrm{i}\eta)$. The branching point $s = 4m^2$ marks the *two-particle threshold* which can be read off the Feynman diagram by the two propagators building the loop. Later in this chapter we shall investigate such analytic properties from first physical principles as are unitarity of the S-matrix and causality.

Now the unitarity of the S-matrix can only survive if the counter terms chosen to make the loop diagrams finite are real. Thus we can set the physical point at which we like to fix the physical coupling only below the threshold. For instance we can chose the symmetric point of the static limit, i.e., $s = t = u = 0$, which means to say $\Gamma^{(4)}(0,0,0,0) = 0$. We have only to introduce another counterterm to compensate for the contribution of the loop integral at this point leading to

$$A_{\mathrm{phys}}(s) = -\frac{\lambda^2}{16\pi^2}\left[1 + \sqrt{\frac{s - 4m^2 - \mathrm{i}\eta}{s + \mathrm{i}\eta}}\,\operatorname{artanh}\left(\sqrt{\frac{s + \mathrm{i}\eta}{s - 4m^2 + \mathrm{i}\eta}}\right)\right] \tag{5.99}$$

[7]Note that in the effective action all three diagrams in figure 5.5 dressed with mean fields appear. Thus the counter term for the coupling has an additional factor 3 compared to the single diagram.

with the counter term for the physical scheme

$$\delta A_{\text{phys}} = \frac{\lambda^2}{32\pi^2}\left[\frac{1}{\epsilon} - \gamma - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] \Rightarrow \delta\lambda_{\text{phys}} = 3\,\delta A_{\text{phys}}. \tag{5.100}$$

Note that here the dependence on the renormalisation scale μ has been moved from the finite part to the counterterm, i.e., (5.99) is independent from μ, while the counterterm becomes explicitly dependent on μ and (more importantly) also on the mass m.

5.4.3 Two-loop self-energy diagrams

To complete the renormalisation of ϕ^4 theory up to order λ^2 we have to consider the self-energy diagrams, which are two-loop diagrams. They are also interesting, because they are examples for diagrams with overlapping subdivergences, i.e., they contain one-loop subdiagrams that are divergent.

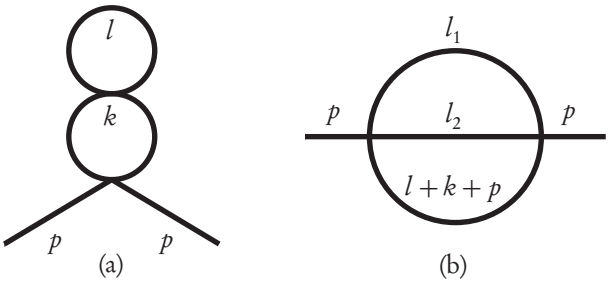


Figure 5.6: *The self-energy contributions at order λ^2.*

Let us start with the quite simple diagram (a) in Fig. 5.6. The Feynman rules lead to the dimensionally regularised expression

$$\begin{aligned}
\Sigma_a^{(2)} &= \frac{\lambda^2\mu^{4\epsilon}}{4}\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\int\frac{\mathrm{d}^{2\omega}l}{(2\pi)^{2\omega}}\frac{1}{(m^2-k^2)^2(m^2-l^2)} \\
&= \frac{\lambda^2\mu^{4\epsilon}}{4}\int\frac{\mathrm{d}^{2\omega}l}{(2\pi)^{2\omega}}\frac{\mathrm{i}\Gamma(\epsilon)}{m^{2\epsilon}}\frac{1}{m^2-l^2} \\
&= -\frac{\lambda^2 m^2}{1024\pi^4}\left(\frac{4\pi\mu^2}{m^2}\right)^{2\epsilon}\Gamma(\epsilon)\Gamma(-1+\epsilon) \\
&= \frac{\lambda^2 m^2}{1024\pi^4}\left[\frac{1}{\epsilon^2} - \left(2\gamma - 1 - 2\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right)\frac{1}{\epsilon}\right] + \text{finite}.
\end{aligned} \tag{5.101}$$

Here we have again used the standard formulae from Appendix C.

Let us now calculate the sun-set diagram (diagram (b) in Fig. 5.6). This calculation is introduced here to present an example for the non-trivial problems concerned with hiding infinities into the Feynman parameterisation which forces us to use some tricks during the calculation. It is also important to realise how important it has been to introduce the Wick rotation for the calculation of the standard integrals and to use the form with the negative propagators mentioned already above in the *very beginning* of the calculation because otherwise there we would have serious problems in the analytic continuation in the dimension due to factors $\propto (-1)^\epsilon$ which are only well defined for $\epsilon \in \mathbb{Z}$.

The Feynman rules lead to the self-energy contribution in dimensional regularisation

$$\Sigma_b^{(2)}(p) = \frac{\lambda^2\mu^{4\epsilon}}{6}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{m^2-k^2}\frac{1}{m^2-l^2}\frac{1}{m^2-(l+k+p)^2}. \tag{5.102}$$

As in our previous calculation of the one-loop vertex diagram we could introduce immediately Feynman-parameter integrals for the three Green's functions but this would cause a new sort or trouble, namely that we *hide divergences in the Feynman parameter integral.* Experience shows that the Feynman parametrization is usually unproblematic for logarithmically divergent integrals. Now from power counting it is clear that the self-energy should be of the general form

$$\Sigma_b^{(2)}(p) = p^2\tilde{\Sigma}_{b1}^{(2)}(p) + m^2\tilde{\Sigma}_{b2}^{(2)}(p), \tag{5.103}$$

where $\tilde{\Sigma}_{b1}$ and $\tilde{\Sigma}_{b2}$ are at most logarithmically divergent since these functions have energy dimension 1. To extract these functions from the Feynman integral we multiply the trivial identity

$$\partial_\mu^{(k)}k^\mu + \partial_\mu^{(l)}l^\mu = 2d = 4\omega \tag{5.104}$$

under the integral and then integrate by parts:

$$\Sigma_b^{(2)}(p) = -\frac{\lambda^2\mu^{4\epsilon}}{24\omega}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\left(k^\mu\partial_\mu^{(k)} + l^\mu\partial_\mu^{(l)}\right)\frac{1}{m^2-k^2}\frac{1}{m^2-l^2}\frac{1}{m^2-(l+k+p)^2}. \tag{5.105}$$

Differentiating explicitly and algebraic manipulations with some linear shifts of the integration variable, which is allowed because the integrals are dimensionally regularised, gives after combining the original integral on the right hand side of the equation with the left hand side:

$$\Sigma_b^{(2)}(p) = -\frac{\lambda^2\mu^{4\epsilon}}{6(2\omega-3)}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{3m^2-p(k+l+p)}{(m^2-k^2)(m^2-l^2)[m^2-(k+l+p)^2]^2}. \tag{5.106}$$

This enables us to split the integral into the desired two parts:

$$\Sigma_b^{(2)}(p) = -\frac{\lambda^2}{6(2\omega-3)}\left[3m^2\tilde{\Sigma}^{(1)}(p) + p^\mu\tilde{\Sigma}_\mu^{(2)}(p)\right]. \tag{5.107}$$

Let us start with the less complicated diagram

$$\begin{aligned} p^\mu\tilde{\Sigma}_\mu^{(2)} &= -\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{p(k+l+p)\mu^{4\epsilon}}{(m^2-k^2)(m^2-l^2)[m^2-(k+l+p)^2]^2} = \\ &= +\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{pk\mu^{4\epsilon}}{(m^2-k^2)^2(m^2-l^2)[m^2-(k+l+p)^2]}. \end{aligned} \tag{5.108}$$

In the last step we have substituted $-k' = k+l+p$ and renamed k' back to k afterwards. Now we introduce Feynman parameters for the integration over l which is the most divergent part of the integral. This is mainly the same calculation as we have done when we calculated the vertex diagram in Sect. 5.4.2. After a shift of the integration variable this integral reads

$$\begin{gathered} p^\mu\tilde{\Sigma}_\mu^{(2)}(p) = \int\frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{\mu^{4\epsilon}kp}{(M_1^2-l^2)^2(m^2-k^2)^2} \\ \text{with } M_1^2 = m^2 - x(1-x)(k+p)^2. \end{gathered} \tag{5.109}$$

Using (C.8) this gives

$$p^\mu \tilde{\Sigma}^{(2)}_\mu(p) = \frac{\mathrm{i}\Gamma(\epsilon)\mu^{4\epsilon}}{(4\pi)^\omega} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{kp}{M_1^{2\epsilon}(m^2-k^2)^2}. \tag{5.110}$$

Now we introduce another Feynman parameter for combining the denominator to the power of a propagator-like function. Thereby it is advantageous for the further calculation to take out a factor $[x(1-x)]^{-\epsilon}$:

$$p^\mu \tilde{\Sigma}^{(2)}_\mu(p) = \frac{\mathrm{i}\Gamma(2+\epsilon)}{(4\pi)^\omega} \mu^{4\epsilon} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{kp(1-y)y^{\epsilon-1}[x(1-x)]^{-\epsilon}}{\left[(m^2-k^2)(1-y) + y\frac{M_1^2}{x(1-x)}\right]^{2+\epsilon}}. \tag{5.111}$$

A shift from k to $k' = k - yp$ in the integration variable and using equations (C.8) and (C.9)[8]. So we end up with the parameter integral

$$\begin{aligned} p^\mu \tilde{\Sigma}^{(2)}_\mu(p) =& \frac{p^2}{(4\pi)^4}\Gamma(2\epsilon) \int_0^1 \mathrm{d}x [x(1-x)]^{-\epsilon} \int_0^1 \mathrm{d}y (1-y) y^\epsilon \left(\frac{4\pi\mu^2}{M_2^2}\right)^{2\epsilon} \\ & \text{with } M_2^2 = m^2\left(1 - y + \frac{y}{x(1-x)}\right) - y(1-y)p^2. \end{aligned} \tag{5.112}$$

Now we take the Laurent expansion of this expression around $\epsilon = 0$ (and now it indeed turns out to be important to have used the quasi-Euclidean sign convention for the propagators):

$$p^2 \tilde{\Sigma}^{(2)}_{b1} = -\frac{\lambda^2}{6(2\omega-3)} p^\mu \tilde{\Sigma}^{(2)}_\mu = -\frac{p^2}{(24\pi)^4 \epsilon} + \text{finite}. \tag{5.113}$$

It is quite easy to see that the Feynman parameter integrals over x and y are finite. But because these are rather complicated dilogarithmic functions (in the literature also known as *Spence's function*) we leave it for numerical integration where a trial shows that an adaptive integrator of low order is the right method.

Now let us turn to the first contribution. With exactly the same steps we have just done for the second contribution before we obtain

$$\begin{aligned} \tilde{\Sigma}^{(1)}(p) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mu^{4\epsilon}}{(m^2-k^2)(m^2-l^2)^2[m^2-(k+l+p)^2]} = \\ &= -\frac{\Gamma(2\epsilon)}{(4\pi)^4} \int_0^1 \mathrm{d}x [x(1-x)]^{-\epsilon} \int_0^1 \mathrm{d}y (1-y) y^{\epsilon-1} \left(\frac{4\pi\mu^2}{M_2^2}\right)^{2\epsilon}. \end{aligned} \tag{5.114}$$

This is a good example for the dangers of the Feynman parameterisation. The y-integral is divergent for $\epsilon \leq 0$ at the boundary $y = 0$ of the integral. This makes it impossible to Laurent expand around $\epsilon = 0$. But this is only a logarithmic divergence[9] and thus is cured easily by a partial integration giving an explicit factor $1/\epsilon$ as expected from the overlapping divergences. Now we can Laurent expand around $\epsilon = 0$ with the result

$$m^2 \tilde{\Sigma}^{(2)}_{b2} = -\frac{\lambda m^2}{2(2\omega-3)} \tilde{\Sigma}^{(1)}(p) = \frac{\lambda^2 m^2}{1024\pi^4} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(3 - 2\gamma + 2\ln\frac{4\pi\mu^2}{m^2}\right)\right] + \text{finite}. \tag{5.115}$$

The finite part is again left for numerical integration.

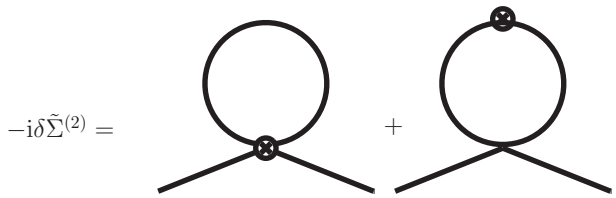


Figure 5.7: *The contributions to the self-energy at order $\mathcal{O}(\lambda^2)$ due to the $\mathcal{O}(\lambda)$ coupling-constant and mass counter-terms (symbolized by a crossed vertex point).*

At order λ^2 are now two more diagrams, which are necessary to subtract the one-loop divergences in the divergent subdiagrams. The diagrams are shown in Fig. 5.7.

The evaluation of the divergent pieces yield

$$\delta\tilde{\Sigma}^{(2)}_{\mathrm{MS}} = \frac{\lambda^2 m^2}{1024\pi^4}\left[-\frac{4}{\epsilon^2}+\frac{1}{\epsilon}\left(-3+4\gamma-4\ln\frac{4\pi\mu^2}{m^2}\right)\right]+\text{finite}. \tag{5.116}$$

To obtain this result we used the usual rules to evaluate the Feynman diagrams, using the $\mathcal{O}(\lambda)$ counterterms (5.98) and (5.88). Adding now (5.101), (5.113) and (5.116) we obtain the remaining "overall" divergent part of the two-loop self-energy,

$$\tilde{\Sigma}^{(2)} = \frac{\lambda^2}{1024\pi^4}\left(-\frac{2m^2}{\epsilon^2}+\frac{m^2}{\epsilon}-\frac{p^2}{6\epsilon}\right)+\text{finite}. \tag{5.117}$$

As we see, thanks to the contributions from the diagrams involving the $\mathcal{O}(\lambda)$ counterterms, the divergent part is no longer explicitly dependent on the renormalization-scale parameter μ and the $\delta Z_{m2,\mathrm{MS}} = \delta m^2_{2,\mathrm{MS}}/m^2$ is also independent of the mass m. The same holds true for the wave-function counterterm:

$$\delta m^2_{2,\mathrm{MS}} = \delta Z_{m2,\mathrm{MS}} m^2, \quad \delta Z_{m2,\mathrm{MS}} = \frac{\lambda^2}{1024\pi^4}\left(\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\right), \quad \delta Z_{2,\mathrm{MS}} = -\frac{\lambda^2}{6144\pi^4\epsilon}. \tag{5.118}$$

As we shall see in the reminder of this chapter concerning renormalization, this mass independence of the dimensionless scaling parameters of the various counterterms holds to any order, i.e., we shall show that it is always possible to choose such a "mass-independent renormalization scheme". This is of great advantage to analyse the independence of the physical content of renormalized perturbation theory from the renormalization scheme and particularly the renormalization scale: The coupling constant, wave-function normalization constant and mass-renormalization constant, must be chosen as dependent on the renormalization scale μ such that the physical quantities do not change under a change of the renormalization scale. This will lead us to the renormalization-group equations in Sect. 5.11.

The mass independence of the minimal-subtraction scheme, which is defined via the dimensional regularisation of the loop integrals, is not obvious, but will become clear in terms of the general analysis of the divergent diagrams and the proper subtraction of the subdivergences before determining the overall divergence of the diagram, which provide the counterterms at the considered order of perturbation theory in terms of the BPHZ formalism in Sect. 5.8. Then we shall

[8] the last one only to show that the integral with pk' in the numerator vanishes because there we have the case $q = 0$ in (C.9).

[9] which was the reason for introducing the "trick" (5.104 in (5.105)

also see that after the proper subtraction of subdivergences the counterterms renormalizing the overall divergences are of the form in the original Lagrangian, i.e., one has only a wave-function, mass-renormalization, and a coupling-constant counterterm. Particularly the counterterms are of the same local structure (e.g., there are no divergent terms $\propto \ln(p^2/\mu^2)$). Heuristically we can understand this feature of the MS scheme by the fact that the here considered divergences are all UV divergences, i.e., they occur from the divergence of the loop integrals for large loop momenta. In this region any finite energy-momentum-mass scales are irrelevant, and thus the divergences are also independent of these scales.

Of course, there are arbitrarily many other renormalization schemes that are not mass-independent schemes. An example is what we called "physical renormalization scheme", where the both the mass-renormalization scale and the coupling-constant counterterm became mass dependent, cf. (5.89) and (5.100. Although such schemes are as legitimate as any other scheme, the corresponding renormalization-group equations, describing the running of the renormalized parameters with the renormalization sacle become more complicated. Sometimes such schemes become also problematic, when massless particles are involved in the theory or occur due to spontaneous symmetry breaking as Nambu-Goldstone modes (see Sect. 5.10). In such cases the choice of a mass-independent renormalization scheme is preferable. As we have also seen, the choice of dimensional regularisation and the minimal-subtraction scheme is particularly convenient in perturbative calculations, because it might be much simpler to just extract the divergent pieces to determine the counterterms than to evaluate the complicated finite parts of the diagram. Sometimes one also subtracts some finite constant terms in addition to the poles in ϵ. This, of course, doesn't change the advantageous features of the MS renormalization scheme. One calls such schemes the modified minimal-subtraction or $\overline{\text{MS}}$ schemes. Such an $\overline{MS}$ scheme is usually applied when dealing with Quantum Chromodynamics, the non-Abelian gauge theory describing the strong interaction between quarks and gluons.

5.5 Power counting

Our experience from the previous section lets us now look for the systematic proof of renormalisability to all orders of perturbation theory. We shall again take ϕ^4-theory as the most simple example of a renormalisable theory.

At first we look on the so called *superficial degree of divergence*. This is obtained by simply counting the powers of loop momenta within the Feynman integral Γ in d-dimensional space time. A diagram with L loops yields an integral over dL momenta. Each internal line stands for a propagator which gives a power of $-2I$ (where I is the number of internal lines). The whole integral has thus a momentum power

$$D_s(d,\Gamma) = Ld - 2I. \tag{5.119}$$

For the convergence of Γ it is obviously *necessary* but by no means sufficient that $D_s(\Gamma) < 0$.

The diagrams in the previous section showed that of course the divergent part had the power law in the external momenta as expected by this simple power counting method but the finite part contains non-trivial logarithms. The powers in momenta where ≤ 2 for $d = 4$. Now it is evident that it is necessary for a theory to be renormalisable that the degree of divergence is negative and that the infinite part can be subtracted with help of a counter term which is of the same form as monomials of fields and its derivatives already contained in the Lagrangian. This means

that necessarily a renormalisable theory can contain only interaction monomials such that the superficial degree of divergence for proper vertex-functions is positive only for a finite set of such vertex functions and exactly those which are already in the Lagrangian.

Now we want to show that ϕ^4 is fulfilling this necessary requirements. This is simply done by substituting E and L in (5.119) instead of I. The conditions are fulfilled if only the 2- and 4-point 1PI vertex functions are superficially divergent. It does not matter if the divergences arise at all orders perturbation theory but the counter terms should only contain polynomials of order $O(p^2)$ for the 2-point function and only of order $O(p^0)$ for the 4-point vertex. The 3-point vertex should be finite and also all n-point vertices for $n \geq 5$. Due to the symmetry under $\phi \to -\phi$ the 3-point vertex vanishes at all.

Now we have to count the number of internal lines in terms of the number of loops and external lines. From momentum conservation at each vertex we have $I - V$ independent momenta but 1 condition is already fulfilled by conservation of the external momenta (the sum of all momenta running into or out of the vertex diagram has to be 0), thus we have

$$L = I - V + 1. \tag{5.120}$$

While (5.119) and (5.120) are valid for any bosonic quantum field theory now we have to use the fact that each vertex of ϕ^4-theory contains exactly 4 legs leading to $I = (4V - E)/2$ (each of the $4V$ legs is connected to an external point linked with another leg. The external legs do not contribute to the internal lines, each of which is the linking of two fields). This leads to the following expression for the superficial degree of divergence

$$D_S^{(d)}(\Gamma) = (d-4)V + d + \left(1 - \frac{d}{2}\right) E. \tag{5.121}$$

For $d = 4$ this reads $D_S^{(4)}(\Gamma) = 4 - E$. This means that the superficial degree of divergence is negative for $E \geq 5$, i.e., the first condition is fulfilled. Now we have to count the naive powers of momentum for the vertex function.

An n-point vertex function has the same naive momentum power as a coupling constant in front of a (fictive or real) ϕ^n-contribution in the Lagrangian. In our system of units the action has dimension $O(p^0)$ and thus from the kinetic term $\partial_\mu \phi \partial^\mu \phi$ we read off that $\phi = O(p^{(d-2)/2})$. This shows that an E-point proper vertex function has the naive dimension $O(p^{E-n(E/2-1)}) = O(p^{D^{(d)}(\Gamma)})$. Thus for $d = 4$ the ϕ^4-theory is really superficially renormalisable because for $E = 2$ the naive momentum power is 2 (we have a mass counter term there to absorb the infinities into the mass) and for $E = 4$ the power is 0 and this gives rise to the counterterm absorbing the infinities into the bare coupling.

As we have always emphasised this ideas are not complete. The so far developed power counting arguments are only necessary but not sufficient conditions for renormalisability. Although a diagram may have a negative naive degree of divergence it need not be finite and even worse the infinities need not be polynomial in the external momenta which seems to introduce non-local interactions into the Lagrangian. It is not a difficult task to explain how this comes about and how this problem can be solved while the mathematical proof is a hard stuff.

So let us first give a heuristic argument how to treat these problems practically.

First take a superficially finite diagram namely the 6-point vertex of which some contributions are shown in fig. 5.8. The first diagram is primitive, i.e. it does not contain any 1PI subdiagrams which can diverge. It is clear how to calculate this diagram qualitatively: One introduces a

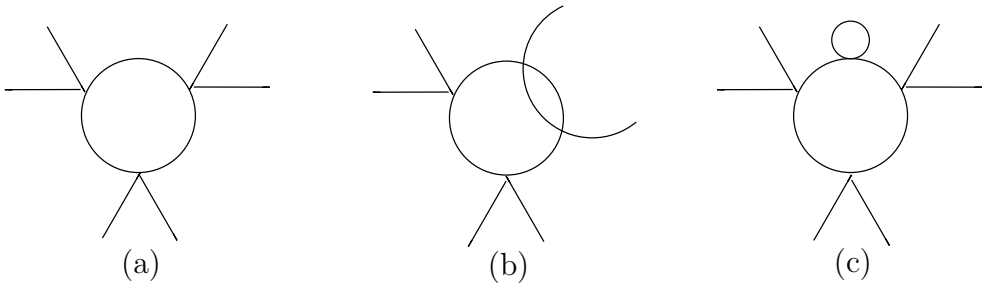


Figure 5.8: *An example for a convergent (a) and two superficially convergent but in fact divergent diagrams*

Feynman-parameter and treats the diagram making use of the standard formulas of appendix C. The result is finite by naive power counting.

But now look on diagram (b) which contains a 1-loop four-point-vertex sub-diagram which we have calculated in just some paragraphs above. This diagram is divergent, but we have also to add a counter-term diagram making use of our result of the counter-term at least in the MS-scheme. This cancels the divergencies which are not local, i.e., which are not polynoms of p^2 which look awkward on the fist look because this would spoil the idea of renormalising the divergencies with local counter terms in the effective quantum action and absorb them to the bare parameters. The same is of course true for diagram (c). The diagram with the sub-divergences subtracted is finite due to our naive power counting. If it was divergent there would remain only local over-all divergences which can be renormalised by absorbing them into the bare parameters of the theory.

But this sort of divergences is not the worst one! In these examples there was only one divergent sub-diagram which could be subtracted unambiguously. The next example, depicted in fig. 5.9 is an example of so called *overlapping divergences*.

Any of the bold lines in the diagrams indicates a sub-divergence. Looking on diagrams (a)-(c) each pair of these divergent sub-diagrams have one line in common. This is what is known as *overlapping* divergences and realizing that there are overlapping divergences also in QED threatened the field theorists of the old days this could spoil the renormalisability completely. To keep the reader calm we mention the solution to this problem. Just don't worry if the divergences are overlapping or not and add the three counter terms from the one-loop four-point result obtained above and magically all difficulties vanish because this procedure leaves us with an overall divergence which is *local*. This can be subtracted and put to the bare wave function normalisation factor and the bare mass of the Lagrangian. This solution also shows that in this case of the setting-sun diagram there are no non-local divergences because the subdivergences are subtracted by tadpole diagrams not dependent on the external momentum.

This is also an example for the most important point of the BPHZ formalism which systematically shows that these conjectures are correct. Although it is a rather hard thing to calculate this diagram we shall treat it as the next example for the power of dimensional regularisation to have a fully calculated example with overlapping divergences at hand when switching to a detailed description of the BPHZ renormalisation.

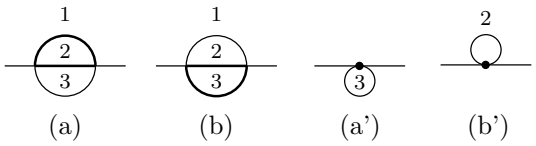


Figure 5.9: *The setting sun diagram (of which we have drawn 2 copies to indicate the sub-divergences by bold lines) in ϕ^4 theory as an example for overlapping sub-divergences: The two divergent sub-diagrams indicated by the bold lines in the diagrams (a) and (b) have one propagator line in common. Additionally the diagram has also an overall divergence indicated by the box around the entire diagram which we have not depicted here. As results from our calculation in the next section by adding the counter terms (a') and (b') for the sub-divergences one ends up with* a local *overall divergence which can be absorbed into the bare parameters of the theory. The dot in the diagrams (a'), (b') stands for the one-loop counter term of a single "dinosaur subdiagram" (which is* $1/3$ *of the counter term to the total one-loop contribution). The rule for such diagrams is to set in the local counter term instead of the sub-diagram, keeping all the rest as is in the full diagram. In our case this rule shows that each of the counter terms (a') and (b') stands for* $-\mathrm{i}\delta\lambda\mathrm{i} \times \int G(l)/3$. *As is shown below this cancels the leading divergence of* $1/\epsilon^2$ *in dimensional regularisation.*

5.6 Weinberg's Theorem

Beginning with this section we prove systematically the renormalisability for superficially (Dyson-) renormalisable quantum field theories without gauge symmetries.

Our experience from the few examples of Feynman diagrams we have calculated above leads to the conjecture that a theory is renormalisable by a finite number of local counterterms if there are only a finite class of diagrams superficially divergent, i.e., if all coupling constants are of a momentum dimension greater or equal to 0. For calculating the overall local counterterm of a given diagram one has first to identify and subtract the divergences of subdiagrams.

The program of our proof will be as follows:

(1) Weinberg's theorem: This theorem states in a mathematical precise way what is meant by divergences and subdivergences and gives a precise criterion when a diagram is convergent.

As an additional result we shall extract the asymptotic behaviour of Feynman diagrams for large space like momenta.

(2) BPH-Renormalisation: Then we set up a recursive definition for the renormalisation procedure and prove the locality of all counterterms of a theory and give the criterion for renormalisability of a theory.

One of the major advantages of the BPH-scheme compared to older schemes is that there are no problems with overlapping divergences.

Further it is shown that the renormalised theory is independent of the regularisation scheme (which we have chosen to be the dimensional regularisation for convenience).

(3) BPHZ-Renormalisation: The final step will be the explicit solution of the recursive BPH-scheme by Zimmermann, the famous *forest formula*.

We should also emphasise that for QED or the whole standard model (or other gauge theories) this is only the first step of the proof that these physically most important theories are renormalisable because for them we have in addition to prove that the counterterms are gauge invariant. This topic will be treated in the next chapter where we set up the mathematical foundation of the standard model and where we shall treat the problem of symmetries and renormalisation.

But now let us start with the formulation of Weinberg's theorem. In rough words we like to show that a Feynman diagram[10] gives a finite result if its superficial degree of divergence as well as the superficial degree of divergence of all its subdiagrams is negative is in fact convergent. But for this purpose we have to give a mathematically feasible criterion for the degree of divergence for the subdiagrams as well as the whole diagram.

The problem is that one has to read off this from the integrand of the Feynman diagram where one can go to infinity with the loop momenta independently from each other as well as with more than one momentum to infinity at different or the same orders for each momentum.

We shall give here Weinberg's original formulation and proof of his theorem [Wei60].

Definition 1. *Let $f : \mathbb{R}^n \to \mathbb{C}$. Then f is said to be in the class A_n if for any*

$$S = span\{\vec{L}_1, \ldots, \vec{L}_m\}$$

[10] Of course it is enough to show this for the Wick rotated diagrams, i.e., in Euclidean quantum field theory. The Minkowskian version of a finite diagram is given by analytic continuation to time like momenta.

with $m \leq n$ independent $\mathbb{R}^n$-vectors and for any compact region $W \subseteq \mathbb{R}^n$ exist numbers

$$\alpha(span\{\vec{L}_1, \cdots, \vec{L}_k\}), \quad \beta(\vec{L}_1, \cdots, \vec{L}_k)$$

such that for every $\vec{C} \in W$:

$$\begin{aligned} f(\vec{L}_1\eta_1 \cdots \eta_m + \vec{L}_2\eta_2 \cdots \eta_m + \cdots + \eta_m\vec{L}_m + \vec{C}) &\underset{\eta_1,\ldots,\eta_m\to\infty}{\cong} \\ \underset{\eta_1,\ldots,\eta_m\to\infty}{\cong} O\left[\prod_{k=1}^{m} \eta_k^{\alpha(span\{\vec{L}_1,\ldots,\vec{L}_k\})} \prod_{j=1}^{m} (\ln \eta_j)^{\beta(\vec{L}_1,\cdots,\vec{L}_j)}\right]. \end{aligned} \tag{5.122}$$

The numbers α and β are called the asymptotic coefficients.

Further we define for any set of independent vectors $\{\vec{L}'_1, \ldots, \vec{L}'_k\}$ the integral

$$f_{\vec{L}'_1,\ldots,\vec{L}'_k}(\vec{P}) = \int_{-\infty}^{\infty} \mathrm{d}y_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}y_k f(\vec{P} + \vec{L}'_1 y_1 + \cdots + \vec{L}'_k y_k). \tag{5.123}$$

to be "existent" if it is absolutely convergent with respect to the iterated integrations as they are written down in the equation. Then Fubini's theorem immediately proves that this integral is dependent up to a factor on $I = \mathrm{span}\{\vec{L}'_1, \ldots, \vec{L}'_k\} \preceq \mathbb{R}^n$[11] not on the specific vectors $\vec{L}'_j$. This factor is due to the Jacobi determinant of the basis transformations from one set of basis vectors for I to another. Thus $f_{\vec{L}'_1,\ldots,\vec{L}'_k}$ exists if and only if the integral over the subspace I

$$f_{\vec{L}'_1,\ldots,\vec{L}'_k}(\vec{P}) = \int_I \mathrm{d}^k\vec{P}' f(\vec{P} + \vec{P}') \tag{5.124}$$

exists, and both integrals have the same asymptotic behaviour.

Further it is clear that the integral (5.124) depends only on the complement of I, i.e., for any $\vec{L} \in I$ we have $f_I(\vec{P} + \vec{L}) = f(\vec{P})$. Thus it is convenient and appropriate for the application of the theorem to let $\mathbb{R}^n = I \oplus E$ where $I, E \preceq \mathbb{R}^n$ with $I \cap E = \{0\}$.

In perturbative Euclidean QFT this is a generalisation of the typical situation that one has to calculate a k-dimensional integral over the internal loop momenta covering the subspace I with external momenta spanning the subspace E of $\mathbb{R}^n$. Our present formulation is in turn more convenient for the general proof of the theorem and because of the above stated equivalence of the convergence and asymptotic behaviour of the integrals (5.123) and (5.124) we can switch between these formulations whenever this is convenient for the proof.

To close this section of definitions before we finally state the theorem we have to define the "projection along a subspace": Let $S \preceq \mathbb{R}^n$ and S' its disjoint complement such that $\mathbb{R}^n = S \oplus S'$. Then for any vector $\vec{V} \in \mathbb{R}^n$ there is a unique decomposition $\vec{V} = \vec{L}_1 + \vec{L}_2$ with $\vec{L}_1 \in S$ and $\vec{L}_2 \in S'$. Then we define the projection of $\vec{V}$ along S to be $\Lambda(S)\vec{V} = \vec{L}_2$.

Further for $S'' \preceq \mathbb{R}^n$ we define the projection of $A \preceq \mathbb{R}^n$ along $B \preceq \mathbb{R}^n$ to be

$$\Lambda(B)A = \mathrm{span}\{\Lambda(B)\vec{V} | \vec{V} \in A\}.$$

Now we can state Weinberg's theorem:

[11] Here and in the following $A \preceq B$ means that A is a subspace of B.

Theorem 4. *Let $f \in A_n$ where $f : \mathbb{R}^n \to \mathbb{C}$ with the asymptotic coefficients $\alpha(S)$ and $\beta[\mathcal{B}(S)]$*[12]*. Further let f be integrable over any finite region of $I \preceq \mathbb{R}^n$ in the sense of (5.124) and let*

$$D_I = \max_{S' \preceq I}[\alpha(S') + dim S'] < 0. \tag{5.125}$$

We call D_I the real degree of divergence *of the diagram.*
Then the following holds for the integral f_I defined in (5.124):

(a) f_I is absolutely convergent (in short we say "it exists").

(b) $f_I \in A_{n-k}$ where the asymptotic coefficients for any $S \preceq E$ is given by

$$\alpha_I(S) = \max_{\Lambda(I)S'=S}[\alpha(S') + dim S' - dim S] \tag{5.126}$$

Before we go into the rather lengthy (but elementary) proof we comment on this with respect to the application of the theorem to perturbative QFT: Clearly D_I is the greatest degree of divergence of any subdiagram contained in the diagram and part (a) is precisely the statement we need for the convergence of a diagram and (a) is thus enough for the general proof of divergence of the renormalised theory as we shall have to show below while for (b) we have to work a little more but it has practical importance for the *operator product expansion* which is needed to draw empirically testable conclusions from QCD for *deep inelastic scattering.*

Weinberg's example

Here we want shortly refer to the example given by Weinberg in the above cited paper. This is very simple but nevertheless also illuminating. It is a typical "diagram" in Euclidean $1+0$-dimensional Yukawa theory, namely

$$\Sigma(p') = \int_{-\infty}^{\infty} \mathrm{d}p'' \underbrace{\frac{p''}{(p''^2 + m^2)[(p'' - p')^2 + \mu^2]}}_{f(p',p'')}. \tag{5.127}$$

Here we have $\mathbb{R}^2$ as the total vector space and $E = \mathrm{span}\{(1,0)\}$ and $I = \mathrm{span}\{(0,1)\}$. Now we want to determine the asymptotic coefficients for f. Of course the logarithmic coefficients β are 0 while we have to determine the α carefully. We start with $\alpha(I)$ which is important for the question of convergence according to (a) of the theorem. To determine $\alpha(I)$ we have to use (5.122) with only one vector $\vec{L} = (0,1)$. We read off $\alpha(I) = -3$ immediately. Since I is one dimensional the only non-trivial subspace $S' \preceq I$ is I itself and thus according to (5.125) the real degree of divergence is $D_I = \alpha(I) + \dim I = -3 + 1 = -2 < 0$ and according to (a) the integral Σ exists because the integral over any finite interval with respect to p'' exists because f is a continuous function in p'' for any p'. To determine the asymptotic behaviour of Σ according to (b) we have to calculate $\alpha_I(S)$ for any subspace $S \preceq E$. Now because E is one-dimensional the only non-trivial subspace is E itself. To calculate $\alpha_I(E)$ we have in turn to calculate $\alpha(S')$ for any $S' \preceq \mathbb{R}^2$ with $\Lambda(I)S' = E$. Of course it is immediately clear that S' can be the whole $\mathbb{R}^2$ and any line different from I. For $S'' = \mathbb{R}^2$ we have only to calculate the behaviour of $f(\eta_2(\eta_1\vec{L_1} + \vec{L_2})$ with respect to $\eta_2 \to \infty$ for any $\eta_1 > 1$ and independent $\vec{L}_1, \vec{L}_2 \in \mathbb{R}^2$ fixed. This

[12] Here we are a little bit more precise than Weinberg, because the "logarithmic coefficients" may depend on the basis $\mathcal{B}(S)$ chosen for the subspace S but this does not change anything for the reasoning of the proof.

gives $O(\eta^{-3})$ and thus $\alpha(\mathbb{R}^2) = -3$. For the $\alpha(S')$ with S' any line different from I we have to look at $f(\eta_1 \vec{L} + \vec{C})$ for any $\vec{C}$ fixed and $\vec{L} \notin I$ which gives

$$\alpha(S') = \begin{cases} -2 & \text{for } S' = \text{span}\{(1,0)\} \\ -1 & \text{for } S' = \text{span}\{(1,1)\} \\ -3 & \text{for all other lines different from } I. \end{cases} \tag{5.128}$$

Thus according to (5.126) we have $\alpha_I(E) = -1$.
An explicit integration gives indeed

$$\Sigma(p') = \frac{\pi p'}{\mu[(\mu+m)^2 + p'^2]} \underset{p' \to \infty}{\cong} O(p'^{-1}). \tag{5.129}$$

5.6.1 Proof of Weinberg's theorem

The proof of Weinberg's theorem is obtained in three steps:

(A) Step of induction: We show that if the theorem is true for $\dim I \leq k$ where $k \geq 1$ then it is true for $\dim I = k + 1$.

(B) Convergence for $\dim I = 1$: We prove that the integral over an arbitrary one-dimensional subspace I is absolutely convergent provided $D_I < 0$.

(C) Covering the one-dimensional subspace I with a finite number subintervals J each of which contributes a certain asymptotic behaviour of the integral.

(D) The sum over the subintervals constructed in (C) is a function of the class A_{n-1} with α_I given by (5.126).

The convergence theorem, i.e., conjecture (a) of the theorem is proven with (A) and (B) which is the simple part of the proof.

Ad (A): Let the theorem be true for any subspace $I \preceq \mathbb{R}^n$ with $\dim(I) \leq k$.

Now let I be a subspace with $\dim I = k + 1$. Let S_1 and S_2 be arbitrary non-trivial disjoint subspaces of I with $I = S_1 \oplus S_2$ (there are arbitrary many such splittings of I!). Then necessarily $\dim S_1 + \dim S_2 = \dim I = k + 1$ and because S_1 and S_2 are both non-trivial they are both of a dimension less then k. By the induction hypothesis the theorem holds for these subspaces. Thus we apply it twice making use of Fubini's theorem:

(a1) f_{S_2} converges absolutely if $D_{S_2}(f) < 0$ where

$$D_{S_2}(f) = \max_{S'' \preceq S_2} [\alpha(S'') + \dim S'']. \tag{5.130}$$

(b1) If $D_{S_2}(f) < 0$ then $f_{S_2} \in A_{n-k_2}$, $k_2 = \dim S_2$, with

$$\alpha_{S_2}(S') = \max_{\Lambda(S_2) S'' = S'} [\alpha(S'') + \dim S'' - \dim S']. \tag{5.131}$$

(a2) If $f_{S_2} \in A_{n-k_2}$ cf. (b1) then f_I converges absolutely if

$$D_{S_1}(f_{S_2}) = \max_{S' \preceq S_1} [\alpha_{S_2}(S') + \dim S'] < 0. \tag{5.132}$$

(b2) If $f_{S_2} \in A_{n-k_2}$ and $D_{S_1}(f_{S_2}) < 0$ then $f_I \in A_{n-k-1}$ with

$$\alpha_I(S) = \max_{\Lambda(S_1)S'} [\alpha_{S_2}(S') + \dim S' - \dim S]. \tag{5.133}$$

(a1) and (a2) together mean that f_I is absolutely convergent if

$$D'_I(f) = \max\{D_{S_2}(f), D_{S_1}(f_{S_2})\} < 0 \tag{5.134}$$

and this can be written as

$$D'_I(f) = \max_{S''}{}^* [\alpha(S'') + \dim S''] \tag{5.135}$$

where $\max^*$ indicates that S'' has to be a subspace of S_2 or such subspaces of $\mathbb{R}^n$ that $\Lambda(S_2)S'' \preceq S_1$. To show that $D'_I(f) = D_I(f)$ we have to prove that this means that S'' is running in fact over all subspaces of $I = S_1 \oplus S_2$.

Let $S' \preceq I$ but not a subspace of S_2. Then let $\vec{V} \in S'$. Because $I = S_1 \oplus S_2$ there exist vectors $\vec{L}_1 \in S_1$ and $\vec{L}_2 \in S_2$ such that $\vec{V} = \vec{L}_1 + \vec{L}_2$. So we have $\Lambda(S_2)\vec{V} = \vec{L}_1 \in S_1$ and thus $\Lambda(S_2)S' \preceq S_1$. Thus since S'' in $\max^*$ runs over all subspaces of S_2 or such subspaces of $\mathbb{R}^n$ for which $\Lambda(S_2)S'' \preceq S_1$ in fact S'' runs over all subspaces of I and thus

$$D'_I(f) = \max_{S'' \preceq I} [\alpha(S'') + \dim S''] = D_I(f) \tag{5.136}$$

Thus we have shown claim (a) of the theorem for f_I.

To show (b) out of the induction hypothesis we have to combine (b1) and (b2) which immediately show that $f_I \in A_{n-k-1}$. Thus we have only to prove (5.126). From (b1) and (b2) we know that

$$\begin{aligned} \alpha_I(S) &= \max_{\Lambda(S1)S'=S} \left[\dim S' - \dim S + \max_{\Lambda(S_2)S''=S'} [\alpha(S'') + \dim S'' - \dim S'] \right] = \\ &= \max_{\Lambda(S_1)S'=S; \Lambda(S_2)S''=S'} [\alpha(S'') + \dim S'' - \dim S]. \end{aligned} \tag{5.137}$$

To complete the induction step we have just to show that we can combine the both conditions for S'' to $\Lambda(I)S'' = S$.

Thus we have to show the equivalence

$$\left[\Lambda(S_2)S'' = S' \wedge \Lambda(S_1)S' = S\right] \Leftrightarrow S = \Lambda(S_1 \oplus S_2)S'' \tag{5.138}$$

Let $S = \mathrm{span}\{\vec{L}_1, \ldots, \vec{L}_r\}$ then $S = \Lambda(S_1)S'$ means that in the above maximum we can restrict S' to such subspaces for which exist $\vec{L}'_1, \ldots, \vec{L}'_r \in S_1$ such that $S' = \mathrm{span}\{\vec{L}_1 + \vec{L}'_1, \ldots, \vec{L}_r + \vec{L}'_r\}$. The same argument gives $S'' = \mathrm{span}\{\vec{L}_1 + \vec{L}'_1 + \vec{L}''_1, \ldots, \vec{L}_r + \vec{L}'_r + \vec{L}''_r\}$ if $\Lambda(S_2)S'' = S'$ with vectors $\vec{L}''_1, \ldots, \vec{L}''_r \in S_2$. But since S_1 and S_2 are disjoint subspaces this is nothing else than the statement that S'' runs over all subspaces with $\Lambda(S_1 \oplus S_2)S'' = S$. This proves (A).

Ad (B): Now we like to show (a) for a one-dimensional subspace. Thus we set $I = \mathrm{span}\{\vec{L}\}$ and look on

$$f_{\vec{L}}(\vec{P}) = \int_{-\infty}^{\infty} f(\vec{P} + y\vec{L}). \tag{5.139}$$

Since $f \in A_n$ by definition

$$f(\vec{P} + \vec{L}y) \underset{y\to\infty}{\cong} O\left(y^{\alpha(I)} (\ln y)^{\beta(\vec{L})}\right) \tag{5.140}$$

This integral converges absolutely if $\alpha(\vec{L})+1<0$ if it exists for any finite integral wrt. y. Now the only non-trivial subset of I is I itself because $\dim I=1$ and thus $D_I(f)=\alpha(I)+1$ which proves part (a) completely.[13]

Ad (C): The rest of the proof is devoted to show that the integral f_I of a function $f\in A_n$ over a one-dimensional subspace $I=\mathrm{span}\{\vec{L}\}$ is of class A_{n-1} provided $D_I(f)<0$.

To show this we use the form (5.123), i.e. we have to study the asymptotic behaviour of

$$f_{\vec{L}}(\vec{P})=\int_{-\infty}^{\infty}\mathrm{d}y f(\vec{P}+y\vec{L}) \tag{5.141}$$

which is the same as that of the function f_I defined by (5.124).

Let $\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_m\}\preceq\mathbb{R}^n$ where the $\vec{L}_j$ for $j=1,\ldots,m$ build a linearly independent set of vectors which are also linearly independent from $\vec{L}$, and $W\subseteq\mathbb{R}^n$ compact. Then we have to show that

$$f_{\vec{L}}(\vec{P})\underset{\eta_1,\ldots,\eta_m\to\infty}{\cong}O\left(\prod_{k=1}^{m}\eta_k^{\alpha_{\vec{L}}(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_k\})}(\ln\eta_k)^{\beta_{\vec{L}}(\vec{L_1},\ldots,\vec{L}_k)}\right) \tag{5.142}$$

with the asymptotic coefficients

$$\alpha_{\vec{L}}(S)=\max_{\Lambda(I)S'=S}[\alpha(S')+\dim S'-\dim S]. \tag{5.143}$$

For this purpose we try to split the integration range $\mathbb{R}$ in (5.141) in regions of definite asymptotic behaviour of the integral. Since we have to find only an upper bound of the integral it is enough to construct a finite coverage which needs not necessarily to be disjoint.

We look on the set of vectors

$$\vec{L}_1+u_1\vec{L},\vec{L}_2+u_2\vec{L},\ldots,\vec{L}_r+u_r\vec{L},\vec{L},\vec{L}_{r+1},\ldots,\vec{L}_m \tag{5.144}$$

with $0\le r\le m$ and $u_1,\ldots,u_r\in\mathbb{R}$.

Now we have to write out what it means that $f\in A_n$ for this set of vectors: There exist numbers $b_l(u_1,\ldots,u_r)>(0\le l\le r)$ and $M(u_1,\ldots,u_r)>0$ such that for all $\eta_l>b_l(u_1,\ldots,u_r)$ and $\vec{C}\in W$ the function fulfils the inequality:

$$\begin{aligned}|f[(\vec{L}_1+u_r\vec{L})\eta_1\cdots\eta_m\eta_0+\cdots+(\vec{L}_r+u_r\vec{L})\eta_r\cdots\eta_m\eta_0+\vec{L}_{r+1}\eta_{r+1}\cdots\eta_m+\cdots+\vec{L}_m\eta_m+\vec{C}]|\\ \le M(u_1,\ldots,u_r)\prod_{k=1}^{r}\eta_k^{\alpha(\mathrm{span}\{\vec{L}_1+u_1\vec{L},\ldots,\vec{L}_k+u_k\vec{L}\})}(\ln\eta_k)^{\beta(\vec{L}_1+u_1\vec{L},\ldots,\vec{L}_k+u_k\vec{L})}\times\\ \times\eta_0^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_r,\vec{L}\})}\ln(\eta_0)^{\beta(\vec{L}_1+u_1\vec{L},\ldots,\vec{L}_r+u_r\vec{L},\vec{L})}\times\\ \times\eta_{r+1}^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_{r+1},\vec{L}\})}(\ln\eta_{r+1})^{\beta(\vec{L}_1+u_1\vec{L},\ldots,\vec{L}_{r+1}+u_{r+1}\vec{L},\vec{L})}\times\cdots\times\\ \times\eta_m^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_m,\vec{L}\})}(\ln\eta_m)^{\beta(\vec{L}_1+u_1\vec{L},\ldots,\vec{L}_r+u_r\vec{L},\vec{L}_{r+1},\ldots,\vec{L}_m,\vec{L})}.\end{aligned} \tag{5.145}$$

Now each $u\in[-b_0,b_0]$ is contained in a closed interval $[u-b_1^{-1}(u),u+b_1^{-1}(u)]$. Because $[-b_0,b_0]$ is compact in $\mathbb{R}$ by the Heine-Borel theorem one can find a finite set of points $\{U_i\}_{i\in J_1}$ (J_1 a

[13] The reader who intends just to understand the convergence conjecture (a) for renormalisation theory may stop here because this part is now completely proved. The rest of the section is devoted to the proof of the asymptotic behaviour for a one dimensional integration space.

finite index set) such that $|U_1| < b_0$ and $0 < \lambda_i \leq b_1^{-1}(U_i)$ such that

$$\bigcup_{i\in J_1} [U_i - \lambda_i, U_i + \lambda_i] = [-b_0, b_0]. \tag{5.146}$$

Next take $i \in J_1$ and the closed interval $[-b_0(U_i), b_0(U_i)$ which can be covered again due to Heine-Borel with a finite set intervals $[U_{ij} - \lambda_{ij}, U_{ij} + \lambda_{ij}]$ ($j \in J_2$, J_2 finite index set) with $0 < \lambda_{ij} \leq b_2^{-1}(U_i, U_{ij})$. This construction we can continue and find m finite sets of points

$$\{U_{i_1}\}_{i_1\in J_1},\ \{U_{i_1 i_2}\}_{i_1\in J_1, i_2\in J_2}, \ldots, \{U_{i_1\ldots i_m}\}_{i_1\in J_1,\ldots,i_m\in J_m} \tag{5.147}$$

and numbers

$$0 < \lambda_{i_1 i_2 \ldots i_r} \leq b_r^{-1}(i_1, \ldots, i_r) \text{ with } r \leq m \tag{5.148}$$

such that

$$\bigcup_{i_r\in J_r} [U_{i_1\ldots i_r} - \lambda_{i_1\ldots i_r}, U_{i_1\ldots i_r} + \lambda_{i_1\ldots i_r}] \subseteq [-b_0(i_1, \ldots, i_{r-1}), b_0(i_1, \ldots, i_{r-1})]. \tag{5.149}$$

Here we have used the abbreviation

$$b_l(i_1, \ldots, i_r) = b_l(U_{i_1}, U_{i_1 i_2}, \ldots, U_{i_1 i_2 \ldots i_r}). \tag{5.150}$$

Now for given $\eta_1, \ldots, \eta_m > 1$ we define intervals $J^{\pm}_{i_1\ldots i_r}(\eta)$ with $r \leq m$ which consist of all y which can be written as

$$y = U_{i_1}\eta_1 \cdots \eta_m + U_{i_1 i_2}\eta_2 \cdots \eta_m + \cdots + U_{i_1\ldots i_m}\eta_m + z\eta_{r+1}\ldots\eta_m \tag{5.151}$$

where z is running over all values with

$$b_0(i_1, \ldots, i_r) \leq |z| = \pm z \leq \eta_r \lambda_{i_1\cdots i_r}. \tag{5.152}$$

For $r = 0$ we define $J^{\pm}(\eta)$ to consist of all y with

$$\pm y = |y| \geq b_0\eta_1 \cdots \eta_m. \tag{5.153}$$

Finally we also define $J^0_{i_1\ldots i_m}$ to consist of all y that can be written as

$$y = U_{i_1}\eta_1 \cdots \eta_m + U_{i_1 i_2}\eta_2 \cdots \eta_m + \cdots + U_{i_1\ldots i_m}\eta_m + z \text{ with } |z| < b_0(i_1, \ldots, i_m). \tag{5.154}$$

Now we show that any $y \in \mathbb{R}$ is contained in at least one of these intervals J. If $y \notin J^{\pm}(\eta)$ then by (5.153) we know that $|y| \leq b_0\eta_1 \cdots \eta_m$ and thus because of (5.146) it exists an $i_1 \in J_1$ such that

$$y \in [\eta_1 \cdots \eta_m(U_{i_1} - \lambda_{i_1}), \eta_1 \cdots \eta_m(U_{i_1} + \lambda_{i_1})]. \tag{5.155}$$

So we can write

$$y = \eta_1 \cdots \eta_m U_{i_1} + y' \text{ with } |y'| \leq \eta_1 \cdots \eta_m \lambda_{i_1}. \tag{5.156}$$

If $\pm y' \geq \eta_2 \cdots \eta_m b_0(i_1)$ then $y \in J^{\pm}_{i_1}(\eta)$ and we are finished. If on the other hand $|y'| \leq \eta_2 \cdots \eta_m b_0(i_1)$ by the same line of arguments using our construction below (5.146) we find a y'' with

$$y = \eta_1 \cdots \eta_m U_{i_1} + \underbrace{U_{i_1 i_2}\eta_2 \cdots \eta_m + y''}_{y'}. \tag{5.157}$$

It may happen that we have to continue this process until the final alternative, which then leads to $y \in J^0_{i_1\cdots i_m}(\eta)$. Thus y must indeed be in one of the sets $J^{\pm}_{i_1\ldots i_m}(\eta)$, $J^{\pm}(\eta)$ or $J^0_{i_1\ldots i_m}$. Thus we have found an upper boundary for $|f_{\vec{L}}(\vec{P})|$ given by

$$|f_{\vec{L}}(\vec{P})| \leq \sum_{\pm}\sum_{r=0}^{m}\sum_{i_1\ldots i_m}\int_{J^{\pm}_{i_1\cdots i_r}(\eta)} \mathrm{d}y|f(\vec{P}+\vec{L}y| + \sum_{i_1\ldots i_m}\int_{J_{i_1\cdots i_m}(\eta)} \mathrm{d}y|f(\vec{P}+y\vec{L})|. \tag{5.158}$$

Ad (D): The final step is to determine the asymptotic behaviour of each term on the right hand side of (5.158) where $\vec{P}$ is given by

$$\vec{P} = \vec{L}_1\eta_1\cdots\eta_m + \vec{L}_2\eta_2\cdots\eta_m + \cdots + \vec{L}_m\eta_m + \vec{C},\ \vec{C}\in W \tag{5.159}$$

where $W \subseteq \mathbb{R}^n$ is a compact region.

(i) Let $y \in J^{\pm}_{i_1\ldots i_r}(\eta)$

According to the definition of $J^{\pm}_{i_1\ldots i_r}(\eta)$ we can combine (5.159) with (5.151) to

$$\begin{aligned}\vec{P}+\vec{L}y &= (\vec{L}_1+U_{i_1}\vec{L})\eta_1\cdots\eta_m + (\vec{L}_2+U_{i_1i_2}\vec{L})\eta_2\cdots\eta_m+\cdots+\\ &+(\vec{L}_r+U_{i_1\ldots i_r}\vec{L})\eta_r\cdots\eta_m + z\vec{L}\eta_{r+1}\cdots\eta_m + \vec{L}_{r+1}\eta_{r+1}\cdots\eta_m+\cdots+\vec{L}_m\eta_m+\vec{C} =\\ &= (\vec{L}_1+U_{i_1}\vec{L})\eta_1\cdots\frac{\eta_r}{|z|}|z|\eta_{r+1}\cdots\eta_m + (\vec{L}_2+U_{i_1i_2}\vec{L})\eta_2\cdots\frac{\eta_r}{|z|}|z|\eta_{r+1}\cdots\eta_m+\cdots+\\ &+(\vec{L}_r+U_{i_1\ldots i_r}\vec{L})\eta_r\cdots\eta_m \pm |z|\vec{L}\eta_{r+1}\cdots\eta_m + \vec{L}_{r+1}\eta_{r+1}\cdots\eta_m+\cdots+\vec{L}_m\eta_m+\vec{C}\end{aligned} \tag{5.160}$$

Now we define

$$\alpha(i_1,\ldots,i_l) = \alpha(\mathrm{span}\{\vec{L}_1+U_{i_1}\vec{L},\vec{L}_1+U_{i_1i_2}\vec{L},\ldots,\vec{L}_l+U_{i_1\ldots i_l}\vec{L}\}),\ 1\leq l\leq m \tag{5.161}$$

and apply (5.145) together with the definition (5.151), (5.152) and (5.148) applied to $\eta_0 = |z|$ with which we find that for

$$\eta_l > b_l(i_1,\ldots,i_r) \text{ for } l\neq r \tag{5.162}$$

the following boundary condition is valid:

$$\begin{aligned}|f(\vec{P}+\vec{L}y)| \leq M(i_1,\ldots,i_r)&\eta_1^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\cdots\eta_{r-1}^{\alpha(i_1,\ldots,i_{r-1})}(\ln\eta_{r-1})^{\beta(i_1,\ldots,i_{r-1})}\times\\ \times\eta_{r+1}^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_{r+1},\vec{L}\})}&(\ln\eta_{r+1})^{\beta(\vec{L}_1,\ldots,\vec{L}_{r+1},\vec{L})}\cdots\eta_m^{\alpha(\mathrm{span}\{\vec{L}_1,\cdots,\vec{L}_m,\vec{L}\})}(\ln\eta_m)^{\beta(\vec{L}_1,\cdots,\vec{L}_m,\vec{L})}\times\\ &\times\left(\frac{\eta_r}{|z|}\right)^{\alpha(i_1,\ldots,i_r)}\left[\ln\left(\frac{\eta_r}{|z|}\right)\right]^{\beta(i_1,\ldots,i_r)}|z|^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_r,\vec{L}\})}(\ln|z|)^{\beta(\vec{L}_1,\ldots,\vec{L}_r,\vec{L})}.\end{aligned} \tag{5.163}$$

Now introducing z according to (5.160) as the integration variable we find by using the boundary condition defining $J^{\pm}_{i_1\ldots i_r}(\eta)$ cf. (5.152) we find

$$\begin{aligned}\int_{J^{\pm}_{i_1\ldots i_r}(\eta)}\mathrm{d}y|f(\vec{P}+y\vec{L})| \leq M(i_1,\ldots,i_r)&\eta_1^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\cdots\eta_{r-1}^{\alpha(i_1,\ldots,i_{r-1})}(\ln\eta_{r-1})^{\beta(i_1,\ldots,i_{r-1})}\times\\ \times\eta_{r+1}^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_{r+1},\vec{L}\})}&(\ln\eta_{r+1})^{\beta(\vec{L}_1,\ldots,\vec{L}_{r+1},\vec{L})}\cdots\eta_m^{\alpha(\mathrm{span}\{\vec{L}_1,\cdots,\vec{L}_m,\vec{L}\})}(\ln\eta_m)^{\beta(\vec{L}_1,\cdots,\vec{L}_m,\vec{L})}\times\\ &\times\eta_{r+1}\cdots\eta_m\times\\ \times\int_{b_0(i_1,\ldots,i_r)}^{\eta_r\lambda_{i_1\cdots i_r}}\mathrm{d}|z||z|^{\alpha(\mathrm{span}\{\vec{L}_1,\ldots,\vec{L}_r,\vec{L}\})}&(\ln|z|)^{\beta(\vec{L}_1,\ldots,\vec{L}_r,\vec{L})}\times\left(\frac{\eta_r}{|z|}\right)^{\alpha(i_1,\ldots,i_r)}\left[\ln\left(\frac{\eta_r}{|z|}\right)\right]^{\beta(i_1,\ldots,i_r)}.\end{aligned} \tag{5.164}$$

Because for our purposes an upper bound of the logarithmic coefficients $\beta(\vec{L},\dots,\vec{L}_l)$ is sufficient we assume without loss of generality that these are positive integers. Then we use the

Lemma 1. *Let $0<\lambda<1$, $b>1$, $\alpha,\alpha'\in\mathbb{R}$ and $\beta,\beta'\in\mathbb{N}_{>0}$. Then the following holds:*

$$\int_b^{\lambda\eta}\mathrm{d}z\left(\frac{\eta}{z}\right)^{\alpha}\left[\ln\left(\frac{\eta}{z}\right)\right]^{\beta}z^{\alpha'}\eta^{\beta'}\underset{\eta\to\infty}{\cong}\begin{cases}\eta^{\alpha}(\ln\eta)^{\beta+\beta'+1} & \text{for } \alpha'+1=\alpha\\ \eta^{\alpha}(\ln\eta)^{\beta} & \text{for } \alpha'+1<\alpha\\ \eta^{\alpha'+1}(\ln\eta)^{\beta'} & \text{for } \alpha'+1>\alpha.\end{cases}\tag{5.165}$$

We give the elementary proof of this lemma at the end of the section. From (5.165) we have with (5.164)

$$\begin{aligned}\int_{J^{\pm}_{i_1\dots i_r}(\eta)}\mathrm{d}y|f(\vec{P}+y\vec{L})| &\le M(i_1,\dots,i_r)\eta_1^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\cdots\eta_{r-1}^{\alpha(i_1,\dots,i_{r-1})}(\ln\eta_{r-1})^{\beta(i_1,\dots,i_{r-1})}\times\\ &\times\eta_{r+1}^{\alpha(\mathrm{span}\{\vec{L}_1,\dots,\vec{L}_{r+1},\vec{L}\})}(\ln\eta_{r+1})^{\beta(\vec{L}_1,\dots,\vec{L}_{r+1},\vec{L})}\cdots\eta_m^{\alpha(\mathrm{span}\{\vec{L}_1,\cdots,\vec{L}_m,\vec{L}\})}(\ln\eta_m)^{\beta(\vec{L}_1,\cdots,\vec{L}_m,\vec{L})}\times\\ &\times\begin{cases}\eta_r^{\alpha(i_1,\dots,i_r)}(\ln\eta_r)^{\beta(i_1,\dots,i_r)+\beta(\vec{L}_1,\dots,\vec{L}_r,\vec{L})+1} & \text{if } \alpha(i_1,\dots,i_r)=\alpha(\mathrm{span}\{\vec{L}_1,\dots,\vec{L}_r,\vec{L}\})+1\\ \eta_r^{\alpha(i_1,\dots,i_r)}(\ln\eta_r)^{\beta(i_1,\dots,i_r)} & \text{if } \alpha(i_1,\dots,i_r)>\alpha(\mathrm{span}\{\vec{L}_1,\dots,\vec{L}_r,\vec{L}\})+1\\ \eta_r^{\alpha(\mathrm{span}\{\vec{L}_1,\dots,\vec{L}_r,\vec{L}\})+1}(\ln\eta_r)^{\beta(\vec{L}_1,\dots,\vec{L}_r,\vec{L})} & \text{if } \alpha(i_1,\dots,i_r)<\alpha(\mathrm{span}\{\vec{L}_1,\dots,\vec{L}_r,\vec{L}\})+1\end{cases}\end{aligned}\tag{5.166}$$

whenever $\eta_l>b_l(i_1,\dots,i_r)$ for $l\neq r$ and $\eta_r>c(i_1,\dots,i_r)$ because cf. (5.152) the integral over the interval $J^{\pm}_{i_1\dots i_r}(\eta)$ contributes only if

$$\eta_r\ge\frac{b_0(i_1,\dots,i_r)}{\lambda_{i_1\dots i_r}}\le b_0(i_1,\dots,i_r)b_r(i_1,\dots,i_r).\tag{5.167}$$

Now we look on the infinite intervals $J^{\pm}(\eta)$. By definition (5.153) of these intervals we find

$$\int_{J^{\pm}(\eta)}\mathrm{d}y|f(\vec{P}+y\vec{L})|=\int_{b_0\eta_1\cdots\eta_m}^{\infty}\mathrm{d}y|f(\vec{P}\pm y\vec{L})|.\tag{5.168}$$

Substitution of $y=z\eta_1\cdots\eta_m$ and (5.159) for $\vec{P}$ gives

$$\int_{J^{\pm}(\eta)}|f(\vec{P}+y\vec{L})|\mathrm{d}y=\eta_1\cdots\eta_m\int_{b_0}^{\infty}\mathrm{d}y|f(\pm\vec{L}z\eta_1\cdots\eta_m+\vec{L}_1\eta_1\cdots\eta_m+\cdots+\vec{L}_m\eta_m+\vec{C})|\tag{5.169}$$

and now we can apply (5.145) for $r=0$ and $\eta_0=z$:

$$\begin{aligned}\int_{J^{\pm}(\eta)}\mathrm{d}y|f(\vec{P}+y\vec{L})| &\le M\eta_1^{\alpha(\mathrm{span}\{\vec{L}_1,\vec{L}\})}(\ln\eta_1)^{\beta(\vec{L}_1,\vec{L})}\cdots\eta_m^{\alpha(\vec{L}_1,\dots,\vec{L}_m,\vec{L})}(\ln\eta_m)^{\beta(\vec{L}_1,\dots,\vec{L}_m,\vec{L})}\times\\ &\times\int_{b_0}^{\infty}z^{\alpha(\mathrm{span}\{\vec{L}\})}(\ln z)^{\beta(\vec{L})}.\end{aligned}\tag{5.170}$$

The integral in the last row is a finite constant since by hypothesis $\alpha(\mathrm{span}\{\vec{L}\})+1<0$.

Now we are left with the interval $J^0_{i_1\dots i_m}$. With its definition (5.154) and (5.159) for $\vec{P}$ we have for these intervals

$$\vec{P}+\vec{L}y=(\vec{L}_1+U_{i_1}\vec{L})\eta_1\cdots\eta_m+\cdots+(\vec{L}_m+U_{i_1\cdots i_m}\vec{L})\eta_m+z\vec{L}\tag{5.171}$$

where $|z| \leq b_0(i_1, \ldots, i_m)$. Now the region

$$R' = \{\vec{C}' | \vec{C}' = \vec{L}z + \vec{C},\ |z| \leq b_0(i_1, \ldots, i_m), \vec{C} \in W\} \preceq \mathbb{R}^n \tag{5.172}$$

is compact. Because $f \in A_n$ by hypothesis there exist numbers

$$M'(i_1, \ldots, i_m) > 0 \text{ and } b'_l(i_1, \ldots, i_m) > 1$$

such that

$$\begin{aligned} |f[(\vec{L}_1 + U_{i_1}\vec{L})\eta_1 \ldots \eta_m + \cdots + (\vec{L}_m + U_{i_1 \ldots i_m}\vec{L})\eta_m + \vec{C}']| \leq \\ \leq M'(i_1, \ldots, i_m)\eta_1^{\alpha(i_1)}(\ln \eta_1)^{\beta(i_1)} \cdots \eta_m^{\alpha(i_1,\ldots,i_m)}(\ln \eta_m)^{\beta(i_1,\ldots,i_m)} \end{aligned} \tag{5.173}$$

for $\vec{C}' \in R'$ and $\eta_l > b'_l(\eta_1, \ldots, \eta_m)$. Thus

$$\begin{aligned} \int_{J_{i_1 \ldots i_m}(\eta)} \mathrm{d}y |f(\vec{P} + \vec{L}y)| \leq 2b_0(i_1, \ldots, i_m)M'(i_1, \ldots, i_m) \times \\ \times \eta_1^{\alpha(i_1)}(\ln \eta_1)^{\beta(i_1)} \cdots \eta_m^{\alpha(i_1,\ldots,i_m)}(\ln \eta_m)^{\beta(i_1,\ldots,i_m)} \end{aligned} \tag{5.174}$$

if $\eta_l \geq b_l(i_1, \ldots, i_m)$ $(1 \leq l \leq m)$.
Now our proof is finished by inspection of (5.166), (5.173) and (5.174) because of the result of the construction in part (C) cf. (5.158). From these estimates of upper bounds for $f_{\vec{L}}(\vec{P})$ we read off that this function is indeed contained in the class A_{n-1} and the asymptotic coefficients are given by

$$\alpha_{\vec{L}}(\mathrm{span}\{\vec{L_1}, \vec{L}_2, \ldots, \vec{L}_r\}) = \max_{i_1,\ldots,i_r} [\alpha(i_1, \ldots, i_r), \alpha(\mathrm{span}\{\vec{L}_1, \ldots, \vec{L}_r, \vec{L}) + 1] \tag{5.175}$$

where $i_1, \ldots, \mathrm{i}_r$ are running over the index sets defined in (5.161) and it remains to show that this is the same as given by (5.126) for our case of integrating over a one-dimensional subspace. By definition (5.161) we can rewrite (5.175) by

$$\begin{aligned} \alpha_{\vec{L}}(\mathrm{span}\{\vec{L_1}, \vec{L}_2, \ldots, \vec{L}_r\}) = \\ = \max_{u_1,\ldots,u_r} [\alpha(\mathrm{span}\{\vec{L}_1 + u_1\vec{L}, \ldots, \vec{L}_r + u_r\vec{L}\}), \alpha(\mathrm{span}\{\vec{L}_1, \ldots \vec{L}_r, \vec{L}\}) + 1]. \end{aligned} \tag{5.176}$$

Here the $u_1, \ldots, u_r$ run over the sets $U_{i_1}, \ldots, U_{i_1 \ldots i_r}$ defined above cf. (5.147) respectively. This expression has to be compared with (5.126).
So let $S = \mathrm{span}\{\vec{L}_1, \ldots, \vec{L}_r\}$ and $S' \preceq \mathbb{R}^n$ such that $\Lambda(\mathrm{span}\{\vec{L}\}S' = S$. Then of course S' could be either $S \oplus \mathrm{span}\{\vec{L}\}$ or for each $\vec{v} \in S$ there must exist a $\vec{w} \in S'$ and $u \in \mathbb{R}$ such that $\vec{v} = \vec{w} + u\vec{L}$. In other words in this there are $u_1, \ldots, u_r \in \mathbb{R}$ such that

$$S' = \mathrm{span}\{\vec{L}_1 + u_1\vec{L}, \cdots, \vec{L}_r + u_r\vec{L}\}. \tag{5.177}$$

In the first case we have $\dim S' = \dim S + 1$ in the second we have $\dim S' = \dim S$ since S' in (5.126) runs over all subspaces disjoint with $\mathrm{span}\vec{L}$ and this shows immediately the equivalence of (5.176) with (5.126) for the case of integration over the one-dimensional subspace $\mathrm{span}\{\vec{L}\}$ and this finally finishes the proof. Q.E.D.

5.6.2 Proof of the Lemma

Now for sake of completeness we prove the above used lemma. (5.165) can be written as:

$$I(\eta) = \eta^\alpha \int_b^{\lambda\eta} \mathrm{d}z \frac{1}{z^{\alpha-\alpha'}} \left[\log\left(\frac{\eta}{z}\right)\right]^\beta (\ln z)^{\beta'} \tag{5.178}$$

where $0 < \lambda < 1$, $b > 1$, $\beta, \beta' \in \mathbb{N}_{>0}$ and $\alpha, \alpha' \in \mathbb{R}$. With the substitution $u = \ln z$ we find

$$I(\eta) = \eta^\alpha \int_{\ln b}^{\ln(\lambda\eta)} \mathrm{d}u \exp[(\alpha' - \alpha + 1)u](\ln\eta - u)^\beta u^{\beta'}. \tag{5.179}$$

Since we have to investigate the asymptotic behaviour of this integral for $\eta \to \infty$ we can assume without loss of generality $\eta\lambda > b$.

(a) $\alpha' - \alpha + \mathbf{1} = \mathbf{0}$

In this case we use the binomial theorem to find

$$I(\eta) = \eta^\alpha \sum_{k=0}^{\beta} \binom{\beta}{k} \frac{u^{\beta'+k+1}}{\beta'+k+1}\Bigg|_{\ln b}^{\ln(\lambda\eta)} (\ln\eta)^{\beta-k} \underset{\eta\to\infty}{\cong} O[\eta^\alpha (\ln\eta)^{\beta+\beta'+a}] \tag{5.180}$$

(b) $\alpha' - \alpha + \mathbf{1} < \mathbf{0}$

Then we can let the upper boundary of the integral (5.179) to infinity making the integral greater. Thus we have the boundary

$$I(\eta) \leq \eta^\alpha \sum_{k=0}^{\beta} (-1)^k (\ln\eta)^{\beta-k} \int_{k=0}^{\beta} \binom{\beta}{k} \int_{\ln b}^{\infty} \mathrm{d}u u^{k+\beta'} \exp[(\alpha' - \alpha + 1)u] \tag{5.181}$$

and because the integrals under the sum are just finite factors we have the asymptotic behaviour

$$I(\eta) \underset{\eta\to\infty}{\cong} O[\eta^\alpha (\ln\eta)^\beta. \tag{5.182}$$

(c) $\alpha' - \alpha + \mathbf{1} > \mathbf{0}$

In this case we can let the lower boundary of the integral (5.179) to 0. Now the integral

$$J_n(a, x) = \int_0^x u^n \exp(au)\mathrm{d}u \tag{5.183}$$

can be calculated by n-fold differentiation of $J_0(a, x)$ with respect to a but this is

$$J_0(a, x) = \frac{1}{a}[\exp(ax) - 1],\ J_n(a, x) = \frac{\partial^n}{\partial a^n} J_0(a, x). \tag{5.184}$$

The asymptotic behaviour for $x \to \infty$ is thus given if we keep only the differentiation in the product law where only the exponential term is differentiated leading to

$$J_n(a, x) \underset{x\to\infty}{\cong} x^n \exp(ax) \tag{5.185}$$

Applying this to (5.179) again using the binomial theorem we find

$$\begin{aligned} I(\eta) &= \eta^{\alpha} \sum_{k=0}^{\beta} (\ln \eta)^{\beta-k} (-1)^k \int_0^{\ln(\lambda\eta)} \mathrm{d}u \exp[(\alpha' + 1 - \alpha)u] u^{\beta'+\beta+k} \underset{\eta\to\infty}{\cong} \\ &\underset{\eta\to\infty}{\cong} O\left\{ \eta^{\alpha'+1} (\ln \eta)^{\beta'} \sum_{k=0}^{\beta} (-1)^k \ln(\eta)^{\beta-k} [\ln(\lambda\eta)]^k \right\} \underset{\eta\to\infty}{\cong} \\ &\underset{\eta\to\infty}{\cong} O[\eta^{\alpha'+1} (\ln \eta)^{\beta'}]. \end{aligned} \tag{5.186}$$

The asymptotic formula (5.182), (5.182) and (5.186) together prove the claim of the lemma. Q.E.D.

5.7 Application of Weinberg's Theorem to Feynman diagrams

After this lengthy mathematical interlude we come back to perturbative quantum theory and the problem of renormalisation. Now we can use Weinberg's theorem to show that a Feynman diagram is finite if its superficial degree of divergence as well as this of all its subdiagrams is negative. For this purpose we have only to specify our graphical idea of a subdiagram in a mathematical manageable way which allows the application of the theorem. Again we shall work in Wick rotated, i.e., in Euclidean quantum field theory and follow Weinberg's original paper but for a general 2ω-dimensional space time.

The typical integrand of a Feynman diagram $\mathfrak{G}$ is of the form

$$F = \gamma(\sigma) \Pi_{j=1}^{M} \Delta_j(\bar{p}_j, \sigma) \tag{5.187}$$

where σ is a collection of discrete variable like spins, polarisations etc. and $\gamma(\sigma)$ is the total factor independent of the external and the internal momenta of the diagram stemming from the vertices. All momenta contained in the vertices have to be included in the Δ_j. The $\bar{p}_j \in \mathbb{R}^{2\omega}$ is the Euclidean momentum in 2ω-dimensional space time associated with the line labelled with j. In (5.187) all momentum conserving δ-distributions are integrated out. The total number of independent internal as well as external momenta is called N.

In order to apply Weinberg's theorem as formulated in the last section we introduce vectors $\vec{V}_{j\mu} \in \mathbb{R}^{2\omega N}$ such that

$$(\bar{p}_j)_\mu = \vec{V}_{j\mu} \vec{P},\ j \in \{1, \ldots, N\}. \tag{5.188}$$

Then the $\vec{V}_{j\mu}$ are linearly independent in $\mathbb{R}^{2\omega N}$. Now for the integrand (5.187) we can write

$$F(\vec{P}, \sigma) = \gamma(\sigma) \prod_{j=1}^{M} \Delta_j(\vec{P}\vec{V}_j, \sigma) \tag{5.189}$$

where $\vec{P}\vec{V}_j$ is a shorthand notation for the 2ω components $(\bar{p}_j)_\mu$ as given by definition (5.188).

Then we define that the external lines of the diagram $\mathfrak{G}$ span the subspace E and the inner ones I which are disjoint and fulfil

$$\mathbb{R}^{2\omega N} = E \oplus I. \tag{5.190}$$

Now $\vec{P} \in E(I)$ if and only if $\vec{P} \perp \vec{V}_j$ for all j running over the inner (external) lines of $\mathfrak{G}$ because in the basis we use here naturally from the Feynman rules $\vec{P}$ has only components different from 0 for external or internal momenta respectively[14].

The $\Delta_j(\bar{p}_j, \sigma)$ are tensor or spinor valued expressions in Euclidean 2ω-dimensional space time and have the asymptotic behaviour

$$\Delta_j(\bar{p}_j, \sigma) \underset{\bar{p}_j^2 \to \infty}{\cong} O(\bar{p}_j^{\alpha_j/2}). \tag{5.191}$$

For instance a Green's function for a scalar particle is

$$\Delta_j(\bar{p}_j, \sigma) = \frac{1}{\bar{p}_j^2 + m_j^2} \Rightarrow \alpha_j = -2 \tag{5.192}$$

while for a Dirac Fermion

$$\Delta_j(\bar{p}_j, \sigma) = \frac{\mathrm{i}\bar{\not{p}}_j + m}{\bar{p}_j^2 + m_j^2} \Rightarrow \alpha_j = -1 \tag{5.193}$$

Lemma 2. *The integrand F associated with a Feynman diagram $\mathfrak{G}$ is of class $A_{2\omega N}$ with asymptotic coefficients for $S \preceq \mathbb{R}^{2\omega N}$ given by*

$$\alpha(S) = \sum_j^{(S)} \alpha_j, \ \beta(S) = 0. \tag{5.194}$$

Here $\sum_j^{(S)}$ runs over all j for which the vectors $\vec{V}_j$ are not perpendicular to the subspace S.

Proof: Let

$$\vec{P}\vec{L}_1\eta_1 \cdots \eta_m + \vec{L}_2\eta_2 \cdots \eta_m + \cdots + \vec{L}_m\eta_m + \vec{C} \tag{5.195}$$

with $\vec{L}_1, \ldots, \vec{L}_m \in \mathbb{R}^{2\omega N}$ arbitrary linearly independent vectors and $\vec{C} \in W$ where W is an arbitrary compact region in $\mathbb{R}^{2\omega N}$. Then

$$\Delta_j(\vec{V}_j\vec{p}, \sigma) = O[(\eta_l \cdots \eta_m)^{\alpha_j}] \tag{5.196}$$

where

$$l = \min\{k | \vec{V}_j\vec{L}_k \neq 0\} \tag{5.197}$$

Thus we find for the asymptotic behaviour of the integrand

$$F(\vec{P}, \sigma) \cong O[\prod_{l=1}^{M} \prod_j{}^{(l)} (\eta_l\eta_{l+1} \cdots \eta_m)^{\alpha_j}] \tag{5.198}$$

where $\prod_j{}^{(l)}$ is running over j satisfying (5.197) for the given l. Collecting the powers for each η we get

$$F(\vec{P}, \sigma) \cong O(\eta_1^{\alpha_{(1)}} \cdots \eta_m^{\alpha_{(m)}}) \text{ with } \alpha_{(r)} = \sum_j{}^{(r)} \alpha_j \tag{5.199}$$

[14]Here and in the following with $\vec{P} \perp \vec{V}_j$ we mean that for the line label j for all $\mu \in \{1, \ldots, 2\omega\}$ we have $\vec{P}\vec{V_{j\mu}} = 0$ where $\vec{A}\vec{B}$ denotes the usual Euclidean scalar product in $\mathbb{R}^{2\omega N}$.

where $\sigma_j{}^{(r)}$ runs over all j satisfying (5.197) for some $l \leq r$, i.e., over all j for which

$$\exists k \in \{1, \ldots, r\} : \ \vec{L}_k \vec{V}_j \neq 0. \tag{5.200}$$

In other words j runs over those j for which the four $\vec{V}_j$ are not simultaneously perpendicular to the subspace span$\{\vec{L}_1, \ldots \vec{L}_r\}$. Thus $\alpha_{(r)}$ depends only on this subspace S and we have proven our lemma. Q.E.D.

From this lemma the convergence theorem is indeed only a application of Weinberg's general theorem (part (a) is sufficient):

Theorem 5. ***Convergence Theorem:*** *A Feynman integral corresponding to a Feynman graph $\mathfrak{G}$ is convergent if its overall superficial degree of divergence, i.e. $\alpha(I) + dimI < 0$ and if all subintegrations are superficially convergent, i.e., if $\alpha(I') + dimI' < 0$ for all $I' \preceq I$.*

To extract the asymptotic behaviour of the Feynman integral by application of part (b) of Weinberg's theorem we have to specify the meaning of the terminus "subgraph" we have given in a previous section diagrammatically by drawing a box around a part $\mathfrak{G}'$ of the Feynman graph $\mathfrak{G}$.

Definition 2. *Let $\mathfrak{G}$ be the Feynman-graph given by its lines*[15]*. Then $\mathfrak{G}' \subset \mathfrak{G}$ is named subgraph of $\mathfrak{G}$ if there is no vertex which is attached to only one line with a vertex in $\mathfrak{G}'$. A subgraph $\mathfrak{G}'$ is therefore composed of paths which begin and end in each other or in external lines but never in a vertex of $\mathfrak{G}$.*

Lemma 3. *Let $S' \preceq \mathbb{R}^{2\omega N}$. Then $\mathfrak{G}' = \{j | \vec{V}_j \not\perp S'\}$ is a subgraph associated with S'. For each subgraph $\mathfrak{G}'$ there is a subspace S' to which it is associated.*

Proof: Suppose the lines $j_1, \ldots, j_r$ join one arbitrary vertex in $\mathfrak{G}$. Then the momentum conservation at this vertex can be written with help of the vectors $\vec{V}_j$ associated with these lines as

$$\pm \vec{V}_{j_1} \pm \vec{V}_{j_2} \pm \cdots \pm \vec{V}_{j_r} = 0 \tag{5.201}$$

with a plus-sign for an incoming and a minus-sign for an outgoing momentum (which are linear combinations of internal and external momenta). Now define $\mathfrak{G}'$ as in the lemma and suppose $j_1, \ldots, j_{r-1}$ label lines not contained in $\mathfrak{G}'$. Then $\vec{V}_{j_1}, \ldots, \vec{V}_{j_r} \perp S'$. Thus cf. (5.201) also $j_r \neq \mathfrak{G}'$. On the other hand if one has given a subgraph $\mathfrak{G}'$ graphically as described in the definition then each line carries a certain momentum $\bar{p}_j \in \mathbb{R}^{2\omega N}$. Call the set of lines contained in $\mathfrak{G}$ $\mathfrak{L}(\mathfrak{G})$ and define $S' = \left(\text{span}\{\vec{V}_j\}_{j \in \mathfrak{L}(\mathfrak{G}')}\right)^{\perp}$. Then by definition the subgraph $\mathfrak{G}'$ is associated to the subset S'. Q.E.D.

Now let $\mathfrak{G}'$ be a subgraph corresponding to a subspace S'. The external lines contained in $\mathfrak{G}'$ are those for which their corresponding $\vec{V}_j$ are not orthogonal to the projection $\Lambda(I)S'$ of S' along the subspace of internal momenta. The number of independent internal momenta in $\mathfrak{G}'$ is thus

$$\dim[\Lambda(E)S'] = \dim S' - \dim[\Lambda(I)S']. \tag{5.202}$$

It follows that the dimensionality

$$D_I(\mathfrak{G}') = \alpha(S') + \dim S' - \dim[\Lambda(I)S'] \tag{5.203}$$

[15] We shall have to specify the diagrammatic elements more precisely according to Zimmermann [Zim69]but for now this shorthand notation should be clear enough.

is the net number of momentum factors (including those of derivative couplings associated with the vertices) of the subgraph $\mathfrak{G}'$ counting α_j for each line j and 2ω for each integration. If the subdiagram $\mathfrak{G}'$ associated to the subspace S' consists only of internal lines of $\mathfrak{G}$ lines then $\Lambda(I)S' = 0$ and thus in this case $D_I(\mathfrak{G}')$ is the superficial degree of divergence of the subdiagram $\mathfrak{G}'$.

Now it is simple to rewrite (5.126) for our special application to Feynman diagrams as

$$\alpha_I(S) = \max_{S \in \mathfrak{G}'} D_I(\mathfrak{G}'). \tag{5.204}$$

According to Lemma 3 in other words we can say that max runs over all subgraphs $\mathfrak{G}'$ containing just that very set $\mathfrak{E}_\infty$ of external lines j for which $\vec{V}_j$ is not perpendicular to S. We can summarise this in the

Theorem 6. ***Asymptotic behaviour of Green's functions:*** *If a set $\mathfrak{E}_\infty$ of external lines of a Feynman diagram $\mathfrak{G}$, which is overall convergent as well as its subgraphs according to theorem 5, goes to infinity in the sense of Weinberg's theorem then the integral associated with $\mathfrak{G}$ has the asymptotic behaviour $O(\eta^{\alpha_I(\mathfrak{E}_\infty)}(\ln \eta)^{\beta_I(\vec{L})}$ where $\alpha_I(\mathfrak{E})$ is the maximum of the dimensionality (defined by (5.203)) of all subgraphs $\mathfrak{G}'$ of $\mathfrak{G}$ including the external lines in $\mathfrak{E}_\infty$ and no more external lines of $\mathfrak{G}$.*

5.8 BPHZ-Renormalisation

Now we have all ingredients to prove the main theorem of perturbation theory, namely that for a theory which has only coupling constants with dimensions $O(p^k)$ with $k \geq 0$ we can renormalise any diagram to any order by adding counterterms to the effective action which are of the same form as in the classical action. This means that only a finite number of parameters (wave function renormalisation constants, masses and coupling constants) have to be fixed at any order of perturbation theory to experiment. We call a theory with only renormalisable interaction terms in the Lagrangian, superficially renormalisable, because we have to show now that they are really renormalisable. The original parameters in the classical Lagrangian, the so-called bare parameters, are infinite due to the infinities of finite diagram but cannot be observed in principle. This is what we call *renormalisability in the strict sense*. The great advantage of such a renormalisable theory is that one needs only a finite number of parameters which have to be fixed to experiment. All other comes out of the model given by the original Lagrangian containing these parameters.

The mathematical task is thus to show that for a theory with renormalisable couplings, i.e. such with coupling constants of momentum order greater or equal 0, one can render any diagram finite by subtracting a polynomial in its external momenta which can be written as a term like that in the original Lagrangian if interpreted in position space.

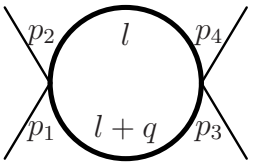
Up to now we have calculated some examples by using the method of dimensional regularisation as an intermediate step to give the integrals a meaning and to find a systematic way to subtract the divergent parts of the integrals. We have also seen that the renormalisation program works for our examples using this regularisation scheme. The great advantages of dimensional regularisation are that it respects most of the symmetries which are so important in quantum field theory, especially that of local gauge invariance (which is important for QED and essential for the more complicated non-abelian gauge theories which build the heart of our nowadays understanding of

elementary particles within the *standard model*). For our present task it is not so appropriate as for the practical calculations.

We come now to the important idea of Boguliubov and Parasiuk who invented a systematic scheme to render the Feynman integrals finite by handling the integrands first without introducing an intermediate step of regularisation. They use directly Weinberg's power counting theorem, which was so hard to prove in the previous section, to subtract polynoms of the external momenta of the divergent diagrams and its divergent sub-diagrams up to the order given by the superficial degree of divergence. If we can show that such a scheme works then we are done with the task to show the renormalisability of superficially renormalisable theories, because then all infinities are rendered finite by subtracting counterterms in the Lagrangian which are local (i.e., they are polynomials of the external momenta momenta which is true by construction) and of the same form as of the terms already present in the original Lagrangian[16].

5.8.1 Some examples of the method

In this section we understand the Feynman rules as rules for building the *integrands* rather than the integrals of the perturbation series. To define the subtraction operator we start again with the simple example of the one-loop contribution to the four point vertex, the dinosaur diagram:

$\Gamma =$ 

with $q = p_1 + p_2 = p_3 + p_4$. (5.205)

According the Feynman rules the integrand of the diagram Γ is

$$I(q,l) = \frac{\lambda}{2} G(l) G(l+p) \text{ with } G(k) = \frac{1}{m^2 - k^2 - \mathrm{i}\eta}. \tag{5.206}$$

Power counting shows that the superficial degree of divergence is

$$D_s(\Gamma) = 0. \tag{5.207}$$

This diagram does not contain any divergent sub-diagrams in the sense of Definition 2 in the previous section. Indeed breaking up any number of lines to extract sub-diagrams gives a tree level diagram which is finite. Such diagrams are called *primitively divergent*.

The idea of Boguliubov and Parasiuk is to render primitively divergent diagrams finite by subtracting the Taylor expansion around 0 with respect to the external momenta of this diagram (in our case it depends only on the one momentum q which is due to the simplicity of the diagram) up to the order given by $D_s(\Gamma)$ from the integrand. The general definition of the *Taylor operator* is given by

$$t^j(q) f(q_1, \ldots, q_k) = \sum_{n=0}^{j} \frac{1}{n!} q_{m_1}^{\mu_1} \cdots q_{m_n}^{\mu_n} \left[\frac{\partial^n}{\partial q_{m_1}^{\mu_1} \cdots q_{m_n}^{\mu_n}} f(q_1, \ldots, q_k) \right]_{q_1 = \ldots = q_k = 0}. \tag{5.208}$$

[16] In this chapter we consider theories in which all superficially renormalisable interaction terms are present and consistent with Poincaré symmetry. One should note that the Poincaré symmetry is no restriction for the renormalisation program since all integrands of Feynman integrals are tensors or contractions of tensors with γ-matrices and thus manifestly Lorentz covariant. The same holds true for the ϕ^4–theory which we take as a simple toy model to have simple examples at hand: Here the superficially renormalisable terms $\propto \phi$ and $\propto \phi^3$ are missing but due to the symmetry of the Lagrangian with respect to transformations $\phi \to -\phi$ all Green's functions with an odd number of external legs vanish.

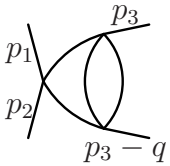
Herein summation over the Lorentz indices μ_l and the independent external momenta labels m_l is understood. For the Taylor operator with respect to the external momenta of the diagram Γ to order $D_s(\Gamma)$ we write t_Γ. Application of the Taylor t_γ operator belonging to a sub-diagram γ is defined to act on the external momenta of γ only and not to affect the complementary diagram $\Gamma \setminus \gamma$. Here we introduce the notation we shall need in this and the next sections: We read the diagram not only as an expression of the Feynman integrals or integrands but also as a set of the lines and vertices it is built off. By definition the external lines are never contained in the diagram.

In our case of the dinosaur the subtraction of the overall divergence is given by

$$R_\Gamma = (1 - t_\Gamma)\Gamma = \frac{\lambda}{2} G(l)[G(l+q) - G(l)]. \tag{5.209}$$

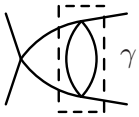
The degree of divergence is -1 and thus the integral over this expression exists, and the power counting theorem tells us that the integral has the asymptotic behaviour $O[(\ln q)^\beta]$. Using the techniques of Feynman parameterisation (C.16) and (C.8) for $\omega = 2$ we find the same result as we have calculated in section 6.4 with the dimensional regularisation technique. The subtraction at external momentum 0 is in this case the same as we defined as the physical scheme. Of course one can change the scheme also in the BPHZ technique by subtracting an arbitrary value so that one may define λ to be the physical coupling on another set of external momenta. We shall investigate the meaning of such changes of the renormalisation scheme in another chapter of these notes when we talk about the renormalisation group.

Now consider the diagram

$\Gamma =$ 

$= \frac{\mathrm{i}\lambda^3}{2} G(l_1) G(l_1 - q) G(l_2) G(l_1 + l_2 - p_3)$ with $q = p_1 + p_2$. (5.210)

which is not only overall divergent (its superficial degree of divergence is $D_s(\Gamma) = 0$, i.e. it is logarithmitically divergent) but contains a divergent subdiagram γ around which we draw a box:

$-t_\gamma\Gamma =$ 

$= I_{\Gamma\setminus\gamma}(-t_\gamma)\gamma = -\frac{\mathrm{i}\lambda^3}{2} G(l_1) G(l_1 - q) G^2(l_2)$ with $q = p_1 + p_2$. (5.211)

We define that a diagram Γ with one (or more disjoined) subdiagrams stand for the diagram were the boxed subgraphs are substituted by the negative of the Taylor operator up to the order of superficial divergence with respect to the external lines of the subdiagram while the remainder $\Gamma \setminus \gamma$ is unaffected to any manipulations[17].

In our case there are no further subdivergent diagrams. Thus we have

$$\bar{R}_\Gamma = (1 - t_\gamma) I_\Gamma = \frac{\mathrm{i}\lambda^3}{2} G(l_1) G(l_2) G(l_1 - q)[G(l_1 + l_2 - p) - G(l_2)]. \tag{5.212}$$

[17] According to the diagram rules one can always write $I_\Gamma = I_{\Gamma\setminus(\gamma_1\cup\ldots\cup\gamma_k)} I_{\gamma_1} \cdots I_{\gamma_k}$ were $\gamma_1, \ldots, \gamma_k$ are arbitrary mutually disjoined 1PI subdiagrams of Γ. Since $\gamma_1, \ldots, \gamma_k$ are mutually disjoined t_{γ_j} (which means that any two diagrams have neither a line nor a vertex in common) acts only on I_{γ_j} while all $I_{\gamma_{j'}}$ for $j \neq j'$ can be taken out of the operator t_{γ_j}

Power counting shows that this expression is $O(l_1^{-5})$ and thus the integration over l_1 is convergent while the result of this integration is of order $O[l_1^{-4}(\ln l_1)^\beta]$. Thus the integral over l_1 is divergent. Here we have introduced the definition of $\bar{R}_\Gamma$ which is the integrand according to the diagram rules for Γ with all subdivergences removed.

Now it is simple to remove also the overall divergence, namely by using the rule for primitively divergent integrals. This is sufficient to render the whole diagram finite because by definition $\bar{R}_\Gamma$ does not contain any subdivergence. Thus in our case the integrand of the completely renormalised diagram reads

$$\begin{aligned} R_\Gamma &= (1-t_\Gamma)\bar{R}_\Gamma = (1-t_\Gamma)(1-t_\gamma)I_\Gamma = \\ &= G(l_1)G(l_2)\{G(l_1-q)[G(l_1+l_2+p_3)-G(l_2)]-G(l_1)[G(l_1+l_2)-G(l_2)]\}. \end{aligned} \tag{5.213}$$

We see that nested divergences are no problem for the BPH-formalism. Two diagrams γ_1 and γ_2 are called nested if $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$[18].

Now the power counting theorem discussed at length at sections 6.7 and 6.8 shows that the so renormalised integrals are convergent and up to logarithms behave for large external momenta asymptotically as given by superficial power counting. Of course the BPH-subtractions of the subdivergences and overall divergences lead to an integrand which all have a superficial degree of divergence less than 0 and according to the theorem are thus perfectly finite.

The only thing we have to do is to give a general prescription how to render *a general diagram* finite by this formalism which task is completed in the next section by giving the explicit solution of the recursive BPH-subtraction scheme which is called *Zimmermann's forest formula*.

5.8.2 The general BPH-formalism

For sake of completeness we start anew with defining the classes of subdiagrams of a given diagram relevant for the BPH-formalism. Let Γ be an arbitrary Feynman diagram. Then two subdiagrams γ_1 and γ_2 are called

- *disjoined* if the do not share any common internal line or vertex. The shorthand notation for this case is $\gamma_1 \cap \gamma_2 = \emptyset$.
- *nested* if the one diagram, e.g. γ_1, is a subdiagram of the other, e.g. γ_2: $\gamma_1 \subseteq \gamma_2$
- *overlapping* if they are neither disjoined nor nested: $\gamma_1 \circ \gamma_2$.

Diagrams and subdiagrams are called *renormalisation parts* if they are superficially divergent, i.e., $d(\gamma) \geq 0$. Suppose we have renormalised all subdivergences of Γ by applying the above described subtraction method (called *R-operation*) to any divergent subdiagram (including all nested subdivergences of the subdiagrams themselves). The corresponding integrand is named $\bar{R}_\Gamma$. Due to Weinberg's power counting theorem then the integrand is either finite (which is the case if Γ is *not a renormalisation part*) or it can be rendered finite by the R-operation applied by subtracting the Taylor expansion with respect to the external momenta of Γ if it is a renormalisation part. Thus this last step can be written compactly as

$$R_\Gamma = (1-t^\Gamma)\bar{R}_\Gamma. \tag{5.214}$$

[18] $\gamma_1 \subset \gamma_2$ means that all lines and vertices of γ_1 are also contained in γ_2 and γ_2 contains lines and/or vertices which are not contained in γ_1.

Here and in the following we define t^Γ as the Taylor expansion with respect to the external momenta around an arbitrary appropriate renormalisation point[19] up to the order given by superficial degree of divergence of the considered (sub)diagram. If the diagram is not superficially divergent, i.e. has a superficial degree of divergence $d(\gamma) < 0$ than t_γ is simply the operator which maps the integrand to 0.

The $\bar{R}_\gamma$ themselves are defined recursively by

$$\bar{R}_\gamma = I_\gamma + \sum_{\{\gamma_1,\ldots,\gamma_c\}} I_{\gamma\setminus\{\gamma_1,\ldots,\gamma_c\}} \prod_{\tau=1}^{c} O_{\gamma_\tau} \tag{5.215}$$

with

$$O_\gamma = -t^\gamma \bar{R}_\gamma. \tag{5.216}$$

The sum has to be taken over all sets $\{\gamma_1,\ldots,\gamma_c\}$ mutually disjoined subdiagrams of γ which are different from γ.

This is the desired general prescription of the finite part of a Feynman diagram. Indeed if one takes any subdiagram of Γ by construction of $\bar{R}_\Gamma$ it has a superficial degree of divergence less then 0 and thus is finite according to Weinberg's theorem. By definition further also the overall superficial degree of divergence of the expression R_Γ is also less than 0 and thus R_Γ is finite, again by applying Weinberg's theorem.

Further we have shown that we had to remove only polynomial expressions in p with the order given by the superficial degree of divergence $\delta(\gamma)$ for any subdiagram γ of Γ. According to (5.121) in the case of ϕ^4-theory for $d = 4$ only the diagrams with $E \leq 4$ external legs are superficially divergent and the degree of divergence is $D_s = 4 - E$. Further since only an even number of fields appears in the Lagrangian also only diagrams with an even number of legs are different from 0. Thus we need to renormalise only the diagrams with 0, 2 and 4 legs. The diagrams with 0 legs are contributions to the vacuum energy which can be subtracted without changing any physical predictions. We do not need to consider them here. The diagrams with 2 legs are of divergence degree 2 and thus only polynomials of order 2 in the external momentum appear as counterterms. Further in our case of a scalar theory all vertex functions and the self-energy are scalars which means that the overall counterterms for self-energy diagrams have the form $\delta\Sigma = \delta Z p^2 - \delta m^2$. We can reinterpret these counterterms as contributions to the effective action which looks the same as those from the original Lagrangian δZ is an infinite contribution to the bare field-normalisation and δm^2 to the bare mass parameter. The remaining divergences are those of four-point vertex functions which are logarithmitically divergent and have local counter terms $\delta\Gamma^{(4)} = \delta\lambda$ which are contributions to the bare coupling constant. This means that any divergent 1PI diagram part can be made finite by infinite contributions to the bare Lagrangian which has the same form as the finite tree-level Lagrangian we started with. The only difference is that the renormalised finite parameters of the theory, which are fixed at certain physical processes and are thus parameters carrying physical information, are substituted with infinite bare parameters which are not observable and thus the infinities are not important.

The only problem which is remaining in this description is that of the case of massless particles. In addition to the so far considered UV-divergences, i.e divergences arising from the integration

[19] In the case that massless particles are present one has to use a space like momentum otherwise one can use $p = 0$ for this so called intermediate renormalisation point. It is called intermediate because of course one can easily choice another renormalisation point and do a finite renormalisation provided. In the practical applications we shall use dimensional regularisation where this procedure is much simplified, especially infrared divergences are cured very naturally with it!

up to infinite momenta in the loop integrals, such theories contain also IR-divergences arising from the very soft momenta in the loop integrals. As we shall see below in the case of ϕ^4-theory they can be renormalised by making use of a different renormalisation scheme, like the above introduced MS (minimal subtraction scheme). Then it will also become clear that we have to introduce the scale parameter μ as explained in the paragraph after eq. (5.85). In the ϕ^4-theory this means we have to substitute $\lambda\mu^{2\epsilon}$ for the coupling constant instead of λ, and all counterterms will depend on μ, which means that especially the renormalised coupling constant will depend on the choice of μ although the bare coupling constant is independent of this momentum scale. For the massless case which contains no dimensionful bare parameters this means that we introduce the parameter μ with the dimension of an energy and we have to adjust the renormalised coupling at a certain value of μ which fixes the momentum scale of the physical process used to fix the coupling. As (5.97) shows if the renormalised coupling λ is small at a certain scale μ it might become large if we look at the same process at a momentum much different in scale from μ. Then the use of perturbation theory is unjustified since the effective coupling becomes large. We shall come back to this issue when we use the scale independence of the bare couplings to derive equations for the "running of the coupling" with μ, the so called *renormalisation group equations* which make it possible to resum *leading logarithmic contributions to the renormalised coupling constant* which can improve the range of applicability of perturbation theory.

5.9 Zimmermann's forest formula

Zimmermann has given an explicit solution of the recursive BPH R-operation formula derived in the last section. We start with his definition of a *forest*. Let Γ be an arbitrary diagram. Then a set U of subdiagrams is called Γ-forest if

- All elements of U are renormalisation parts of Γ.
- If $\gamma_1, \gamma_2 \in U$ then they are *not* overlapping.
- U may be the empty set $\emptyset$.

Now it is easy to prove that the recursive BPH-scheme described in the last section can be explicitely solved for a given diagram Γ with help of *Zimmerman's forest formula*. In the following we denote the set of all forests of a given digram Γ with $\mathfrak{F}(\Gamma)$. Then the integrand for the renormalised Feynman integral corresponding to Γ is given by

$$R_\Gamma = \sum_{U \in \mathfrak{F}(\Gamma)} \left[\prod_{\gamma \in U}{}' (-t^\gamma) \right] I_\Gamma, \tag{5.217}$$

where I_Γ is the integrand given by applying the naive Feynman rules for Γ. The product of the Taylor operators t^γ is ordered such that if $\gamma_1 \subseteq \gamma_2$ then t^{γ_1} as to be applied before t^{γ_2}. Otherwise γ_1 and γ_2 are disjoined and thus the corresponding Taylor operators are commuting.

To prove the forest formula we introduce the set $\bar{\mathfrak{F}}(\Gamma)$ of all *normal forests*. Thereby a forest U is called normal if $\Gamma \notin U$; otherwise its called a *full forest*. Now we use the simple fact that each full forest U is given by the union of a normal forest $\bar{U} = U \setminus \{\Gamma\}$ with $\{\Gamma\}$ which is a special full forest.

Now we define

$$\bar{R}'_\gamma = \sum_{U\in\bar{\mathfrak{F}}(\Gamma)} \left[\prod_{\gamma'\in U}(-t^{\gamma'})\right] I_\gamma. \tag{5.218}$$

Now we like to show that $\bar{R}'_\gamma$ is identical with $\bar{R}_\gamma$ defined by (5.215-5.216). For that purpose it is of course enough to show that they fulfil the recursion relation (5.215). But for any forest $U \in \bar{F}(\gamma)$ we can join together the biggest nested parts which are mutually disjoined and call them $\gamma'_1, \ldots, \gamma'_k$. Of course we have also to take into account the empty forest which means

$$\sum_{U\in\bar{\mathfrak{F}}(\Gamma)} \left[\prod_{\gamma'\in U}(-t^{\gamma'})\right] = 1 + \sum_{\{\gamma'_1,\ldots,\gamma'_k\}} \sum_{\tau=1}^{c} \left\{(-t^{\gamma_\tau})\left[\sum_{U_\tau\in\bar{\mathfrak{F}}(\gamma_\tau)} \prod_{\gamma''_\tau\in U_\tau}(-t^{\gamma''_\tau})\right]\right\} \tag{5.219}$$

where the sum runs over all sets of mutually disjoined nested parts $\gamma'_1, \ldots, \gamma'_k$ of γ. In the bracket on the right hand side is of the same form as (5.218) but for the γ'_τ instead of γ. Since these γ'_τ are mutually disjoined by definition we can write

$$I_\gamma = I_{\gamma/\{\gamma'_1,\ldots,\gamma_k\}} \prod_{\tau=1}^{k} I_{\gamma'_\tau}. \tag{5.220}$$

Thus (5.218) can be written in the form

$$\bar{R}'_\gamma = I_\gamma + \sum_{\{\gamma'_1,\ldots,\gamma'_k\}} I_{\gamma/\{\gamma'_1,\ldots,\gamma_k\}} \prod_{\tau=1}^{k}(-t^{\gamma'_\tau})\bar{R}'_{\gamma'_\tau}, \tag{5.221}$$

which is exactly the same recursion as (5.215). This shows that $\bar{R}'(\gamma) = \bar{R}(\gamma)$. From the definition of the forests of Γ we see immediately that also the overall subtraction is given in the right way:

$$R_\Gamma = (1-t^\Gamma)\bar{R}_\Gamma = \sum_{U\in\bar{\mathfrak{F}}(\Gamma)} (1-t^\Gamma) \prod_{\gamma\in U}(-t^\gamma) I_\Gamma, \tag{5.222}$$

but the right hand side is identical with that of (5.217). Indeed if Γ is overall divergent then it is a member of all full forests which in turn can again be written as the union of a normal forest with $\{\Gamma\}$ as already mentioned above, and thus (5.222) is indeed identical with (5.217). If on the other hand Γ is not a renormalisation part $t^\Gamma = 0$ by definition and (5.217) is also correct since then Γ is not contained in any forest.

Now we are ready to combine the minimal subtraction scheme defined with help of the dimensional regularisation scheme with the BPHZ renormalisation description which is given in condensed form by the Forest formula (5.217). The only difference is how the subtraction of the divergences is defined. The only necessity for the proof of the renormalisability with local counterterms was the fact that in the BPH scheme the $t^\gamma I_\gamma$ are polynomials of the external momenta of the order given by the superficial degree of divergence.

Now the minimal subtraction scheme is defined that one has to subtract only the pole parts in the analytical dimension plane which are proportional to $1/\epsilon^k$.[20] The BPHZ-formalism and especially the forest formula shows that this divergent part of a Feynman integral must be a polynomial in the external momenta if all its sub-divergences have been removed before. Since

[20] Remember that the physical space time dimension $d = 4 - 2\epsilon$ is given by $\epsilon = 0$!

the forest formula exactly provides this task of removing all subdivergences before the overall divergences we can simply read t^γ in (5.217) as the operator which subtracts the pole parts in ϵ from the dimensionally regularised Feynman integral corresponding to the diagram γ, which completely defines the *minimal subtraction renormalisation scheme*. Of course we are free to switch to other schemes by subtracting further finite polynomials of the divergent vertices.

Let us now look a last time on the explicit example of ϕ^4-theory to clarify this rather abstract ideas a little bit. Thereby we also solve the problem of the massless theory within dimensional regularisation and the MS scheme. While the renormalised Lagrangian is given by

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda \mu^{2\epsilon}}{24}\phi^4, \tag{5.223}$$

the bare one reads

$$\mathscr{L}_0 = \mathscr{L} + \delta\mathscr{L} = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{24}\phi_0^4. \tag{5.224}$$

In our dimensional regularisation scheme with the counterterm approach we express all quantities in terms of the renormalised quantities. Concerning the massless case the important point within this scheme is to write the counterterm Lagrangian in the form

$$\delta\mathscr{L} = \frac{\delta Z}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\delta Z_m}{2} m^2 \phi^2 - \frac{\delta Z_\lambda \mu^{2\epsilon}}{24}\lambda \phi^4. \tag{5.225}$$

Now δZ, δZ_m and $\delta\lambda$ are dimensionless quantities which also do not contain any derivatives. In the momentum representation this means that these quantities are independent of the external momenta of the n-point vertex functions and dimensionless.

Now we investigate the relation between the dimensionally regularised bare and the renormalised quantities:

$$(1+\delta Z)\phi^2 := Z\phi^2 = \phi_0^2, \quad m_0^2 = \frac{1+\delta Z_m}{Z} m^2, \quad \lambda_0 = \frac{1+\delta Z_\lambda}{Z^2}\mu^{2\epsilon}\lambda. \tag{5.226}$$

Since Z, Z_m and Z_λ are dimensionless and the bare coupling λ_0 is independent of μ they are of the form

$$\begin{aligned} Z &= 1 + \sum_{k=1}^{\infty} \frac{Z_k(\lambda)}{\epsilon^k}, \\ Z_m &= 1 + \sum_{k=1}^{\infty} \frac{Z_{mk}(\lambda)}{\epsilon^k}, \\ Z_\lambda &= 1 + \sum_{k=1}^{\infty} \frac{Z_{\lambda k}(\lambda)}{\epsilon^k}. \end{aligned} \tag{5.227}$$

Thus *in the minimal subtraction scheme* the renormalisation constants Z, Z_m and Z_λ are independent of the renormalised mass m and thus for $m^2 = 0$ there is no singularity at all. This leads to the conclusion that for $m^2 = 0$ also the bare mass vanishes such that also the massless theory is renormalisable.

5.10 Global linear symmetries and renormalisation

We come now back to the issue of symmetries. We have shown in section 4.6.5 that symmetries which are implemented by linear representations on the fields survive the quantisation process provided that the path integral measure used to define the effective action is unchanged under the linear symmetry operation.

From the point of view of the Lagrangian a symmetry means a restriction for possible terms which can appear in it. In this chapter so far we have shown that a field theory is renormalisable if the Lagrangian is built by polynomials in the fields and its derivatives such that all constants appearing in the Lagrangian have an energy dimension less or equal to 4. Since a scalar field has dimension 1 and each derivative adds also 1 dimension the most general Lorentz invariant Lagrangian for a theory with N scalar fields reads

$$\mathscr{L} = \frac{1}{2}A_{jk}(\partial_\mu\phi^j)(\partial^\mu\phi^k) + \frac{1}{2}B_{jk}\phi^j\phi^k + \frac{C_{jkl}}{3!}\phi^j\phi^k\phi^l + \frac{D_{jklm}}{4!}\phi^j\phi^k\phi^l\phi^m + E_j\phi^j. \tag{5.228}$$

The coefficients are arbitrary but can be taken without loss of generality to be totally symmetric in the indices. The discussion in this chapter so far has shown that this defines a renormalisable theory, i.e., all infinities can be compensated by rescaling the coefficients (with mass independent) factors. For the scalar theory there is a speciality concerning the three-point vertices: On the first glance these should be linearly divergent but since they are scalars for the purely scalar theory they cannot contain a contribution proportional to a four-momentum. Thus in the same way Lorentz symmetry and power counting forbids us to write down a derivative coupling for a renormalisable scalar theory it saves us from the need of a counterterm with a derivative coupling. Thus the three-point vertex is only logarithmically and not linearly divergent as naive power counting would suggest. This shows that symmetries can lead to a reduction of the order of divergence compared to pure power counting.

Now we like to find out whether a linearly realised symmetry makes trouble for the renormalisation program. As an example we want to look on the O(N)-symmetric version of the theory given by (5.228), i.e., we want the classical action

$$S[\phi] := \{\mathscr{L}_1\}_1 \tag{5.229}$$

to be symmetric under all transformations of the form

$$x' = x,\ \vec{\phi}'(x') = \hat{A}\vec{\phi}(x),\ \text{with}\ \hat{A}^{-1} = \hat{A}^t \in \mathbb{R}^{N\times N}, \tag{5.230}$$

where we have used the notation introduced in (3.33). An infinitesimal transformation is given by

$$\delta\phi_j = \delta\omega_{jk}\phi_k \tag{5.231}$$

where here and in the following we suppress the space-time argument which is not important as long as we investigate only the here considered *global symmetries*. It is clear that the orthogonality conditions translate into the antisymmetry of $\delta\omega_{jk} = -\delta\omega_{kj}$. Now we can count the number of linearly independent generators of infinitesimal O(N) transformations: The diagonal elements of the generators must be 0 and we have $N(N-1)/2$ independent off-diagonal elements. An basis of the Lie algebra o(N) is given by

$$(\tau^{lm})_{jk} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{5.232}$$

and the most general infinitesimal transformation can be written in the form

$$\frac{1}{2}\delta\omega_{lm}\hat{\tau}^{lm}. \tag{5.233}$$

Now we are ready to investigate the restrictions for the Lagrangian (5.228) if we want it to be invariant under all infinitesimal O(N) transformations. It is clear from simple dimensional analysis that each monomial itself must be invariant. We have only to apply the general infinitesimal basis transformation which leads to the condition

$$\forall l,m \in \{1,\ldots,N\},\ \vec{\phi} \in \mathbb{R}^N :\ E_j \tau_{jk}^{lm} \phi_k = E_l \phi_m - E_m \phi_l = 0 \tag{5.234}$$

Now for each l we may $\phi_l = 0$ keeping all other components of $\vec{\phi}$ non-zero which leads to $E_l = 0$ for all $l \in \{1,\ldots,N\}$.

By analogous arguments we find that the most general O(N)-symmetric renormalisable Lagrangian for a scalar N-let reads

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \frac{m^2}{2}\vec{\phi}^2 - \frac{\lambda}{8}(\vec{\phi}^2)^2. \tag{5.235}$$

We have written the renormalised Lagrangian in the usual way with the free propagator residuum normalised to 1. Now from the discussion in chapter 4.6.4 we know that also the dimensionally regularised effective quantum action $\Gamma[\vec{\varphi}]$ is symmetric under global O(N) transformations, which of course does not depend on the space-time dimension. Since $1/\hbar$ is an overall factor in the exponential of the path-integral the effective action is also O(N)-symmetric to any loop order L. Now suppose we have renormalised the effective action to all loop order smaller than or equal to L. Then we know that at loop order $L+1$ all proper sub-divergences of any diagram are of loop order $\leq L$ and thus are renormalised with the symmetric counterterms by assumption. Now the BPHZ-renormalisation formalism tells us that the remaining overall divergences of the diagrams of loop order $L+1$ are local polynomials up to order 4 since we have restricted ourselves. Further we know that the regularised action is symmetric to any orders. From this we immediately conclude that also for the overall divergences we need counterterms which are only of the same form as the most general *symmetric Lagrangian*. This means we have

$$\delta\mathscr{L} = \frac{\delta Z_\phi}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \frac{\delta Z_m m^2}{2}\vec{\phi}^2 - \frac{\delta Z_\lambda \lambda}{8}(\vec{\phi}^2)^2 \tag{5.236}$$

where all δZ_k for $k \in \{\phi, m, \lambda\}$ are independent of the mass in the minimal subtraction schemes MS or $\overline{\text{MS}}$. This shows that also the bare Lagrangian must be O(N) symmetric, i.e., of the form (5.235).

Now we can answer the question what happens if we chose $m^2 = -\tilde{m}^2 < 0$. On the first side we would conclude that we would describe "tachyons", i.e., particles with a space-like on-shell momentum. On the other hand since the vacuum is translationally invariant in space and time, the vacuum expectation value of the fields must be space-time independent, and on the classical level we need to inspect the potential

$$V(\phi) = -\mathscr{L}(\phi)|_{\phi=\text{const}} = \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{8}(\vec{\phi}^2)^2. \tag{5.237}$$

If now $m^2 = -\tilde{m}^2 < 0$ we see that the solution $\vec{\phi} = 0$ of the static and homogeneous equations of motion is a maximum of the potential and not a minimum and thus not a stable solution. Especially it is not stable against quantum fluctuations in the quantised theory! On the other hand

we find an continuously degenerated solution of minima of V, namely those field configurations which obey

$$\frac{\lambda}{2}\vec{\phi}^2 = \tilde{m}^2. \tag{5.238}$$

We are free to chose any vector $\vec{v}$ as a solution. This means that although the equations of motion by construction fulfil the restriction of O(N) symmetry leading to a conserved Noether current, but the stable homogeneous solution is not $\vec{\phi} = 0$ but any vector $\vec{v}$ different from 0 which is of course not O(N)-invariant since it defines a preferred direction in the Noether-charge space. This is known as the *the spontaneous breakdown of the symmetry.*

Now suppose the same holds true for the regularised effective quantum action too, i.e., the stable solution of the equation of motion

$$\left.\frac{\delta\Gamma[\vec{\varphi}]}{\delta\vec{\varphi}}\right|_{\vec{\varphi}=\vec{v}} = 0 \tag{5.239}$$

is different from 0 then we can conclude immediately from its O(N)-symmetry that any O(N) transformation of $\vec{v}$ also is a possible vacuum expectation value. Since the δZ_k in (5.236) are all independent of the mass m we can renormalise also this case of a negative mass parameter $m^2 = -\tilde{m}^2$ with symmetric counterterms. Thus also the renormalised theory may have vacuum expectation values different from 0 and the symmetry is spontaneously broken for the renormalised quantum theory. In this case one also says the symmetry is realised in the *Nambu-Goldstone mode*. The name comes from the seminal papers about the subject of spontaneous symmetry breaking in quantum field theories [NJL61, Gol61]. The here given more modern discussion making use of the effective action goes back to [JL64]. If the vacuum expectation value itself is invariant under the symmetry, i.e., in our case if the stable solution is given by $\vec{v} = 0$ then one calls the symmetry realised in the *Wigner-Weyl mode*.

To investigate the conclusions we can draw from the spontaneous symmetry breaking we give a derivation of the the so called *Ward-Takahashi* identities for the O(N) symmetry. We only use the fact that the classical action is symmetric under infinitesimal symmetry transformations. For sake of convenience we label the basis elements of the Lie algebra o(N) with $\hat{\tau}^a$ where $a \in \{1, 2, \ldots, N(N-1)/2$. Then (5.231) can be written in the more general form

$$\delta\vec{\phi} = \delta\chi_a \hat{\tau}^a \vec{\phi}. \tag{5.240}$$

Now we use the invariance of the path-integral measure in the definition (4.152) for the generating functional Z for disconnected Green's functions. The integration over the field momenta is a Gaussian path integral, which leads up to an indefinite factor which we lump again in the overall normalisation N:

$$Z[\vec{J}] = \int \mathrm{D}\vec{\phi} \exp\left[\mathrm{i}\left\{\mathscr{L}(\vec{\phi}(x)) + \mathrm{i}\vec{J}(x)\vec{\phi}(x)\right\}_x\right]. \tag{5.241}$$

The invariance of the path integral under general field translations reads:

$$Z[\vec{J}] = \int \mathrm{D}\vec{\phi} \exp\left[\mathrm{i}\left\{\mathscr{L}[\vec{\phi}(x) + \delta\vec{\phi}(x)] + \vec{J}(x)(\vec{\phi}(x) + \delta\vec{\phi}(x))\right\}_x\right], \tag{5.242}$$

and for infinitesimal transformations

$$0 = \left\{\int \mathrm{D}\vec{\phi}\left[\frac{\delta S}{\delta\vec{\phi}(x')}\delta\vec{\phi}(x') + \vec{J}(x')\delta\vec{\phi}\right] \exp\left[\mathrm{i}\left\{\mathscr{L}(\vec{\phi}(x)) + \mathrm{i}\vec{J}(x)\vec{\phi}(x)\right\}_x\right]\right\}_{x'}. \tag{5.243}$$

Now we use (5.240) and the invariance of $S[\vec{\phi}]$ under the infinitesimal O(N) transformations:

$$\begin{aligned} 0 &= \delta\chi_a \left\{ \int \mathrm{D}\vec{\phi} \left[\vec{J}(x')\hat{\tau}^a\vec{\phi} \right] \exp\left[\mathrm{i} \left\{ \mathscr{L}(\vec{\phi}(x)) + \mathrm{i}\vec{J}(x)\vec{\phi}(x) \right\}_x \right] \right\}_{x'} = \\ &= \delta\chi_a \left\{ \vec{J}(x')\hat{\tau}^a \frac{\delta}{\delta \mathrm{i}\vec{J}(x)} \right\} Z[\vec{J}]. \end{aligned} \tag{5.244}$$

This tells us nothing else than that Z is a functional of $\vec{J}$ which is invariant under O(N) transformations of the external auxiliary current $\vec{J}$. This is what we call the Ward-Takahashi identity (WTI) for the generating functional Z although usually this name is used for the analogous situation in the case of abelian local gauge theories like QED which we shall discuss in detail in the next chapter.

Since (5.244) contains first functional derivatives only the same holds true for the generating functional of connected Green's functions (4.204):

$$W[\vec{J}] = -\mathrm{i}\ln Z[\vec{J}],\ \delta\chi_a \left\{ \vec{J}(x')\hat{\tau}^a \frac{\delta}{\delta \mathrm{i}\vec{J}(x)} \right\} W[\vec{J}] = 0. \tag{5.245}$$

In our case that is a triviality since, if Z is invariant under O(N) transformations the same holds true for $W = -\mathrm{i}\ln Z$. Finally making use of (4.216) we obtain the relation

$$\delta\chi^a \left\{ \frac{\delta\Gamma}{\delta\vec{\varphi}(\vec{x})} \hat{\tau}^a \vec{\varphi}(x) \right\}_x = 0 \tag{5.246}$$

which again reproduces the result we obtained in (4.228) in a more explicit form for the more restricted case of a *linearly realized global symmetry*: The (symmetrically regularised) effective quantum action is invariant under the same realisation of the symmetry group as the effective action and, as we have shown by our discussion above, this also holds true for the renormalised effective quantum action.

The power of the compact formulation of the symmetry property (5.246) lays in the fact that by taking further functional derivatives and setting $\vec{\varphi} = \vec{v}$ we can derive an infinite set of relations between the proper n-point vertex functions which are the WTIs for the vertex functions which play a even more crucial role in the case of gauge theories (including QED) than here in the case of global symmetries.

We shall give only the most important conclusion from the WTI for the two-point function, which is the inverse exact propagator as was already shown in (4.217). For that we just take the derivative of (5.246) with respect to $\vec{\phi}(x')$ where we switch for sake of uniqueness to a component notation:

$$\delta\chi_a \left\{ (G^{-1})_{1i,2l} \tau^a_{ik} \right\}_1 v_k = 0. \tag{5.247}$$

Using the independence of v_k from the space-time coordinates and the fact that G^{-1} must be a function of the coordinate difference $x_1 - x_2$, which both facts follow from the translation invariance of the theory, we find by inserting the Fourier representation

$$(G^{-1})_{1i,2l} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} [G^{-1}(p)]_{il} \exp[-\mathrm{i}p(x_1 - x_2)] \tag{5.248}$$

into (5.247):

$$\delta\chi^a [G^{-1}(p=0)]_{il} \tau^a_{ik} v^k = 0. \tag{5.249}$$

Now physically this means the following. First we note that $G^{-1}(p=0)_{il}$ is the bilinear mass-matrix in the effective quantum potential. Second we remember that we defined v to minimise the effective quantum potential in order to be a stable solution. This means for the mass matrix that it must be *positive semi-definite*. This means all our particles which are identified with the excitations around the vacuum $\vec{v}$ have positive (or zero) squared masses, i.e., they are no tachyons as it should be. This means that $\vec{v}$ is the stable vacuum which therefore is sometimes also called the "*true vacuum*".

If $\vec{v} \neq 0$, i.e., if the global symmetry is linearly realised in the Nambu-Goldstone mode of a quantum field theory and $\vec{v}$ breaks the symmetry spontaneously, than for any linearly independent infinitesimal symmetry operation there exists a massless field degree of freedom. Such massless fields are called *Nambu-Goldstone modes*.

In our case it is simple to count the Goldstone modes explicitly: It is clear that the symmetry group which leaves the vacuum invariant is the $\mathrm{O}(N-1)$-subgroup of $\mathrm{O}(N)$ which operates only on the field degrees of freedom perpendicular to the direction defined by $\vec{v}$. The number of Goldstone bosons is thus given by

$$\dim \mathrm{O}(N) - \dim \mathrm{O}(N-1) = \frac{N(N-1) - (N-1)(N-2)}{2} = N-1. \tag{5.250}$$

Indeed there from the N field degrees of freedom there is only one in the direction parallel to $\vec{v}$ and thus there must remain $N-1$ perpendicular to it, and those are exactly the $N-1$ massless Nambu-Goldstone modes.

We want to mention here that for the case $N=4$ the three massless Goldstone bosons where identified with the three sorts of pions and the fourth massive field with the σ-mesons. Nowadays we know that this is only one possible realisation of the *chiral symmetry* of the light-quark sector of *Quantum chromodynamics* or *QCD*, the theory describing the strong interactions of elementary particles which are called *quarks*. We know that there are two light quarks, the up- and down-quark (if one likes one can also treat the strange quark as light too). Now free quarks were never observed in experiments so far, which is called the *confinement* but only as bound states of an anti-quark and a quark (which are called *mesons*) and a bound state of three quarks (called *baryons*). Both sorts of strongly interacting composed particles are called the *hadrons*. The quark structure can only be dissolved at higher scattering energies where the quarks inside the hadrons behave as nearly free particles. At low scattering energies the quarks are confined in the hadrons hand are not dissolved by the scattering experiments. At low (scattering) energies we can thus describe the hadrons in good approximation by elementary fields of a quantum field theory with the appropriate symmetries. As we shall discuss in chapter 7 in detail in the limit of vanishing masses of the light quarks the QCD-Lagrangian shows a so called chiral symmetry $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ which is spontaneously broken to a simple $\mathrm{SU}(2)$. Within an effective quantum field theory model for the scalar sector of the possible composite quark anti-quark bound states, describing scalar spin-0-bosons one can realise this symmetry by a linearly operating $\mathrm{O}(4)$-symmetry for four scalar field degrees of freedom. As we have seen this symmetry property together with renormalisability uniquely determines the to be of the form (5.235). Thus all our conclusions are valid for this case. The conclusion is that the pions are the three Goldstone bosons from the spontaneous breakdown of the chiral $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ to a simple $\mathrm{SU}(2)$ in the light quark sector of QCD. For sake of completeness we close the chapter with the remark that the pions have a mass of around 140MeV which leads to the conclusion that the above mentioned chiral symmetry is realised only approximately in nature due to small but non-zero masses of the light quarks. We shall come back in detail to all this questions in chapter 7 where we discuss Quantum

chromodynamics.

5.10.1 Example: 1-loop renormalisation

To illustrate the general considerations of the previous section we shall calculate the divergent one-loop diagrams for the general case of a non-vanishing constant mean field $\vec{v} \neq 0$ for the O(N) linear Σ-model.

First we have to derive the Feynman rules. For the propagator we introduce the two projectors

$$P_{\parallel}^{jk} = \frac{v^j v^k}{\vec{v}^2}, \quad P_{\perp}^{jk} = \delta^{jk} - \frac{v^j v^k}{\vec{v}^2}. \tag{5.251}$$

The inverse free propagator is the second functional derivative of the classical action with respect to the fields taken at $\vec{\varphi} = \vec{v}$:

$$\hat{\Delta}^{-1}(k) = \Delta_{\parallel}^{-1}(k)\hat{P}_{\parallel} + \Delta_{\perp}^{-1}(k)\hat{P}_{\perp} \tag{5.252}$$

with

$$\begin{aligned} \Delta_{\parallel}(k) &= \frac{1}{k^2 - m_{\parallel}^2 + \mathrm{i}\eta}, \quad \Delta_{\perp}(k) = \frac{1}{k^2 - m_{\perp}^2 + \mathrm{i}\eta}, \\ m_{\parallel}^2 &= \frac{3}{2}\lambda v^2 - \tilde{m}^2, \quad m_{\perp}^2 = \frac{\lambda}{2}v^2 - \tilde{m}^2. \end{aligned} \tag{5.253}$$

Then we have a three-point and a four-point vertex, which read, with our method to determine the symmetry factors:

$$\begin{aligned} &\begin{array}{ccc} & & l \\ & \diagup & \diagdown \\ j & & k \end{array} = -\frac{\mathrm{i}\lambda}{3!}(v^j\delta^{kl} + v^k\delta^{jl} + v^l\delta^{jk}) = -\frac{\mathrm{i}\lambda}{3!}\delta^{jklm}v_m, \\ &\begin{array}{ccc} i & & l \\ j & \times & k \end{array} = -\frac{\mathrm{i}\lambda}{4!}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) = -\frac{\mathrm{i}\lambda}{4!}\delta^{ijkl}. \end{aligned} \tag{5.254}$$

We start to calculate the one-point function which has to vanish according to the equations of motion for $\vec{\varphi} = \vec{v}$. The one-loop part of this function reads according to the Feynman rules

$$\mathrm{i}[\Gamma_1^{(1)}]_j = \begin{array}{c} \bigcirc \\ | \\ j \end{array} = \frac{\lambda}{2}v_j\mu^{2\epsilon}\int\frac{\mathrm{d}^d l}{(2\pi)^d}[3\Delta_{\parallel}(l) + (N-1)\Delta_{\perp}(l)]. \tag{5.255}$$

With help of (C.8) we obtain with our usual abbreviation $d = 4 - 2\epsilon$ after an Laurent expansion around $\epsilon = 0$:

$$[\Gamma_1^{(1)}]_j = \frac{\lambda v_j}{64\pi^2}\frac{\lambda(8+N)v^2 - 2(N+2)\tilde{m}^2}{\epsilon} + O(\epsilon^0). \tag{5.256}$$

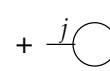
Here we are only interested in the part of the integrals which are infinite for $\epsilon \to 0$ since we like to show the validity of our considerations concerning renormalisation within the minimal-subtraction scheme which subtracts only the pole terms for $\epsilon \to 0$.

According to (5.236) we have as the counterterms of the *minimal subtraction scheme*

$$\begin{aligned} \delta_{\mathrm{MS}}[\Gamma_1^{(1)}] &= (\delta Z_m^{(\mathrm{MS})})_1\tilde{m}^2 v_j - \frac{(\delta\lambda^{(\mathrm{MS})})_1}{2}v^2 v_j, \\ (\delta Z_m^{(\mathrm{MS})})_1 &= \frac{\lambda(N+2)}{32\pi^2\epsilon}, \quad (\delta\lambda^{(\mathrm{MS})})_1 = \frac{\lambda^2(N+8)}{32\pi^2\epsilon} \end{aligned} \tag{5.257}$$

Thus already (5.256) *determines completely the mass- and coupling constant counterterm.* This shows again how strict the constraints are which are forced by the O(N)-symmetry of the theory.

Now we come to the self-energy part. This is according to the Feynman rules

$$\mathrm{i}[\Gamma_1^{(2)}(p)]_{jk} = -\mathrm{i}[\Sigma_1(p)]_{jk} =$$  $$+$$  $$. \tag{5.258}$$

Due to the O(N)-symmetry the self-energy decomposes like the free propagator in a parallel and a transverse part:

$$[\Sigma_1(p)]_{jk} = \Sigma_{\parallel 1}(p) P_{\parallel jk} + \Sigma_{\perp 1}(p) P_{\perp jk}. \tag{5.259}$$

From the Feynman rules we read off the analytic expressions

$$\begin{aligned}
\Sigma_{\parallel 1}(p) =& \frac{\mathrm{i}\lambda}{2}\mu^{2\epsilon}\int\frac{\mathrm{d}^d l}{(2\pi)^d}[3\Delta_\parallel(l) + (N-1)\Delta_\perp(l)] + \\
&+ \frac{\mathrm{i}\lambda^2 v^2}{2}\mu^{4\epsilon}\int\frac{\mathrm{d}^d l}{(2\pi)^d}[9\Delta_\parallel(l+p)\Delta_\parallel(l) + (N-1)\Delta_\perp(l)\Delta_\perp(l+p)], \\
\Sigma_{\perp 1}(p) =& \frac{\mathrm{i}\lambda}{2}\mu^{2\epsilon}\int\frac{\mathrm{d}^d l}{(2\pi)^d}[\Delta_\parallel(l) + (N+1)\Delta_\perp(l)] + \\
&+ \frac{\mathrm{i}\lambda^2 v^2}{2}\mu^{4\epsilon}\int\frac{\mathrm{d}^d l}{(2\pi)^d}[2\Delta_\parallel(l+p)\Delta_\perp(l)].
\end{aligned} \tag{5.260}$$

The result reads

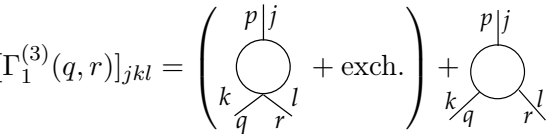
$$\begin{aligned}
\Sigma_{\parallel 1}(p) &= \frac{\lambda}{64\pi^2}\frac{2(N+2)\tilde{m}^2 - 3\lambda(N+8)v^2}{\epsilon} + O(\epsilon^0), \\
\Sigma_{\perp 1}(p) &= \frac{\lambda}{64\pi^2}\frac{2(N+2)\tilde{m}^2 - \lambda(N+8)v^2}{\epsilon} + O(\epsilon^0).
\end{aligned} \tag{5.261}$$

In this case there is no wave-function counterterm at the here considered one-loop level since the tadpole diagram is constant in p^2 and the exchange loop contains only the vertex counterterm already determined by (5.257). Indeed the MS-counterterms for the self-energy are given by the counterterms already determined by (5.257):

$$\begin{aligned}
(\delta_{\mathrm{MS}}\Sigma_\parallel)_1 &= -(\delta Z_m^{(\mathrm{MS})})_1\tilde{m}^2 + \frac{3}{2}(\delta\lambda^{(\mathrm{MS})})_1 v^2 \\
(\delta_{\mathrm{MS}}\Sigma_\perp)_1 &= -(\delta Z_m^{(\mathrm{MS})})_1\tilde{m}^2 + \frac{1}{2}(\delta\lambda^{(\mathrm{MS})})_1 v^2
\end{aligned} \tag{5.262}$$

as it should be for counterterms that are consistent with the O(N) symmetry of the original Lagrangian.

Now we come to the three-point function. The Feynman diagrams read

$$\mathrm{i}[\Gamma_1^{(3)}(q,r)]_{jkl} =$$  $$. \tag{5.263}$$

The reminder "exch." means that one has to add the two crossed versions of the first diagram with the index-momentum pairs (j,p), (q,k), (r,l) interchanged.

Only the first sort of diagrams gives rise to logarithmic (rather than linear, which is another consequence of the underlying O(N)-symmetry!) divergences. We have to show that these are compensated by the already determined coupling-constant counterterm:

$$\delta_{\mathrm{MS}}[\Gamma_1^{(3)}(q,r)]_{jkl} = -\delta\lambda\delta_{jklm}v^m. \tag{5.264}$$

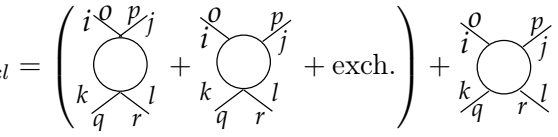
The Feynman rules give for this divergent part (including the exchange terms):

$$[\Gamma_1^{(3)}(q,r)]_{jkl} = \frac{\lambda(N+8)}{32\pi^2\epsilon}\delta_{jklm}v^m + O(\epsilon^0), \tag{5.265}$$

which pole contribution is indeed cancelled by the counterterm (5.264).

Finally we have the four-point vertex itself. Again first we draw the Feynman diagrams:

$$\mathrm{i}[\Gamma_1^{(4)}(o,p,q)]_{ijkl} =$$



(5.266)

Again it is clear that only the first diagram contains logarithmic divergences and thus we have to calculate only this diagram to get the terms in $1/\epsilon$. We have to prove that the counterterm is given by

$$\delta_{\mathrm{MS}}[\Gamma_1^{(4)}(o,p,q)]_{ijkl} = -\delta\lambda\delta_{ijkl}. \tag{5.267}$$

Evaluation of the Feynman diagram including the crossed channels, again denoted by "exch." in (5.266) gives

$$[\Gamma_1^{(4)}(o,p,q)]_{ijkl} = \frac{\lambda(N+8)}{32\pi^2\epsilon}\delta_{jklm} + O(\epsilon^0), \tag{5.268}$$

while the terms $\propto v_i v_j/v^2 g_{kl}$, analogous terms with permuted indices, and $\propto v_i v_j v_k v_l/v^4$ are all finite and thus give no ϵ-pole contributions. Thus again the counterterm (5.267) cancels the divergent piece of (5.268).

We have shown explicitly that the one-loop level can be rendered finite by subtracting mass and coupling constant counterterms compatible with the O(N) symmetry of the classical Lagrangian. For higher order corrections there will be of course also *wave-function renormalisations*, but these are also compatible with the symmetry as we have shown by our inductive argument in the previous section.

5.11 Renormalisation group equations

In this section we investigate the dependence of the vertex functions on the chosen renormalisation scheme. In fact, the outcome for physical observables should not depend on the renormalisation scheme, and this is achieved by the running of the wave function normalisation, the mass and to coupling with the renormalisation scale. The renormalisation group equations (RGEs) describe this running.

We shall see that, although the various functions, entering the RGEs, have to be calculated perturbatively, the solution of the RGEs can improve perturbation theory, since they resum leading logarithmic contributions to the running of the coupling constant.

5.11.1 Homogeneous RGEs and modified BPHZ renormalisation

The derivation of the RGEs becomes most clear in BPHZ-like renormalisation techniques, since there no intermediate regularisation is introduced. First, we shall look at the *mass independent renormalisation schemes* (MIR schemes). We follow here [Kug77, Kug97] with some deviations in the treatment. Especially we avoid the introduction of a cut-off parameter which breaks Lorentz invariance. In the next section we shall show that the MS and $\overline{\text{MS}}$ schemes of the previous section, which are defined through dimensional regularisation, are special cases of such MIR schemes. Such renormalisation schemes, based on dimensional regularisation, is especially well suited for the perturbative investigation of gauge theories although the dependence on the renormalisation scale is less intuitive.

For simplicity's sake we look again at simple ϕ^4 theory with the renormalised Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5.269}$$

The bare Lagrangian is given in terms of the bare fields and parameters

$$\mathscr{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{m^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4. \tag{5.270}$$

Now, we write the counterterm Lagrangian in the form

$$\delta\mathscr{L} = \frac{\delta Z}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\delta Z_m}{2} m^2\phi^2 - \frac{\delta m^2}{2}\phi^2 - \frac{\delta\lambda}{4!}\phi^4. \tag{5.271}$$

In the MIR scheme we introduce a *mass renormalisation scale* M and use the fact that we always can subtract at the symmetric off-shell point $\tilde{p} = 0$ for $m^2 > 0$. Here $\tilde{p}$ denotes an arbitrary set of independent external momenta of an n-point function. We will also choose invariant parameters to characterise these external momenta, for instance the invariant $s = p^2$ for the self-energies or the Mandelstam variables s, t, u for the four-point function.

We also ignore vacuum contributions, corresponding to closed diagrams without external fields, to the effective action, i.e., we understand the action to be normalised such that $\Gamma[\phi = 0] = 0$.

Since the superficial degree of divergence for a $\Gamma^{(n)}$-graph is $\delta(\Gamma^{(n)}) = 4 - n$, we need to impose renormalisation conditions only for $n = 2$ (corresponding to wave-function and mass renormalisation) and for $n = 4$ (corresponding to coupling-constant renormalisation). All the $\Gamma^{(n)}$ with odd n vanish due to field-reflection symmetry.

We need a mass renormalisation scale, because any integral of superficial divergence degree ≤ 0 is not only UV- but also IR divergent for $m^2 = 0$ at the point $\tilde{p} = 0$. Thus, if we like to avoid a space-like momentum subtraction point, and choose the symmetric point $\tilde{p} = 0$ of BPHZ, we have to subtract all divergences with divergence degree 0 at a point $m^2 = M^2 > 0$. The finite renormalised quantities then remain well-defined also for $m^2 = 0$ or even at $m^2 < 0$, where the ϕ^4-model shows spontaneous breaking of the $\mathbb{Z}_2$ field-reflection symmetry. For now, we stay at $m^2 \geq 0$. We thus impose the renormalisation conditions

$$\begin{aligned}
&\Sigma(p^2 = 0, m^2 = 0, \lambda; M^2) = 0 \;\Rightarrow\; \delta m^2 = 0,\\
&[\partial_{m^2}\Sigma(p^2, m^2, \lambda; M^2)]_{m^2=M^2, p^2=0} = 0 \;\Rightarrow\; \delta Z_m,\\
&[\partial_{p^2}\Sigma(p^2, m^2, \lambda; M^2)]_{m^2=M^2, p^2=0} = 0 \;\Rightarrow\; \delta Z,\\
&\Gamma^{(4)}(s = t = u = 0, m^2 = M^2, \lambda; M^2) = -\lambda \;\Rightarrow\; \delta\lambda.
\end{aligned} \tag{5.272}$$

One should keep in mind that the self energy is related to $\Gamma^{(2)}$ by

$$\Gamma^{(2)}(p^2, m^2, \lambda, M^2) = p^2 - m^2 - \Sigma(p^2, m^2, \lambda; M^2). \tag{5.273}$$

The renormalisation conditions (5.272) make it immediately clear, that δZ_m, δZ, and $\delta\lambda$ cannot depend on m: Since all external momenta are set to 0 at the renormalisation point, there is no other dimensionful quantity in the game than the renormalisation scale M^2, and thus, since these quantities are of mass dimension 0, they can only depend on λ. Thus, within the MIR schemes, the M^2 dependence of the logarithmic quantities is only implicit through the M^2 dependence of the running coupling λ.

The choice of the first condition provides a speciality of this particular MIR scheme, namely $\delta m^2 = 0$: Since the integrand of the self-energy without BPHZ-subtractions does not depend on M^2, there is no dimensionful quantity at the point $m^2 = 0$. Thus the dependence of δm^2 on M^2 can be only through λ. But δm^2 is itself of mass dimension 2. So, having no dimensionful quantity at hand, imposing the first condition in Eq. (5.272), makes necessarily δm^2 vanish. It is the special choice of the first condition in (5.272) which is the common feature of this special MIR scheme and the MS and $\overline{\text{MS}}$-scheme.

If we choose another scheme, like an off-shell scheme with momentum renormalisation scale Λ (the so called *momentum subtraction scheme* or MOM scheme) we need an additional mass counterterm δm^2 which is not proportional to m^2 but to Λ^2.

There are also MIR schemes for which $\delta m^2 \neq 0$, as we shall see in Sect. 5.12.2. The only common feature of the MIR schemes is that the counter terms are independent of m^2 and thus especially well suited for renormalisation of the massless case or for $m^2 < 0$ which leads to spontaneous breakdown of the $\mathbb{Z}_2$ field-reflection symmetry.

The effective quantum action Γ, i.e., the generating functional for 1PI truncated Green's functions (proper vertex functions) has a certain (finite!) value, independent of whether we express it in terms of bare or renormalised quantities:

$$\Gamma_0[\phi_0, m_0^2, \lambda_0] = \Gamma[\phi, m^2, \lambda; M^2]. \tag{5.274}$$

The bare quantities do not depend on M^2 which enters the game only through renormalisation. Thus, taking the derivative with respect to M yields 0. So we find

$$\left(M\frac{\partial}{\partial M} + M\frac{\partial\lambda}{\partial M}\frac{\partial}{\partial\lambda} + M\frac{\partial m^2}{\partial M}\frac{\partial}{\partial m^2} + M\int \mathrm{d}^4x \frac{\partial\phi(x)}{\partial M}\frac{\delta}{\delta\phi(x)}\right)\Gamma[\phi, m^2, \lambda; M^2] = 0. \tag{5.275}$$

Now the first task is to determine the dimensionless functions

$$\beta(\lambda) := M\frac{\partial\lambda}{\partial M}, \quad \gamma_m(\lambda) := -\frac{1}{m^2}M\frac{\partial m^2}{\partial M}, \quad \gamma_\phi(\lambda) := -\frac{1}{\phi}M\frac{\partial\phi}{\partial M}. \tag{5.276}$$

It is clear that these quantities in principle can be expressed in terms of the renormalisation factors Z, Z_m, and Z_λ, introduced in (5.235). We shall come back to this issue when we use dimensional regularisation, because the Z's are not available in BPHZ like renormalisation schemes, since we have no meaningful definitions for the Z-factors without some kind of regularisation. What becomes clear from (5.276) is, that nevertheless these quantities are finite. It is also clear that γ_ϕ does not depend on x.

With (5.276) the RGE (5.276) reads:

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} - \gamma_m m^2\frac{\partial}{\partial m^2} - \gamma_\phi\int \mathrm{d}^4x\phi(x)\frac{\delta}{\delta\phi(x)}\right]\Gamma[\phi, m^2, \lambda; M^2] = 0. \tag{5.277}$$

Applying this to the series expansion

$$\Gamma[\phi, m^2, \lambda; M^2] = \sum_{n=2}^{\infty} \frac{1}{n!} \left\{ \Gamma^{(n)}_{12\ldots n}(m^2, \lambda; M^2)\phi_1\phi_2\cdots\phi_n \right\}_{12\ldots n} \tag{5.278}$$

gives the *homogeneous renormalisation group equations*

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial \lambda} - \gamma_m m^2 \frac{\partial}{\partial m^2} - n\gamma_\phi \right] \Gamma^{(n)}[\tilde{p}, m^2, \lambda; M^2] = 0, \tag{5.279}$$

where we have switched to the momentum space representation.

The functions β, γ_m and γ_ϕ can be determined by calculating the $\Gamma^{(n)}$ for $n = 2$ and $n = 4$ perturbatively. The RGE (5.279) for the functions $\Gamma^{(2)}(p^2, m^2, \lambda; M^2)$, $\partial_{p^2}\Gamma^{(2)}(p^2, m^2, \lambda; M^2)$ and $\Gamma^{(4)}(s, t, u, m^2, \lambda; M^2)$ at the points $p^2 = s = t = u = 0$ gives a set of linear equations for RGE coefficients β, γ_m and γ_ϕ.

As an example we take the one-loop approximation for ϕ^4-theory. The subtracted (and thus finite) integrals are easily calculated with help of the standard formulae in the appendix which here are used for $d = 4$:

$$\begin{aligned} \Gamma^{(2)}(p^2, m^2, \lambda; M^2) &= p^2 - m^2 + \frac{\lambda m^2}{32\pi^2}\left[1 - \ln\left(\frac{m^2}{M^2}\right)\right]\hbar + O(\hbar^2), \\ \Gamma^{(4)}(s = t = u = 0, m^2, \lambda; M^2) &= -\lambda - \frac{3\lambda^2}{32\pi^2}\ln\left(\frac{m^2}{M^2}\right)\hbar + O(\hbar^2). \end{aligned} \tag{5.280}$$

We reintroduced $\hbar$ in these equations, in order to indicate the loop expansion, applied here.

The renormalisation group equations for $\Gamma^{(2)}$, $\partial_{p^2}\Gamma^{(2)}$ and $\Gamma^{(4)}$ read, after setting $p^2 = 0$, $s = t = u = 0$ and $m^2 = M^2$:

$$\begin{aligned} &\gamma_m M^2 + 2\gamma_\phi M^2 + \left(\frac{\beta M^2}{32\pi^2} + \frac{\lambda M^2}{16\pi^2} - \frac{\gamma_\phi \lambda M^2}{16\pi^2}\right)\hbar + O(\hbar^2) = 0, \\ &-2\gamma_\phi + O(\hbar^2) = 0, \\ &-\beta + 4\gamma_\phi\lambda + \left(\frac{3\lambda^2}{16\pi^2} + \gamma_m\frac{3\lambda^2}{32\pi^2}\right)\hbar + O(\hbar^2) = 0. \end{aligned} \tag{5.281}$$

Its solution is

$$\beta = \frac{3\lambda^2}{16\pi^2} + O(\hbar^2), \quad \gamma_M = -\frac{\lambda\hbar}{16\pi^2} + O(\hbar^2), \quad \gamma_\phi = O(\hbar^2). \tag{5.282}$$

5.11.2 The homogeneous RGE and dimensional regularisation

The above formalism, using only the renormalised Γ functional, or equivalent to that, the renormalised proper vertex functions, is more complicated than necessary as long as we stick to the perturbative calculation of the vertex functions and the RGE coefficients β, γ_m and γ_ϕ. In particular, for gauge theories, where the MIR scheme is (due to possible infrared problems) not so convenient, the dimensional regularisation method provides a much more convenient possibility to calculate these functions.

As we shall see, the MS and $\overline{\text{MS}}$ renormalisation schemes are also a mass-independent renormalisation scheme. Here a renormalization scale enters the game, because in space-time dimensions

different from 4 the coupling becomes a dimensionful quantity: Its mass dimension is $4-d=2\epsilon$. Thus in dimensional regularisation we introduce a mass scale μ by writing $\mu^{2\epsilon}\lambda$, such that the renormalised coupling parameter λ is kept dimensionless. We shall see that, in the limit $\epsilon \to 0$ which by construction can be taken for renormalised quantities, this scale μ^2 corresponds to M^2 of the MIR scheme with the special first renormalisation condition in (5.272) which makes δm^2 vanishing.

The advantage of the use of a regularisation before the renormalisation is that we have an explicit expression for the Z-factors at hand. From them we can directly derive the RGE coefficients β, γ_m and γ_ϕ as derivatives with respect to μ at fixed bare parameters. As we have shown above, the RGE coefficients are finite[21], so that the limit $d \to 4$ can be taken at the end of the calculation.

The idea is to calculate the *dimensionally regularised* Z-factors, defined by (5.226). For that we need only the terms, divergent for $\epsilon \to 0$ of the renormalisation parts Σ and $\Gamma^{(4)}$. To one-loop order we have

$$\begin{aligned} \Sigma_{\mathrm{DR}} &= -\frac{\lambda\hbar}{32\pi^2}\frac{1}{\epsilon}m^2 + O(\hbar^2) + \text{finite}, \\ \Gamma^{(4)}_{\mathrm{DR}} &= -\frac{3\lambda\hbar}{32\pi^2}\frac{1}{\epsilon} + O(\hbar^2) + \text{finite}. \end{aligned} \tag{5.283}$$

The MS scheme is defined to subtract only the divergent terms $\propto 1/\epsilon$ which leads to the following Z-factors

$$Z^{(\mathrm{MS})} = 1 + O(\hbar^2), \quad Z_m^{(\mathrm{MS})} = 1 + \frac{\lambda\hbar}{32\pi^2}\frac{1}{\epsilon} + O(\hbar^2), \quad Z_\lambda^{(\mathrm{MS})} = 1 + \frac{3\lambda\hbar}{32\pi^2}\frac{1}{\epsilon} + O(\hbar^2). \tag{5.284}$$

From (5.226) we have

$$\lambda_0 = \frac{Z_\lambda}{Z^2}\mu^{2\epsilon}\lambda. \tag{5.285}$$

To find β, we derive with respect to μ. Since the bare quantities do not depend on μ, this yields

$$0 = \left[\frac{Z_\lambda}{Z^2} + \lambda\frac{\partial}{\partial\lambda}\left(\frac{Z_\lambda}{Z^2}\right)\right]\beta + 2\epsilon\lambda\frac{Z_\lambda}{Z^2}. \tag{5.286}$$

Solving for β gives

$$\beta(\lambda) = \lim_{\epsilon\to 0}\beta_\epsilon = -\lim_{\epsilon\to 0}\frac{2\epsilon\lambda}{1+\lambda\frac{\partial}{\partial\lambda}\ln\left(\frac{Z_\lambda}{Z^2}\right)} \tag{5.287}$$

From the previous section we know that the limit exists. If we calculate the renormalisation factors perturbatively, of course, this is true only up to the calculated order of the expansion parameter. With (5.283) we find in the MS scheme up to order $\hbar$:

$$\beta^{(\mathrm{MS})}(\lambda) = \lim_{\epsilon\to 0}\left[-2\epsilon\lambda + \frac{3\lambda^2\hbar}{16\pi^2} + O(\hbar^2)\right] = \frac{3\lambda^2\hbar}{16\pi^2} + O(\hbar^2). \tag{5.288}$$

This is the same as (5.282). This needs not to be true to all orders since in the MS (as well as the $\overline{\mathrm{MS}}$) scheme the finite parts from the loop corrections are changing the renormalisation conditions (5.272). Within the MS or $\overline{\mathrm{MS}}$ schemes, the values of the functions at the renormalisation points, given in (5.272) are known only after the calculation of the vertex functions.

[21] This is, of course, true only up to the calculated order of perturbation theory.

In the same way as (5.287) one derives

$$\gamma_\phi = \lim_{\epsilon\to 0} \frac{\beta_\epsilon}{2} \frac{\partial}{\partial\lambda} \ln Z, \quad \gamma_m = \lim_{\epsilon\to 0} \beta_\epsilon \frac{\partial}{\partial\lambda} \ln\left(\frac{Z_m}{Z}\right). \tag{5.289}$$

To also find the other renormalization-group coefficients we note that

$$\phi_0^2 = Z\phi^2 \;\Rightarrow\; 0 = \mu\frac{\partial}{\partial\mu}\phi_0^2 = \mu\frac{\partial Z}{\partial\mu}\phi^2 + 2\phi Z\mu\frac{\partial\phi}{\partial\mu} \;\Rightarrow\; \gamma_\phi = -\mu\frac{\partial\phi}{\partial\mu} = \lim_{\epsilon\to 0}\frac{\beta_\epsilon}{2}\frac{\partial \ln Z}{\partial\lambda} \tag{5.290}$$

and

$$m_0^2 = \frac{Z_m}{Z}m^2 \;\Rightarrow\; 0 = \mu\frac{\partial}{\partial\mu}m_0^2 \;\Rightarrow \gamma_m = -\frac{1}{m^2}\mu\frac{\partial}{\partial\mu}m^2 = \lim_{\epsilon\to 0}\beta_\epsilon\frac{\partial}{\partial\lambda}\ln\left(\frac{Z_m}{Z}\right). \tag{5.291}$$

Using (5.284) we find, after expansion up to order $\mathcal{O}(\hbar)$, i.e., in one-loop approximation we obtain the indeed finite results

$$\gamma_\phi^{(\mathrm{MS})} = O(\hbar^2), \quad \gamma_m^{(\mathrm{MS})} = -\frac{\lambda\hbar}{16\pi^2} + O(\hbar^2). \tag{5.292}$$

We can also draw conclusions about the general structure of the running coupling and then, via (5.290) and (5.291), the wave-function and mass-renormalization constants Z and Z_m at arbitrary orders of perturbation theory [Col74, CM74]. Since MS and $\overline{\mathrm{MS}}$ schemes are mass-independent schemes and the dimensionally regularized divergent parts of the vertex functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ thus are poles in ϵ with coefficients that depend only on the dimensionless renormalized coupling λ, i.e., not explicitly on the renormalized mass m and the renormalization scale μ, the relation between the bare and renormalized coupling is

$$\lambda_0 = \mu^{2\epsilon}\left(\lambda + \sum_{k=1}^{\infty}\frac{a_k(\lambda)}{\epsilon^k}\right). \tag{5.293}$$

Taking the derivative with respect to μ yields

$$0 = \mu\frac{\partial\lambda_0}{\partial\mu} = 2\epsilon\left(\lambda + \sum_{k=1}^{\infty}\frac{a_k(\lambda)}{\epsilon^k}\right) + \beta_\epsilon\left(1 + \sum_{k=1}^{\infty}\frac{a_k'(\lambda)}{\epsilon^2}\right). \tag{5.294}$$

Here β_ϵ is an analytic function of ϵ in a neighbourhood of $\epsilon = 0$, and the $a_k(\lambda)$ are independent of ϵ. Thus we have by coefficient comparison on both sides of the equation

$$\beta_\epsilon = 2\epsilon\lambda + \beta. \tag{5.295}$$

Plugging this into (5.294) we find by further comparing the coefficients of the Laurent series on both sides of the equation

$$\beta = 2(\lambda a_1' - a_1), \quad \lambda a_{k+1}' - a_{k+1} = (\lambda a_1' - a_1)a_k'. \tag{5.296}$$

The β function thus is determined by the residue of the $\mathcal{O}(1/\epsilon)$ pole of $\lambda Z_\lambda/Z^2$ only. Indeed for the one-loop approximation we find from (5.284)

$$\lambda\frac{Z_\lambda}{Z^2} = \lambda + \frac{3\lambda^2\hbar}{32\pi^2\epsilon} + \mathcal{O}(\hbar^2) \;\Rightarrow\; a_1(\lambda) = \frac{3\lambda^2\hbar}{32\pi^2} \tag{5.297}$$

and thus

$$\beta = \frac{3\lambda^2\hbar}{16\pi^2} + \mathcal{O}(\hbar^2) \tag{5.298}$$

as we have already found in (5.269) by the direct calculation.

The coefficients a_k are all determined from a_1 via (5.296) and the condition that $a_k(0) = 0$.

For the mass we have

$$m_0^2 = m^2 \left(1 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k}\right). \tag{5.299}$$

Taking the derivative with respect to μ and applying (5.295) one finds by comparison of coefficients

$$\gamma_m(\lambda) = -\frac{\mu}{m^2}\frac{\partial}{\partial\mu}m^2 = -2\lambda b_1'(\lambda), \quad \gamma_m b_j = -2\lambda b_{j+1}' + \beta b_j'. \tag{5.300}$$

For the wave-function renormalization factor we have

$$Z = 1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k} \tag{5.301}$$

and from

$$\phi_0^2 = Z\phi^2 \tag{5.302}$$

we find with the same techniques

$$\gamma_\phi(\lambda) = -\frac{\mu}{\phi}\frac{\partial\phi}{\partial\mu} = -\lambda c_1'(\lambda), \quad \lambda C_{j+1}' = \left(\frac{\beta}{2} - \gamma_\phi\right) c_j. \tag{5.303}$$

We check these relations for the expansion with respect to powers in λ, which we have available up to order λ^2. From (5.88) and (5.118) we have

$$Z = 1 \underbrace{-\frac{\lambda^2}{6144\pi^4}}_{c_1}\frac{1}{\epsilon} + \mathcal{O}(\lambda^3), \tag{5.304}$$

$$m_0^2 = \frac{Z_m}{Z}m^2 = m^2\left[1 + \underbrace{\frac{\lambda^2}{512\pi^4}}_{b_2}\frac{1}{\epsilon^2} + \underbrace{\left(\frac{\lambda}{32\pi^2} - \frac{5\lambda^2}{6144\pi^4}\right)}_{b_1}\frac{1}{\epsilon}\right] + \mathcal{O}(\lambda^3), \tag{5.305}$$

and from (5.90) together with (5.98)

$$\lambda_0 = \frac{Z_\lambda}{Z^2}\lambda\mu^{2\epsilon} = \left(\lambda + \underbrace{\frac{3\lambda^2}{32\pi^2}}_{a_1}\frac{1}{\epsilon}\right)\mu^{2\epsilon}\lambda. \tag{5.306}$$

Now with (5.296), (5.301) and (5.303) we find

$$\beta = 2(\lambda a_1' - a_1) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^2) \tag{5.307}$$

$$\gamma_m = -2\lambda b_1' = \frac{\lambda^2}{3} = -\frac{\lambda}{16\pi^2} + \frac{5\lambda^2}{1536\pi^4} + \mathcal{O}(\lambda^3), \tag{5.308}$$

$$\gamma_\phi = -\lambda c_1' = \frac{\lambda^2}{3072\pi^4} + \mathcal{O}(\lambda^3). \tag{5.309}$$

We can also check that the recursion relations for the coefficients a_k, b_k and c_k in the Eqs. (5.296), (5.301) and (5.303) are fulfilled up to order λ^2.

5.11.3 Solutions to the homogeneous RGE

Now we have found two ways to calculate perturbatively the functions β, γ_ϕ and γ_m to the homogeneous RGE (5.275). Given these functions, it is easy to obtain solutions for this equation since it is a functional extension to the flow equation

$$\frac{\partial}{\partial t} S(t,\vec{x}) + \vec{v}(\vec{x}) \nabla_{\vec{x}} S(t,\vec{x}) = 0 \tag{5.310}$$

which describes the conservation of the quantity S along the orbit of a volume element of a fluid $\vec{x}(t;\vec{x}_0)$ with the initial condition $\vec{x}(0) = \vec{x}_0$. Indeed, from (5.310), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} S[t,\vec{x}(t,\vec{x}_0)] = 0, \quad \frac{\mathrm{d}\vec{x}(t,\vec{x}_0)}{\mathrm{d}t} = \vec{v}[\vec{x}(t,\vec{x}_0)]. \tag{5.311}$$

This means

$$S[t,\vec{x}(t,\vec{x}_0)] = S[0,\vec{x}_0]. \tag{5.312}$$

For the RGE (5.275) Γ corresponds to S, the renormalised parameters m^2, ϕ, λ to $\vec{x}$, the functions γ_m, γ_ϕ, β to $\vec{v}(\vec{x})$. Instead of ∂_t, in (5.310) we have $M\partial_M$. Thus we set $\bar{M} = M\exp(\tau)$. Then we have $M\partial_M = \partial_\tau$, and the defining equations (5.276) for the RGE functions β, γ_ϕ and γ_m read

$$\partial_\tau \phi = -\gamma_\phi(\lambda)\phi, \quad \partial_\tau m^2 = -\gamma_m(\lambda) m^2, \quad \partial_\tau \lambda = \beta(\lambda). \tag{5.313}$$

Given the solution to the third equation,

$$\int_\lambda^{\bar{\lambda}} \frac{\mathrm{d}\lambda'}{\beta(\lambda')} = \int_0^\tau \mathrm{d}\tau' = \tau \quad \text{with } \bar{\lambda} = \lambda(\bar{M}),\ \lambda = \lambda(M), \tag{5.314}$$

the solutions for the first equation is given by integration:

$$\bar{\phi} = \phi \exp\left\{ -\int_0^\tau \mathrm{d}\tau' \gamma_\phi[\lambda(\tau')] \right\} = \phi \exp\left[-\int_\lambda^{\bar{\lambda}} \mathrm{d}\lambda' \frac{\gamma_\phi(\lambda')}{\beta(\lambda')} \right] := \bar{Z}^{1/2}\phi. \tag{5.315}$$

Here $\bar{Z}$ is a *finite* renormalisation constant, i.e., (5.315) describes the change of the field's normalisation due to the change of the mass renormalisation scale from M to $\bar{M} = M\exp\tau$. Of course, also the mass has to change according to the second equation of (5.313):

$$\bar{m}^2 = m^2 \exp\left[-\int_\lambda^{\bar{\lambda}} \mathrm{d}\lambda' \frac{\gamma_m(\lambda')}{\beta(\lambda')} \right] := \bar{Z}_m m^2 \tag{5.316}$$

Then we can apply (5.312) to the RGE (5.277):

$$\Gamma[\phi, m^2, \lambda; \mu^2] = \Gamma[\bar{\phi}, \bar{m}^2, \bar{\lambda}, \bar{\mu}^2]. \tag{5.317}$$

This equation is no surprise, since the generating functional for vertex functions Γ is always equal to the corresponding bare functional and thus its value cannot change when the renormalisation scale M is changed. With (5.314-5.316) we have defined how the coupling and the renormalised mass have to be scaled in order to ensure this independence of the renormalisation scale.

From (5.317) one immediately obtains the corresponding behaviour of the vertex functions under change of the renormalisation scale:

$$\Gamma^{(n)}_{1\dots n}(m^2,\lambda;M^2) = \frac{\delta^n}{\delta\phi_1\cdots\delta\phi_n}\Gamma[\phi,m^2,\lambda,M^2]\bigg|_{\phi=0} = \bar{Z}^{n/2}\Gamma^{(n)}_{1\dots n}(\bar{m}^2,\bar{\lambda};\bar{M}^2). \tag{5.318}$$

Here we have used the chain rule for functional derivatives together with (5.315).

A Fourier transformation shows that the same equation holds true in the momentum space representation:

$$\Gamma^{(n)}(\tilde{p},m^2,\lambda;M^2) = \bar{Z}^{n/2}\Gamma^{(n)}(\tilde{p},\bar{m}^2,\bar{\lambda};\bar{M}^2). \tag{5.319}$$

5.11.4 Independence of the S-Matrix from the renormalisation scale

From (5.319) we can immediately see that S-Matrix elements are independent of the renormalisation scale. Due to the LSZ reduction theorem, to calculate transition matrix elements, as the first step we have to build (connected) Green's functions from the vertices (1PI truncated Green's functions) and to amputate the external legs again. This is done by connecting $\Gamma^{(n)}$ functions with Green's functions $G^{(2)} = 1/\Gamma^{(2)}$. Rescaling M to $\bar{M}$, according to (5.319), each Green's function gets a factor $1/Z$. This compensates the two factors $\sqrt{Z}$ of $\Gamma^{(n)}$ corresponding to the legs which are connected. Thus the truncated n-point Green's function obeys the same scaling law (5.318):

$$W^{(n)}_{\text{trunc}}(\tilde{p},m^2,\lambda;M^2) = \bar{Z}^{n/2}W^{(n)}_{\text{trunc}}(\tilde{p},\bar{m}^2,\bar{\lambda};\bar{M}^2). \tag{5.320}$$

To obtain the S-matrix element, for the truncated leg, labelled with k, we have to multiply with the corresponding asymptotically free wave function $\varphi_k(p)$ in momentum space, corresponding to the single-particle states in the in- and out-multi particle states. On the right-hand side we write $\varphi_k(p_k) = \bar{\varphi}_k(p_k)Z^{-1/2}$. Since this happens to each of the n external legs, finally we have

$$S_{fi} = \prod_{k=1}^{n}\varphi_k(p_k)W^{(n)}_{\text{trunc}}(\tilde{p},m^2\lambda;M^2) = \prod_{k=1}^{n}\bar{\varphi}_k(p_k)W^{(n)}_{\text{trunc}}(\tilde{p},\bar{m}^2,\bar{\lambda};\bar{M}^2). \tag{5.321}$$

That means, the S-matrix element does not change with the renormalisation scale as it should be for physical quantities.

It is clear that, for perturbatively calculated matrix elements, again this holds true only up to the order of the expansion parameter taken explicitly into account.

5.12 Asymptotic behaviour of vertex functions

Now we use the fact that in momentum space representation the mass dimension of $\Gamma^{(n)}(p)$ is $4-d$. So these functions must fulfil the following scaling law

$$\Gamma^{(n)}(\mathrm{e}^{\tau}\tilde{p},\mathrm{e}^{2\tau}m^2,\lambda,\bar{M}^2) = \exp[(4-n)\tau)]\Gamma^{(n)}[\tilde{p},m^2,\lambda;M^2]. \tag{5.322}$$

On the left-hand side, we have used the definition $\bar{M} = \mathrm{e}^{\tau}M$. Making use of (5.318) we find

$$\Gamma^{(n)}(\mathrm{e}^{\tau}\tilde{p},\mathrm{e}^{2\tau}m^2,\lambda;\bar{M}^2) = \exp[\tau(4-n)]\bar{Z}^{n/2}\Gamma^{(n)}(\tilde{p},\bar{m}^2,\bar{\lambda};\bar{M}^2). \tag{5.323}$$

If we substitute m^2 with $\mathrm{e}^{-2\tau}m^2$, according to (5.316) also $\bar{m}^2 = \mathrm{e}^{-2\tau}\bar{m}^2$. The same time we can write M^2 instead of $\bar{M}^2$ on both sides of (5.323):

$$\Gamma^{(n)}(\mathrm{e}^{\tau}\tilde{p}, m^2, \lambda; M^2) = \exp\left[\tau(4-n) - n\int_{\lambda}^{\bar{\lambda}} \mathrm{d}\lambda' \frac{\gamma_\phi(\lambda')}{\beta(\lambda')}\right] \Gamma^{(n)}(\tilde{p}, \mathrm{e}^{-2\tau}\bar{Z}_m m^2, \bar{\lambda}; M^2). \quad (5.324)$$

This means, a scaling of the external momenta $\tilde{p}$ can be compensated by a redefinition of m^2 with $\mathrm{e}^{-2\tau}\bar{Z}_m m^2$ and of λ by $\bar{\lambda}$. Here we have to take $\bar{m}$ and $\bar{\lambda}$ as functions of m and λ. The scaling behaviour of the vertex functions is not the naively expected, i.e., it does not scale with the canonical dimension $4-n$ but with one corrected by the factor in (5.324). Also m^2 does not rescale in the canonical way, namely just by a factor $\exp(-2\tau)$ but with an additional factor $\bar{Z}_m$. Since λ is dimensionless, naively we expect it to be unchanged by a rescaling of the renormalisation mass scale, but as we learnt now instead it is multiplied by a factor $\bar{Z}_\lambda/\bar{Z}^2$. For this reason, the RGE coefficients β, γ_ϕ and γ_m often are also called *anomalous dimensions*.

We like to check (5.324) for the one-loop result. As an example, we take as $\Gamma^{(4)}$. Since $\gamma_\phi = O(\hbar^2)$ the overall scaling factor is simply 1. Thus we need to calculate $\bar{\lambda}$ and $\bar{Z}_m$ only. Using (5.289) and (5.292) yields

$$\begin{aligned}
\bar{\lambda} &= \frac{16\lambda\pi^2}{16\pi^2 - 3\hbar\lambda\tau + O(\hbar^2)} = \lambda + \frac{3\lambda^2\hbar}{16\pi^2}\tau + O(\hbar^2),\\
\bar{Z}_m &= \left(\frac{\bar{\lambda}}{\lambda}\right)^{1/3} = 1 + \frac{\lambda\hbar\tau}{16\pi^2},\\
\bar{Z} &= 1 + O(\hbar^2).
\end{aligned} \quad (5.325)$$

The perturbative expression for the four-point function is (in the MIR scheme)

$$\Gamma^{(4)}(s,t,u,m^2,\lambda;M^2) = -\lambda + \hbar\frac{\lambda^2}{2}[A(s) + A(t) + A(u)] \quad (5.326)$$

with

$$A(s) = \frac{1}{16\pi^2}\left[2 + \ln\left(\frac{M^2}{m^2}\right) - 2\frac{\sqrt{s(s-4m^2)}}{s}\operatorname{artanh}\left(\frac{s}{\sqrt{s(s-4m^2)}}\right)\right]. \quad (5.327)$$

After some algebra, one can verify the relation

$$\Gamma^{(4)}(\mathrm{e}^{2\tau}s, \mathrm{e}^{2\tau}t, \mathrm{e}^{2\tau}u, m^2, \lambda; M^2) = \Gamma^{(4)}(s,t,u,\bar{Z}_m^2 m^2, \bar{\lambda}; M^2) \quad (5.328)$$

to order $\hbar$ which is identical with 5.324.

5.12.1 The Gell-Mann-Low equation

For sake of completeness, we like to derive also the RGEs for other renormalisation schemes. We start with a MOM scheme (momentum-subtraction scheme). Here the renormalisation conditions usually are given by

$$\begin{aligned}
&\Sigma(p^2 = m^2, m^2, \lambda; M^2) = 0,\\
&\partial_{p^2}\Sigma(p^2, m^2, \lambda; M^2)|_{p^2=-M^2} = 0,\\
&\Gamma^{(4)}(\tilde{p}, m^2, \lambda; M^2)|_{s=t=u=-4/3M^2} = -\lambda
\end{aligned} \quad (5.329)$$

The subtraction point $p^2 = -M^2$ for the second condition, determining the wave-function renormalisation, should be space-like and independent of m for the case we like to study the massless theory, i.e., $m = 0$. For $m > 0$, one can choose a value for $M^2 \in]-4m^2, \infty[$. The first condition chooses the physical mass of the particle to be given by m. Using a BPHZ renormalisation, modified to fulfil (5.329), in (5.271) again $\delta m^2 = 0$. Here also γ_m vanishes since the unrenormalised integrand for Σ does not depend on M and the subtractions for Σ are taken at $p^2 = m^2$, according to the first line in (5.329), the overall counterterms $\delta Z_m m^2$ to the mass do not depend on M^2 either. So here we have only two RGE coefficients, namely

$$\beta(\lambda, m/M) = M\partial_M \lambda, \quad \gamma_\phi(\lambda, m/M) = -\frac{M}{\phi}\partial_M \phi. \tag{5.330}$$

The corresponding renormalisation group equation is derived in the same way as (5.275), i.e., from

$$\begin{aligned} &\Gamma[\phi_0^2, m_0^2, \lambda_0] = \Gamma[\phi, m^2, \lambda; M^2] \;\Rightarrow \\ &\left[M\partial_M + \beta(\lambda, m/M)\partial_\lambda - \gamma_\phi(\lambda, m/M)\int \mathrm{d}^4x \phi(x)\frac{\delta}{\delta\phi(x)}\right]\Gamma[\phi, m^2, \lambda; M^2] = 0. \end{aligned} \tag{5.331}$$

This is the *Gell-Mann-Low RGE*. Expanding with respect to the field ϕ one obtains the Gell-Mann-Low RGE for the proper vertex functions in the same way as we derived (5.279):

$$[M\partial_M + \beta(\lambda, m/M)\partial_\lambda - n\gamma_\phi(\lambda, m/M)]\,\Gamma^{(n)}(\tilde{p}, m^2, \lambda; M^2) = 0. \tag{5.332}$$

It shows that also in this scheme the S-matrix elements are independent of the choice of the momentum renormalisation scale M^2. The proof is exactly the same as given in Sect. 5.11.4 for the homogeneous RGE.

The difference to the homogeneous renormalisation group equation is that, in general, its solution is not so easy since the coefficients β and γ_ϕ depend on both, λ and explicitly on m^2/M^2. Only in the limit $m \to 0$ the equation becomes solvable as easy as the homogeneous one.

For the perturbative calculation of β and γ_ϕ we can use the same technique as shown in Sect. 5.11.1. One only has to use (5.332) for the renormalisation parts $\Gamma^{(2)}$ and $\Gamma^{(4)}$ and solve for the linear system of equations for β and γ_ϕ. For ϕ^4-theory, we obtain in lowest $\hbar$-order

$$\begin{aligned} \gamma_\phi^{(\mathrm{MOM})} &= O(\hbar^2), \\ \beta^{(\mathrm{MOM})} &= \frac{3\lambda^2\hbar}{16\pi^2}\left[1 - \frac{3m^2}{\sqrt{3m^2M^2 + M^4}}\,\mathrm{artanh}\left(\frac{M^2}{\sqrt{3m^2M^2 + M^4}}\right)\right] + O(\hbar^2). \end{aligned} \tag{5.333}$$

5.12.2 The Callan-Symanzik equation

In the original BPHZ renormalisation scheme, the renormalised mass has to be fixed to $m > 0$. The renormalisation conditions read

$$\begin{aligned} \Sigma_{\mathrm{BPHZ}}(s = 0, m^2, \lambda) &= 0, \\ \partial_s \Sigma_{\mathrm{BPHZ}}(s, m^2, \lambda)|_{s=0} &= 0, \\ \Gamma^{(4)}_{\mathrm{BPHZ}}(s, t, u, m^2, \lambda)|_{s=t=u=0} &= -\lambda. \end{aligned} \tag{5.334}$$

Here we have neither a momentum- nor a mass-renormalisation scale as in the MOM or MIR schemes. Here the physical mass of the particle has to be defined by the zero of

$$\Gamma^{(2)}_{\mathrm{BPHZ}}(m^2_{\mathrm{phys}}, m^2, \lambda) = 0.$$

Thus, within the BPHZ-scheme we interpret m^2 as a free mass scale, which can be varied without changing the theory, provided the wave function and coupling is renormalised such that

$$\Gamma_{\mathrm{BPHZ}}[\phi, m^2, \lambda] = \Gamma_{\mathrm{BPHZ}}[\phi, \bar{m}^2, \bar{\lambda}]. \tag{5.335}$$

Instead of deriving the corresponding RGE from scratch, it is more convenient to use the homogeneous RGE (5.275). For this, we modify our MIR scheme conditions (5.272) slightly and denote the new scheme with $\overline{\mathrm{MIR}}$:

$$\begin{aligned}
&\Sigma_{\overline{\mathrm{MIR}}}(p^2 = 0, m^2, \lambda; M^2)|_{m^2=M^2} = 0 \;\Rightarrow\; \delta m^2 = M^2 \delta Z'_m, \\
&\partial_{m^2} \Sigma_{\overline{\mathrm{MIR}}}(p^2 = 0, m^2, \lambda; M^2)|_{m^2=M^2} = 0 \;\Rightarrow\; \delta Z_m, \\
&\partial_{p^2} \Sigma_{\overline{\mathrm{MIR}}}(p^2 = 0, m^2 = M^2, \lambda; M^2) = 0 \;\Rightarrow\; \delta Z(\lambda), \\
&\Gamma^{(4)}_{\overline{\mathrm{MIR}}}(s = t = u = 0, m^2 = M^2, \lambda; M^2) = -\lambda \;\Rightarrow\; \delta\lambda(\lambda).
\end{aligned} \tag{5.336}$$

The only difference is that $\delta m^2 \neq 0$, but it is independent of m. The reasoning is the same as with the original MIR scheme: The only dimensionful quantity appearing in the non-subtracted Feynman integrands at the renormalisation point for the self-energy $m^2 = M^2$, $p^2 = 0$ is M^2 and thus $\delta m^2 \propto M^2$. The factor can only depend implicitly on M over the dependence on λ.

In the following we denote the mass parameter in the BPHZ scheme with $\tilde{m}$, that of the $\overline{\mathrm{MIR}}$ scheme with m. Then we have

$$\Gamma_{\mathrm{BPHZ}}[\phi, \tilde{m}^2, \lambda] = \Gamma_{\overline{\mathrm{MIR}}}[\phi, m^2, \lambda; M^2]|_{m=M=\tilde{m}}. \tag{5.337}$$

The homogeneous RGE for the $\overline{\mathrm{MIR}}$ scheme reads the same as before, but the RGE coefficients β, γ_ϕ and γ_m are different from those of the MIR scheme. Thus we write $\bar{\beta}$, $\bar{\gamma}_m$, $\bar{\gamma}_\phi$:

$$\left[\tilde{m}\frac{\partial}{\partial \tilde{m}} + \bar{\beta}(\lambda)\frac{\partial}{\partial \lambda} - \bar{\gamma}_m(\lambda) m^2 \frac{\partial}{\partial m^2} - \bar{\gamma}_\phi(\lambda) \int \mathrm{d}^4x \phi(x) \frac{\delta}{\delta\phi(x)}\right] \Gamma_{\overline{\mathrm{MIR}}}[\phi, m^2, \lambda; \tilde{m}^2] = 0. \tag{5.338}$$

Now we define

$$\Gamma^{(k)}_{\mathrm{BPHZ}}[\phi, \tilde{m}, \lambda] = \left(\frac{\partial}{\partial m^2}\right)^k \Gamma_{\overline{\mathrm{MIR}}}[\phi, m^2, \lambda; \tilde{m}^2]|_{m^2=\tilde{m}^2}\Bigg|_{m^2=\tilde{m}^2}. \tag{5.339}$$

For $k = 0$, according to (5.334), this is simply Γ_{BPHZ}. We shall come back to these functions at the end of this section and prove that it corresponds to the same diagrams of the original functional $\Gamma^{(k)}_{\mathrm{BPHZ}}$, but with k insertions of an auxiliary two-point vertex with the the value $-\mathrm{i}\kappa/2$ divided by $1/\kappa^k$. As we shall see, for the derivation of the renormalisation group equation for the BPHZ scheme, we need only the fact that each n-point function

$$\Gamma^{(n,k)}_{\mathrm{BPHZ}12\ldots n}[\tilde{m}, \lambda] = \frac{\delta^n}{\delta\phi_1 \cdots \delta\phi_n} \Gamma^{(k)}_{\mathrm{BPHZ}}[\phi, \tilde{m}, \lambda]\Big|_{\phi=0} \;\Rightarrow\; \Gamma^{(n,k)}_{\mathrm{BPHZ}}[\tilde{p}, \tilde{m}^2, \lambda] \tag{5.340}$$

in momentum representation has the dimension (or superficial degree of divergence)

$$\delta\left[\Gamma^{(n,k)}_{\mathrm{BPHZ}}\right] = 4 - n - 2k. \tag{5.341}$$

Then we can expand $\Gamma_{\overline{\mathrm{MIR}}}$ around $m^2 = \tilde{m}^2$:

$$\Gamma_{\overline{\mathrm{MIR}}}[\phi, m^2, \lambda; \tilde{m}^2] = \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma^{(k)}_{\mathrm{BPHZ}}[\phi, \tilde{m}^2, \lambda](m^2 - \tilde{m}^2)^k. \tag{5.342}$$

From (5.338) we find after some algebra

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{k!}(m^2-\tilde{m}^2)^k\left[\tilde{m}\frac{\partial}{\partial\tilde{m}}+\bar{\beta}\frac{\partial}{\partial\lambda}-\bar{\gamma}_\phi\int \mathrm{d}^4x\phi(x)\frac{\delta}{\delta\phi(x)}\right]\Gamma^{(k)}_{\mathrm{BPHZ}}[\phi,\tilde{m},\lambda]\\ =&\sum_{k=0}^{\infty}\frac{1}{k!}(m^2-\tilde{m}^2)^k(2\tilde{m}^2+\bar{\gamma}_m m^2)\Gamma^{(k+1)}_{\mathrm{BPHZ}}[\phi,\tilde{m}^2,\lambda].\end{aligned} \tag{5.343}$$

Setting $m=\tilde{m}$ we find the *Callan-Symanzik equation* or CS equation:

$$\left[\tilde{m}\frac{\partial}{\partial\tilde{m}}+\bar{\beta}\frac{\partial}{\partial\lambda}-\bar{\gamma}_\phi\int \mathrm{d}^4\phi(x)\frac{\delta}{\delta\phi(x)}\right]\Gamma_{\mathrm{BPHZ}}[\phi,\tilde{m}^2,\lambda]=2\tilde{m}^2\alpha(\lambda)\Gamma^{(1)}_{\mathrm{BPHZ}}[\phi,\tilde{m}^2,\lambda]. \tag{5.344}$$

Here we have introduced

$$\alpha(\lambda)=1+\frac{\bar{\gamma}_m}{2}. \tag{5.345}$$

We see that the CS equation is of the type of a functional flow equation like the homogeneous RGE, but it is *inhomogeneous* due to the right-hand side. It cannot be solved as easily as the homogeneous RGE, although the approximated coefficients $\bar{\beta}$, $\bar{\gamma}_\phi$ and $\bar{\gamma}_m$ are given explicitly from the $\overline{\mathrm{MIR}}$ scheme.

The equation for the n-point function is derived as usual by expanding the functionals in powers of ϕ. With (5.340) the CS equations for the n-point vertex functions read

$$\left[\tilde{m}\frac{\partial}{\partial\tilde{m}}+\bar{\beta}\frac{\partial}{\partial\lambda}-n\bar{\gamma}_\phi\right]\Gamma^{(n)}_{\mathrm{BPHZ}}[\tilde{p},\tilde{m}^2,\lambda]=2\tilde{m}^2\alpha\Gamma^{(n,1)}_{\mathrm{BPHZ}}[\tilde{p},\tilde{m}^2,\lambda] \tag{5.346}$$

From (5.341) we see that the only additional renormalisation part is the two-point function with one κ insertion, i.e., $\Gamma^{(2,1)}_{\mathrm{BPHZ}}$ which is logarithmically divergent. In our approach the renormalisation condition is completely fixed by the second of the $\overline{\mathrm{MIR}}$ conditions (5.336) which can be rewritten in terms of $\Gamma^{(2)}_{\overline{\mathrm{MIR}}}$:

$$\partial_{m^2}\Gamma^{(2)}_{\overline{\mathrm{MIR}}}[p^2=0,m^2,\lambda,M^2]|_{m^2=M^2}=-1. \tag{5.347}$$

Setting herein $M^2=\tilde{m}^2$ yields

$$\Gamma^{(2,1)}_{\mathrm{BPHZ}}(p^2=0,\tilde{m}^2,\lambda)=-1. \tag{5.348}$$

Then the CS equation (5.346) for $m=2$ gives, by setting $p^2=0$ the important relation

$$1-\bar{\gamma}_\phi=\alpha\ \Rightarrow\ \bar{\gamma}_m=-2\bar{\gamma}_\phi, \tag{5.349}$$

where we made use of (5.345). Clearly, the latter relation can be derived directly by specialising the homogeneous RGE (5.279) to the case $n=2$ and $s=0$, but now using the $\overline{\mathrm{MIR}}$- instead of the MIR scheme.

Deriving (5.343) j times with respect to m^2 and setting afterwards $m^2=\tilde{m}^2$, we find the generalised CS equation

$$\left[\tilde{m}\frac{\partial}{\partial\tilde{m}}+\bar{\beta}\frac{\partial}{\partial\lambda}-(n-2l)\gamma_\phi\right]\Gamma^{(n,l)}_{\mathrm{BPHZ}}(\tilde{p},\tilde{m}^2,\lambda)=2\alpha\tilde{m}^2\Gamma^{(n,l+1)}_{\mathrm{BPHZ}}[\tilde{p},\tilde{m}^2,\lambda]. \tag{5.350}$$

Again, (5.349) has been used.

For sake of completeness we shall prove the graphical meaning of the functional (5.339). Since the counterterms of the $\overline{\text{MIR}}$ scheme are independent of m^2, we can calculate the somehow regularised unrenormalised functional Γ. Taking the derivative is not affected by the counterterms, and we have just to renormalise $\Gamma^{(n,k)}[\tilde{p}] = \partial_{m^2}\Gamma^{(n)}[\tilde{p}]$ for $k = 0$ and $n \in \{2, 4\}$ (by the conditions 5.336) and for $n = 2, k = 1$ (by the condition 5.348) to obtain all the functions (5.339). We can write

$$\Gamma^{(k)}[\phi, m^2, \lambda] = \partial_\kappa^k \Gamma[\phi, m^2 + \kappa, \lambda]\Big|_{\kappa=0}. \tag{5.351}$$

Thus we define

$$Z[J, \kappa] = \int \mathrm{d}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{J_1\phi_1\right\}_1 - \frac{\mathrm{i}\kappa}{2}\left\{\phi_1^2\right\}\right]. \tag{5.352}$$

From this we obtain $\Gamma[\varphi, \kappa]$ in the same way as $\Gamma[\varphi]$ from Z, namely as the functional Legendre transform of $W = -\mathrm{i}\ln Z$. But this doesn't help us to make the κ-dependence explicite. To get this, we write (5.352) as

$$Z[J, \kappa] = \exp\left[-\mathrm{i}\int \mathrm{d}^{2\omega}x_1 \frac{\kappa}{2}\frac{\delta^2}{(\mathrm{i}\delta J_1)^2}\right] Z[J]. \tag{5.353}$$

The contribution of order κ^k in an expansion with respect to κ is obviously given by

$$Z^{(k)}[J, \kappa] = \frac{1}{k!}\left(-\mathrm{i}\frac{\kappa}{2}\right)^k \prod_{j=1}^{k} \frac{\delta^2}{(\mathrm{i}\delta J_j)^2} Z[J]. \tag{5.354}$$

Now, in the perturbative expansion of $Z[J]$, each of the k double derivatives takes two J_j away, joining the two propagators, connecting the external points of these external currents J with the (disconnected) Green's functions, together at a new vertex point x_j, and this vertex point stands for the expression $-\mathrm{i}\kappa/2$. Deriving k times by κ simply yields

$$\left(\frac{\partial}{\partial\kappa}\right)^k Z[J, \kappa]\Big|_{\kappa=0} = \frac{k!}{\kappa^k} Z^{(k)}[J, \kappa = 0]. \tag{5.355}$$

That the same construction rule holds true for connected and for 1PI Green's functions is shown in the same way as in section 4.6 for the W- and Γ functionals.

The same result can be obtained from a diagrammatic point of view from the fact that the counterterms are independent of m^2, as the original vertices. A derivative ∂_{m^2} acts only on each propagator line $1/(p^2 - m^2 + \mathrm{i}\eta)$, which becomes $1/(p^2 - m^2 + \mathrm{i}\eta)^2$. Each time deriving a diagram, thus one inserts a κ vertex into each propagator and sums up all these contributions.

This diagrammatical meaning of the derivative ∂_{m^2} shows that one obtains the correct subtractions to a diagram contributing to $\Gamma^{(n,k)}_{\overline{\text{MIR}}}$, because one applies k times the insertion procedure to each *renormalised* diagram with no insertions contributing to the renormalised function $\Gamma^{(n)}_{\overline{\text{MIR}}}$, i.e., the bare diagram together with the complete subtractions of all subdivergences and all overall divergences, due to the BPHZ theorem (corresponding to all forests of the digram). Since especially the overall counterterms of $\Gamma^{(2)}_{\overline{\text{MIR}}}$ are cancelled by taking a derivative ∂_{m^2}, because they do not contain any propagator line, which could suffer an insertion, only the correct overall subtractions for $\Gamma^{(2,1)}_{\text{BPHZ}}$ are generated. All other diagrams with $k \geq 1$ do not have any overall counterterms as it must be due to their dimension $4 - n - 2k$ and the BPHZ description[22].

[22] Note that we have $\Gamma^{(n=0,k)} \equiv 0$ due to our convention of setting $\Gamma[\phi = 0] = 0$

Chapter 6

Quantum Electrodynamics

6.1 Gauge Theory

As we have seen in section 4.1 the massless vector field which describes electromagnetic radiation in classical electromagnetics[1] is necessarily a gauge field which means that there is not a function Hilbert space which realizes the massless helicity-one representation of $\mathrm{P}^{\uparrow}_{+}$ but the quotient space

$$\mathscr{H}(0,1,\pm) = \{A^\mu | \Box A^\mu - \partial^\mu \partial_\nu A^\nu = 0\} / \{A^\mu | \partial_\mu A_\nu - \partial_\nu A_\mu = 0\}. \tag{6.1}$$

This means in our case of $\mathbb{R}^{(1,3)}$ as space-time (which is simply-connected) that for any represent A^μ in a class of solutions of the wave equations the other members of this class are given by

$$A'_\mu = A_\mu + \partial_\mu \chi. \tag{6.2}$$

Now it is rather complicated to deal with the classes of fields whenever we try to quantise it. In terms of path integral quantisation, which we shall adopt in these notes, the problem is to integrate over each class rather than over each representative. Since the action is independent of the representatives of a class for each class there is a redundant integration over the pure gauge fields $\partial\chi$ which is an unphysical (spurious) degree of freedom.

Nevertheless let us start to bring this in the Lagrangian form. From classical electromagnetics we know that for free fields all can be formulated with help of the antisymmetric second-rank tensor[2]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.3}$$

which is called *the Faraday tensor*. The field equation defining the quotient space (6.1) can be expressed in terms of the Faraday tensor as

$$\partial^\mu F_{\mu\nu} = 0 \Leftrightarrow \Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0. \tag{6.4}$$

On the other hand for a correct quantum treatment we need the Lagrangian formalism which can be set up better with help of the vector field A^μ itself. It is also shown experimentally that in the quantum case the $F_{\mu\nu}$ does not contain the full information about the field[3].

[1] We prefer the name "electromagnetics" compared to "electrodynamics" since relativistic covariance shows that there is only one electromagnetic field. The splitting in electric and magnetic field is dependent on the reference frame it is done.

[2] It is important to keep in mind that in the Lagrangian formulation of classical electro-magnetics the vector field enters the game because of the coupling to charges. We shall explain this phenomenon in the next section from the point of view of gauge theory.

[3] Aharonov-Bohm effect

Since this equation is gauge-invariant, the solution is not unique. This is ambiguous within the generating functional formalism even in the so far considered free particle case, because it is not enough to fix the propagator for these free particles by the causal (Feynman-Stueckelberg) boundary conditions to result in a unique propagator. This is clear, since only the whole class of gauge fields is a correct description of the physical degrees of freedom, not the gauge field itself. Thus we have to fix the gauge at least such that there exists a unique solution of the inhomogeneous equation for the propagator with causal boundary conditions.

This is shown as follows. The Green's function is defined by the equation

$$\partial^\mu[\partial_\mu \Delta_\nu{}^\rho(x) - \partial_\nu \Delta_\mu{}^\rho(x)] = \delta(x)\delta^\rho_\nu. \tag{6.5}$$

Trying to solve this equation of motion by the usual Fourier representation ansatz

$$\Delta_\mu{}^\rho(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp(-\mathrm{i}kx)\Delta_\mu{}^\rho(k) \tag{6.6}$$

we obtain in momentum space

$$(p_\nu p^\mu - p^2 \delta^\mu_\nu)\Delta_\mu{}^\rho \Delta^\rho_\mu = \delta^\rho_\nu. \tag{6.7}$$

Now the linear combination

$$p^\nu(p_\nu p^\mu - p^2\delta^\mu_\nu) = 0 \tag{6.8}$$

vanishes for all fixed p. Thus the solution of (6.8) cannot be unique. Clearly this ambiguity is not cured by supposing boundary conditions. If $\Delta_{\mu\rho}$ is a solution of (6.7) then any function of the form $\Delta'_{\mu\rho}(k) = \Delta_{\mu\rho}(k) + k_\mu f_\rho(k)$ with an arbitrary vector field $f_\rho(k)$ is also a solution. This reflects the gauge invariance of the equation in terms of the propagator.

All this shows that we have to fix the gauge with help of some constraint like the *Lorentz gauge constraint* $\partial_\mu A^\mu = 0$. In classical electromagnetics we are ready with this in the case of fields interacting with a conserved current (current conservation is a consistency condition for the inhomogeneous Maxwell equations $\partial^\mu F_{\mu\nu} = j_\nu$).

But now we have to quantise the theory and we have also to fix the gauge in some sense but we have to be careful in order to keep the results gauge-invariant because gauge invariance is necessary in order to be consistent with the space-time structure of special relativity as we have shown in appendix B and chapter 4.

We start as usual with the classical Lagrangian which describes the equation of motion by the stationarity of the corresponding action functional. It is easy to see that the Euler-Lagrange equations for the Lagrangian

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \text{ with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.9}$$

are the desired gauge-independent equations of motion (6.4). Since the Lagrangian is built with the Faraday tensor only this leads to trouble if we like to quantise the theory because there exists no unique propagator for the fields, because of gauge invariance.

We use the so called *Faddeev-Popov quantisation* to get rid of this problem. The main idea is to integrate over one representative of each gauge class or in other words over each element of the quotient function space (6.1) which means to integrate over the physical meaningful degrees of freedom only. This is done by fixing the gauge with help of a gauge-fixing condition

$$g[A, x] = h(x) \tag{6.10}$$

where g is a functional of A. For technical reasons we shall choose a linear one, but in general the only condition is that the constraint (6.10) fixes the gauge completely. This fixing of gauge is forced in the path integral formalism by making use of a functional δ-distribution put into the path integral and integrating out the spurious degrees of freedom coming from gauge invariance. On the other hand we like to have a gauge-independent path integral measure to satisfy gauge invariance for the physical meaningful results (S-matrix elements).

For this purpose we have to look on the problem from another point of view which will become clearer in the next section and chapter 7, namely to identify the gauge invariance as the invariance against local $U(1)$-transformations. In this Abelian case this is very simple, as we can write the gauge transformation (6.2) with help of an exponential as follows:

$$A_\mu^U(x) = A_\mu(x) - \frac{\mathrm{i}}{g} U(x)\partial_\mu U^{-1}(x) \text{ with } U(x) = \exp[-\mathrm{i}g\chi(x)]. \tag{6.11}$$

The main idea is that one can find an invariant measure for integrations over an arbitrary compact Lie group G. This means that for all $U' \in G$:

$$\int \mathrm{d}U f(U) = \int \mathrm{d}U f(UU') = \int \mathrm{d}U f(U'U). \tag{6.12}$$

In the case of path integrals we apply this so-called *Haar measure* on each point of space-time. As we have seen in the last chapter we have always an indefinite constant factor. Thus we can use the real coordinates of the group defined by $g = \exp(\alpha^k \tau_k)$, where τ_k is an arbitrary basis of the Lie algebra $\mathcal{L}G$. Then we may integrate over the whole space $\mathbb{R}^n$ (with $n = \dim G$) instead of the finite subset which covers the group only one time.

Now we apply this idea to our case of $U(1)$ gauge symmetry for QED. As we have clarified the naively written path integral

$$Z[J_\mu] = N \int \mathrm{D}A_\mu \exp\{\mathrm{i}S[A_\mu] + \mathrm{i}\,\langle J_\mu A^\mu \rangle\} \tag{6.13}$$

does not help us in calculating the action functional because the free photon propagator does not exist due to gauge invariance.

As we have seen above we can cure this problem by fixing the gauge. But on the other hand we have to be sure to save gauge invariance for the physical S-matrix elements. For this purpose we define

$$\Delta_g^{-1}[A_\mu, h] = \int \mathrm{D}U \delta[g[A_\mu^U, x] - h]. \tag{6.14}$$

Herein g is a gauge-fixing functional and h an independent function. From the assumed invariance of the measure at each space-time point, we obtain

$$\Delta_g^{-1}[A_\mu^{U'}] = \int \mathrm{D}U \delta[g[A^{U'U} - h]] = \int \mathrm{D}U'' \delta[g[A_\mu^{U''}] - h], \tag{6.15}$$

which is obtained by substituting $U'' = U'U$. This shows that the functional Δ is gauge-invariant.

Now we plug a factor 1 into the naive path integral (6.13):

$$Z[J] = N \int \mathrm{D}A_\mu \int \mathrm{D}U \Delta_g[A, h] \delta[g[A^U, x] - h] \exp\{\mathrm{i}S[A] + \mathrm{i}\,\langle J_\mu A^\mu \rangle \tag{6.16}$$

and interchange the order of integrations and substitute A^U as the integration variable over the gauge field. Using the gauge invariance of the action functional and Δ we obtain

$$Z[J] = N \int \mathrm{D}U \int \mathrm{D}A_\mu \Delta_g[A,h]\delta[g[A,x]-h] \exp\{\mathrm{i}S[A] + \mathrm{i}\left\langle J^\mu A_\mu^{U^{-1}}\right\rangle\}. \tag{6.17}$$

By construction the whole expression is independent of the auxiliary field h, so that we may integrate over it with an arbitrary weight. In order to obtain an exponential which can be treated with the usual perturbation techniques we chose $\exp\{-\mathrm{i}/(2\xi)\left\langle g^2[A,x]\right\rangle_x\}$. This leads to

$$Z[J] = N \int \mathrm{D}U \int \mathrm{D}A \Delta_g[A,g[A]] \exp\left\{\mathrm{i}S - \frac{\mathrm{i}}{2\xi}\left\langle g^2[A,x]\right\rangle + \mathrm{i}\left\langle J^\mu A_\mu^{U^{-1}}\right\rangle\right\} \tag{6.18}$$

and we got rid of the δ-function.

Now the only dependence on U is in the external source term in the exponential. For $J^\mu = 0$ the inner path integral is gauge-invariant. Since we calculate the S-matrix elements for $J^\mu = 0$ we expect that those are gauge-invariant and that the integration over the gauge group indeed gives only an indefinite constant factor which may be put into the overall normalisation constant N.

This can be formally shown by making use of the LSZ reduction formalism. We can write eq. (4.199) by just plugging in for the source term

$$\langle J_\mu(x) A^\mu(x)\rangle_x = \left\langle a^\nu(x')\Delta^{-1}_{\nu\mu}(x',x) A_\mu^{U^{-1}}(x)\right\rangle_{x',x}, \tag{6.19}$$

where a^μ is an arbitrary spin-0 function fulfilling $g[A] = h$ and $\Delta_{\mu\nu}$ is the free photon propagator which is derived from the photon field equation of motion with a minimally coupled external current (where the gauge is again fixed by the gauge condition $g[A] = h$. Then the current

$$j_\mu(x) = \left\langle a^\nu(x')\Delta^{-1}_{\nu\mu}(x',x)\right\rangle_{x'} \tag{6.20}$$

is conserved due to the fact that it fulfils the equation of motion

$$\partial_\nu f^{\nu\mu} = j^\mu \text{ with } f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{6.21}$$

by construction of the photon propagator. Thus we have

$$\left\langle j^\mu A_\mu^{U^{-1}}(\chi)\right\rangle = \langle j^\mu (A_\mu - \partial_\mu)\chi\rangle = \langle j^\mu A_\mu\rangle\,. \tag{6.22}$$

Thus the generating functional for S-matrix elements is indeed gauge-independent as it should be, and we can just fix the gauge in the external source part in (6.18). The integration over the gauge group then leads only to an additional infinite constant factor which is put into the normalisation constant N. Thus the correct gauge-fixed generating functional for disconnected Green's functions is given by

$$Z[J_\mu] = N \int \mathrm{D}A \Delta_g[A,g[A]] \exp\left\{\mathrm{i}S - \frac{\mathrm{i}}{2\xi}\left\langle g^2[A,x]\right\rangle + \mathrm{i}\left\langle J^\mu A_\mu\right\rangle\right\}. \tag{6.23}$$

Since with this step the gauge was fixed arbitrarily the generating functional and the Green's functions are gauge-dependent. But as we have shown explicitly above the S-matrix elements are gauge-independent as it should be for physical quantities.

The next step is to eliminate the functional Δ which disturbs us in proceeding further with calculating the generating functional. For this purpose we change the variable of integration to g in the defining formula (6.14). This is possible because g fixes the gauge by assumption:

$$\Delta_g^{-1}[A,h] = \int \mathrm{D}g \det\left(\frac{\delta U}{\delta g}\right) \delta[g-h] = \det\left(\frac{\delta U}{\delta g}\right)\Big|_{g=h \Leftrightarrow U=1} \tag{6.24}$$

Now we chose the coordinates χ for integrating over the gauge group at each point in space-time. The invariant measure is in that case $\mathrm{d}\chi$ as one can prove immediately by taking into account an infinitesimal gauge transformation. We also integrate for χ over the whole real axes which only gives another indefinite factor absorbed into N. We find

$$\det\left(\frac{\delta g[A^\chi,x]}{\delta\chi}\right)\Big|_{\chi=0} = N\int \mathrm{D}\bar\eta\mathrm{D}\eta \exp\left(-\mathrm{i}\left\langle \bar\eta(x)\,\frac{\delta g[A^\chi,x]}{\delta\chi(y)}\Big|_{\chi=0}\eta(y)\right\rangle_{x,y}\right), \tag{6.25}$$

where $\bar\eta$ and η are two independent Grassmann fields making use of eq. (4.165).

Now all is calculable by functional differentiation. We take the Lorentz gauge functional

$$g[A^\chi,x] = \partial_\mu A^\mu(x) + \Box\chi(x). \tag{6.26}$$

Differentiating this with respect to $\chi(y)$ we obtain the local result

$$\frac{\delta g[A^\chi,x]}{\delta\chi(y)} = \Box_y \delta^{(4)}(x-y). \tag{6.27}$$

Inserting this to (6.25) we find

$$\det\left(\frac{\delta g[A^\chi]}{\delta\chi}\right)\Big|_{\chi=0} = N\int \mathrm{D}\bar\eta\mathrm{D}\eta \exp\left[\mathrm{i}\left\langle(\partial_\mu\bar\eta(x))(\partial^\mu\eta(x))\right\rangle_x\right]. \tag{6.28}$$

Using this in (6.23) we see that the so-called *Feynman ghosts* $\bar\eta$ and η decouple completely from the rest of the path integral such that it is leading only to another field and external source-independent factor which is again absorbed into N.

Thus the final result for the generating functional for free photons is

$$Z[J_\mu] = N\int \mathrm{D}A \exp\left[\underbrace{\mathrm{i}S[A] - \frac{\mathrm{i}}{2\xi}\left\langle(\partial_\mu A^\mu)^2\right\rangle}_{\mathrm{i}S_{\mathrm{eff}}[A]} + \mathrm{i}\left\langle J_\mu A^\mu\right\rangle\right]. \tag{6.29}$$

Now we can calculate the free generating functional for photons as usual, namely finding the propagator. This is a simple algebraic calculation in momentum space. We have only to satisfy again the causal boundary conditions with help of the iϵ-description. The effective classical action is given by

$$S_{\mathrm{eff}}[A] = \int \mathrm{d}^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]. \tag{6.30}$$

In momentum space this leads to the algebraic equation of motion for the propagator

$$\left[-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]\Delta_{\nu\rho}(k) = \delta^\mu_\rho \tag{6.31}$$

with the usual convention for the Fourier transform

$$\Delta_{\nu\rho}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \exp(-\mathrm{i}kx)\Delta_{\mu\nu}(k). \tag{6.32}$$

For reasons of relativistic covariance (we have chosen a covariant gauge!) the propagator must be a symmetric second-rank tensor field. This tensor can only be built with help of the momentum k_μ and the metric $g_{\mu\nu}$. Thus we make the ansatz

$$\Delta_{\nu\rho}(k) = A(k)\Theta_{\nu\rho}(k) + B(k)\frac{k_\nu k_\rho}{k^2} \text{ with } \Theta_{\mu\nu}(k) = \frac{k_\nu k_\rho}{k^2} - g_{\nu\rho}. \tag{6.33}$$

Here Θ is the (not normalised) projector to the k-transverse part of the tensor. Now we can write the equation of motion (6.34) in the form

$$\left[k^2\Theta^{\mu\nu}(k) - \frac{1}{\xi}k^\mu k^\nu\right]\Delta_{\nu\rho}(k) = \delta^\mu_\rho. \tag{6.34}$$

With a little bit of algebra we find by putting in the ansatz (6.33)

$$-k^2A(k)\Theta^\mu_\rho(k) - \frac{1}{\xi}L(k)k^\mu k_\rho = \delta^\mu_\rho. \tag{6.35}$$

Comparing the coefficients in front of δ^μ_ρ and $k_\rho k^\mu$ we find the solution for the propagator of free photons

$$\Delta_{\mu\nu}(k) = \left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}\right]\frac{1}{k^2+\mathrm{i}\epsilon}, \tag{6.36}$$

where we have used the iϵ-description for the denominators in order to fix the correct causal boundary conditions for the propagator.

The class of gauges may be reduced further by choosing a certain value for the gauge constant ξ. The most convenient choice is obtained by $\xi = 1$ which is known as the *Feynman gauge* and leads to the propagator

$$\Delta_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2+\mathrm{i}\epsilon} \tag{6.37}$$

which shows a nice high energy behaviour (it is the same k^2 behaviour as for a scalar boson).

Another common choice is the *Landau gauge*, i.e., $\xi = 0$ where the propagator is k-transverse:

$$\Delta_{\mu\nu}(k) = -\frac{\Theta_{\mu\nu}(k)}{k^2+\mathrm{i}\epsilon}. \tag{6.38}$$

On the other hand it is a simple check when calculating S-matrix elements to keep ξ as a parameter, because ξ has to vanish in the final results for physical matrix elements. This is clear from our derivation where the gauge constant came in only by averaging over the auxiliary field h the path integral did not depend of from the very beginning.

6.2 Matter Fields interacting with Photons

We now come to the main subject of this chapter, namely the description of the interaction of electrons with photons.

At first we shall give another point of view of the gauge invariance which can be extended to the general non-Abelian case in chapter 7.

For this purpose we write down again the Lagrangian for a free spin-1/2 field already given in eq. (4.60):

$$\mathscr{L}_{0F} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi. \tag{6.39}$$

This Lagrangian is invariant under the transformation

$$\psi'(x) = \exp(\mathrm{i}e\alpha)\psi(x),\ \bar{\psi}'(x) = \bar{\psi}(x)\exp(-\mathrm{i}e\alpha), \tag{6.40}$$

$$\delta\psi(x) = \mathrm{i}e\delta\alpha\psi(x),\ \delta\bar{\psi} = -\mathrm{i}e\delta\alpha \tag{6.41}$$

where we have given the infinitesimal version of the transformation in the second line. Due to Noether's theorem this leads to the conservation of the current

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi. \tag{6.42}$$

In the quantised version the conserved operator of the Noether charge is given by eq. (4.75)

$$\mathbf{Q} = -e\int \mathrm{d}^3\vec{x} : \bar{\boldsymbol{\psi}}(x)\gamma^0\psi(x) := -e\sum_{\sigma=\pm 1/2}\int \mathrm{d}^3\vec{p}[\mathbf{N}_-(\vec{p},\sigma) - \mathbf{N}_+(\vec{p},\sigma)]. \tag{6.43}$$

Here we have changed the label for the "a-particle" in chapter 4 to $-$ and of this of the antiparticle to $+$. We shall call the $-$-particle electron and the $+$-particle positron although the QED is valid for all leptons as long as other interactions than electromagnetism (as are weak and strong interaction) can be neglected. From (6.43) we read off that an electron carries a charge $-e$ and a positron a charge $+e$.

The phase transformation (6.40) is global in the sense that the phase is independent of space and time, i.e., the change of the phase is the same instantaneously on each space-time point. Now field theory is based on locality in space and time and thus we try to find a possibility to keep the Lagrangian invariant under *local phase transformations* which means to set $\alpha = \chi(x)$ in 6.40. But in this case the Lagrangian (6.39) cannot be invariant under this transformation because

$$\psi'(x) = \exp[\mathrm{i}e\chi(x)]\psi(x) \Rightarrow \partial_\mu\psi'(x) = [\mathrm{i}e(\partial_\mu\chi(x)) + \partial_\mu]\psi(x)\exp[\mathrm{i}e\chi(x)]. \tag{6.44}$$

But now this problem can be cured by substituting a *covariant derivative* instead of the simple partial derivative into the Lagrangian.[4] The covariant derivative will be of the form

$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu. \tag{6.45}$$

Looking on (6.44) we find that we have to transform the vector field A_μ in addition to the matter fields ψ and $\bar{\psi}$ as follows

$$\psi'(x) = \exp[\mathrm{i}e\chi(x)]\psi(x),\ \bar{\psi}'(x) = \bar{\psi}(x)\exp[-\mathrm{i}e\chi(x)],\ A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x). \tag{6.46}$$

Indeed one finds by a simple calculation

$$D'_\mu\psi'(x) = \exp[\mathrm{i}e\chi(x)]D_\mu\psi(x). \tag{6.47}$$

[4]We shall come back to the geometry underlying this technique when describing the case of general gauge theories.

Thus writing

$$\mathscr{L}_F = \bar{\psi}(\mathrm{i}\not{D} - m)\psi \text{ with } D = \partial_\mu - \mathrm{i}eA_\mu \tag{6.48}$$

we obtain a Lagrangian which is invariant under the *local gauge transformation* (6.46) which is the extension of the global phase invariance to a local one. The vector field A_μ is called the *gauge field*.

Up to now the gauge field is no dynamical degree of freedom because there is no kinetic term for it in the Lagrangian (6.48). In order to keep this Lagrangian gauge-invariant we can add the Lagrangian for a free massless vector field (a naively added mass term would violate the gauge invariance). This simple procedure leads to the famous QED-Lagrangian

$$\mathscr{L}_{\mathrm{QED}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \text{ with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.49}$$

There is no reason not to add other matter fields, spin-1/2 or scalar fields, with the partial derivatives substituted by the covariant ones.

Since this Lagrangian is gauge-invariant all the considerations for quantising the gauge field also applies to this case (including the considerations concerning the external currents when building the generating functional for for the Green's function. In that case the only difference is that the asymptotically free matter field is gauge-independent by itself such that the S-matrix elements are proven to be gauge-invariant also when coupling matter fields to the photons. It is also clear that in general the Green's functions will not be gauge-independent but only the S-matrix elements.)

The result is the effective Lagrangian

$$\mathscr{L}_{\mathrm{eff}} = \underbrace{\bar{\psi}(\mathrm{i}\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\mathscr{L}_0} \underbrace{+e\bar{\psi}\not{A}\psi}_{\mathscr{L}_I} \tag{6.50}$$

which can be quantised with the same rules outlined in the previous chapter. Introducing the external Grassmann sources $\bar{\eta}$ and η for the matter fields and the c-number sources J_μ for the photon field we give the path integral formula for the generating functional for the disconnected QED-Green's functions:

$$Z[J_\mu, \bar{\eta}, \eta] = N \int \mathrm{D}A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left\{\mathrm{i}S_{\mathrm{eff}}[A_\mu, \bar{\psi}, \psi] + \mathrm{i}\left\langle J_\mu A^\mu \right\rangle + \mathrm{i}\left\langle \bar{\eta}\psi \right\rangle + \mathrm{i}\left\langle \bar{\psi}\eta \right\rangle\right\}. \tag{6.51}$$

Now we can read off the Feynman rules in the usual way to be given as in fig. 6.1. The Feynman

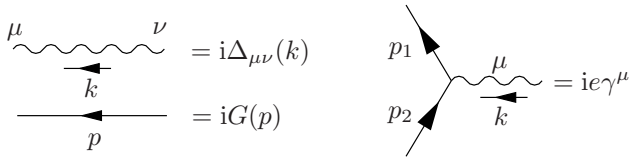


Figure 6.1: *Feynman Rules for spinor quantum electrodynamics.*

rules for calculating Green's functions are derived from the free generating functional as shown in section 4.7 for a simpler toy model with fermions as follows

1. Draw all topologically distinct diagrams with a given number of vertices. The diagrammatical elements have the meaning given in fig. 6.1. The diagram has to be read against the direction of the arrows.

2. There is a factor $1/n!$ for a diagram with n vertices from the Dyson-Wick series. But in the case of QED these factors are cancelled completely by the number of permutations of the vertices because there is only one type of vertex with three distinct lines running in and out. Thus in QED there are no symmetry factors to take into account as long as one calculates Green's functions. (Graphs which contain vacuum-to-vacuum amplitudes are to be neglected by normalisation.)

3. On each vector one has to obey four-momentum conservation as well as for the whole diagram. Over each independent loop momentum not fixed by this rule is to integrate with measure $\mathrm{d}^4 l/(2\pi)^4$.

4. Each loop consisting of a connected line of fermion propagators gives an additional factor -1. In this case one has to take the trace over the corresponding spinor matrices. The overall sign for a Green's function is arbitrary but the relative sign of two diagrams is to be taken due to the relative permutation of external fermion lines.

5. For calculating S-matrix elements one has to amputate the external legs and substitute them by normalised one-particle amplitudes according to the diagrammatical rules for external legs given in fig. 6.2. All external momenta have to be taken on-shell for the given particle mass.

   Here we have used the usual convention for the free Dirac field amplitudes by defining

$$u(p,\sigma) = u_+(p,\sigma),\ v(p,\sigma) = u_-(-p,\sigma), \tag{6.52}$$

   where the $u_\pm$ are the amplitudes defined in chapter 4, where this convention is better to show the correct meaning of the Feynman Stückelberg formalism for finding the right interpretation of the negative energy states.

6.2.1 Ward-Takahashi identities

Now we derive the consequences of the local gauge invariance of the original action starting from the generating functional (6.51). We split the effective action in a gauge-invariant and the gauge-fixing part rewriting (6.50) in the form

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{inv}} + \mathscr{L}_{\text{fix}} \tag{6.53}$$

with

$$\mathscr{L}_{\text{inv}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6.54}$$

$$\mathscr{L}_{\text{fix}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{6.55}$$

Now we write (6.51) by integrating over gauge transformed fields with an infinitesimal gauge transformation,

$$A'_\mu = A_\mu + \partial_\mu \delta\chi, \quad \psi' = (1 + \mathrm{i}e\delta\chi)\psi, \quad \bar{\psi}' = (1 - \mathrm{i}e\delta\chi)\bar{\psi}. \tag{6.56}$$

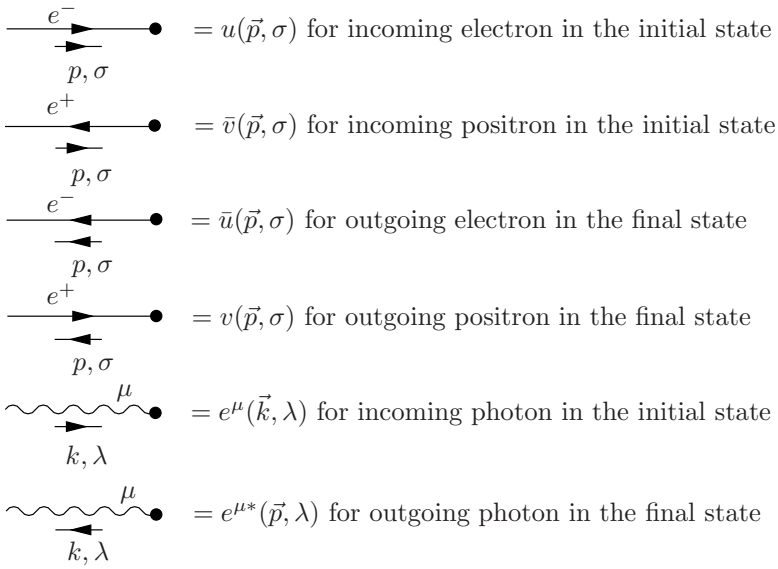


Figure 6.2: *Feynman rules for the external lines when calculating S-matrix elements*

This, of course does not change the value of the functional, but now we can use the invariance of the part S_{inv} of the action, defined by the Lagrangian (6.54). We also note that since the gauge field is just shifted by a field that is independent of the fields we integrate over and the transformations of the Dirac fields is just a (local) redefinition of its overall phase, also the path-integral measure is invariant. Thus we have

$$\begin{aligned} Z[J_\mu, \bar{\eta}, \eta] &= N \int \mathrm{D}A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[\mathrm{i}S_{\text{inv}} + \mathrm{i}(S_{\text{fix}} + \delta S_{\text{fix}})\right] \\ &\quad \times \exp\left[\mathrm{i}\left\langle J_\mu(A^\mu + \delta A^\mu) + \bar{\eta}(\psi + \delta\psi) + (\bar{\psi} + \delta\bar{\psi})\eta\right\rangle\right] \\ &= N \int \mathrm{D}A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[\mathrm{i}S_{\text{inv}} + \mathrm{i}S_{\text{fix}} + \mathrm{i}\left\langle J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta\right\rangle\right] \\ &\quad \times \left[1 + \mathrm{i}\delta S_{\text{fix}} + \mathrm{i}\left\langle J_\mu \delta A_\mu + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta]\right\rangle\right], \end{aligned} \tag{6.57}$$

where all arguments of the functionals have to be taken as the orginal fields A^μ, $\bar{\psi}$ and ψ. Now we subtract (6.51) and obtain

$$\begin{aligned} \int \mathrm{D}A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[\mathrm{i}S_{\text{inv}} + \mathrm{i}S_{\text{fix}} + \mathrm{i}\left\langle J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta\right\rangle\right] \\ \times \left[\delta S_{\text{fix}} + \left\langle J_\mu \delta A_\mu + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta]\right\rangle\right] = 0. \end{aligned} \tag{6.58}$$

Now with (6.56) the square bracket reads

$$\delta S_{\text{fix}} + \left\langle J_\mu \delta A_\mu + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta]\right\rangle = \int \mathrm{d}^4x \left(-\frac{1}{\xi}\Box\partial_\mu A^\mu - \partial_\mu J^\mu + \mathrm{i}e\bar{\eta}\psi - \mathrm{i}e\bar{\psi}\eta\right)\delta\chi. \tag{6.59}$$

Since this holds true for arbitrary scalar fields $\delta\chi$, we obtain the local relation

$$\begin{aligned} \int \mathrm{D}A_\mu \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[\mathrm{i}S_{\text{inv}} + \mathrm{i}S_{\text{fix}} + \mathrm{i}\left\langle J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta\right\rangle\right] \\ \times \left[-\frac{1}{\xi}\Box\partial_\mu A^\mu - \partial_\mu J^\mu + \mathrm{i}e\bar{\eta}\psi - \mathrm{i}e\bar{\psi}\eta\right] = 0. \end{aligned} \tag{6.60}$$

This we can easily express as an identity for the generating functional:

$$-\frac{\mathrm{i}}{\xi}\Delta\partial_\mu\frac{\delta Z}{\delta\xi_\mu}-e\bar{J}\left(\frac{\delta Z}{\delta\bar{\eta}}\right)_L+e\left(\frac{\delta Z}{\delta\eta}\right)_R J-\mathrm{i}\partial_\mu J^\mu Z=0. \tag{6.61}$$

This is the **Ward-Takahashi identity (WTI)** for the generating functional for (unconnected) Green's functions. Writing $Z=\exp(\mathrm{i}W)$ immediately leads to the WTI for the generating functional of the connected Green's functions

$$-\frac{\mathrm{i}}{\xi}\Box\partial_\mu\frac{\delta W}{\delta J_\mu}-e\bar{\eta}\left(\frac{\delta W}{\delta\bar{\eta}}\right)_L+e\left(\frac{\delta W}{\delta\eta}\right)_R\eta-\mathrm{i}\partial_\mu J^\mu=0. \tag{6.62}$$

Finally, we translate this into a WTI for the effective action, the generating functional for truncated one-particle irreducible Green's functions. It is defined by the functional Legendre transformation of W

$$\Gamma[A^\mu,\bar{\psi},\psi]=W-J^\mu A_\mu+\bar{\psi}\eta+\bar{\eta}\psi \tag{6.63}$$

with

$$A_\mu=\frac{\delta W}{\delta J^\mu},\quad \bar{\psi}=\left(\frac{\delta W}{\delta\eta}\right)_R,\quad \psi=\left(\frac{\delta W}{\delta\bar{\eta}}\right)_L \tag{6.64}$$

and

$$J^\mu=-\frac{\delta\Gamma}{\delta A_\mu},\quad \eta=-\left(\frac{\delta\Gamma}{\delta\bar{\psi}}\right)_L,\quad \bar{\eta}=-\left(\frac{\delta\Gamma}{\delta\psi}\right)_R. \tag{6.65}$$

Plugging (6.64) and (6.65) in (6.62) we finally get the WTI for the effective action

$$-\frac{\mathrm{i}}{\xi}\Box\partial_\mu A^\mu+e\left(\frac{\delta\Gamma}{\delta\psi}\right)_R\psi-e\bar{\psi}\left(\frac{\delta\Gamma}{\delta\bar{\psi}}\right)_L+\mathrm{i}\partial_\mu\frac{\delta\Gamma}{\delta A_\mu}=0. \tag{6.66}$$

Taking derivatives of this identity with respect to the "mean fields" and setting $\psi=\bar{\psi}=A_\mu=0$, which are the solutions of the equations $J^\mu=\eta=\bar{\eta}=0$, given in terms of (6.65) as equations of motion which determine the stationary points of the effective action, yields the the WTIs for the corresponding proper vertex functions. E.g., taking the derivative of (6.66) with respect to $A_\nu(y)$ and then setting $\psi=\bar{\psi}=A_\mu=0$ gives the WTI for the exact interacting Green's function $D^{\mu\nu}$ of the photon,

$$\frac{\partial}{\partial x^\mu}[(D^{-1}(x,y)]^{\mu\nu}=\frac{\partial}{\partial x^\mu}\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(y)}=-\frac{1}{\xi}\Box_x\frac{\partial}{\partial x_\nu}\delta^{(4)}(x-y). \tag{6.67}$$

In terms the momentum-space Green's functions this reads

$$-\mathrm{i}k_\mu[D^{-1}(k)]^{\mu\nu}=\frac{\mathrm{i}}{\xi}k^2k^\nu. \tag{6.68}$$

For the free propagator we have according to (6.31)

$$[\Delta^{-1}]^{\mu\nu}=-k^2g^{\mu\nu}+\left(1-\frac{1}{\xi}\right)k^\mu k^\nu\ \Rightarrow k_\mu[\Delta^{-1}]^{\mu\nu}=-\frac{1}{\xi}k^2k^\nu, \tag{6.69}$$

i.e., the longitudinal part of the photon propagator does not change through the interactions. Since by definition

$$[D^{-1}(k)]^{\mu\nu}=[\Delta^{-1}(k)]^{\mu\nu}-\Pi^{\mu\nu}(k), \tag{6.70}$$

where $\Pi^{\mu\nu}$ is the self-energy (or polarization tensor) of the photon, we conclude from (6.68) and (6.69) that only the transverse components of the photon are interacting, i.e.,

$$k_\mu \Pi^{\mu\nu}(k) = 0. \tag{6.71}$$

Another important application of the WTIs is the formal prove that the S matrix is unitary in the subspace of physical states, particularly only the physical transverse photon-polarization states. The S-matrix of the gauge-fixed theory is unitary, but it is not trivial that the projection to the physical subspace $S_{\text{phys}} = P_{\text{phys}} S P_{\text{phys}}$ obeys $S_{\text{phys}} S_{\text{phys}}^\dagger = P_{\text{phys}}$. In QED this is the case if in all calculations of S-matrix elements we can sum over all four polarization states, of which only the transverse ones are physical. The WTI reveals that in the S-matrix elements (with all electron-positron lines on the mass shell), which for a photon of four-momentum k is of the form $\epsilon_\mu \mathcal{M}^\mu(k,\ldots)$, obeys the Ward identity $k_\mu \mathcal{M}^\mu(k,\ldots) = 0$, which implies that (for on-shell photons) the contributions of the time-like and the longitudinal polarization states cancel, i.e., the S-matrix for physical gauge-invariant observables involves only the physical polarization states.

For the connected S-matrix elements one also has to build the tree diagrams from the proper vertices, connected with exact photon propagators. The WTIs also guarantee that the gauge-dependent piece of the propagator does not contribute.

The most simple example is the full $\gamma\bar{\psi}\psi$ vertex. To derive its WTI one just takes a left derivative with respect to $\bar{\psi}$ and a right derivative with respect to ψ of the WTI (6.66) and then sets all the fields to 0. This leads to

$$-\mathrm{i}\partial_{1\mu} \Gamma^{(3)\mu}_{A\bar{\psi}\psi}(x_1,x_2,x_3) = eG^{-1}(x_1-x_2)\delta^{(4)}(x_1-x_3) - eG^{-1}(x_3-x_1)\delta^{(4)}(x_1-x_2), \tag{6.72}$$

which translates via a Fourier transformation to the corresponding expression

$$-q_\mu \Gamma^{(3)\mu}_{A,\bar{\psi},\psi}(q,p_1,p_2) = eG^{-1}(p_1-q) - eG^{-1}(p_2+q), \tag{6.73}$$

where G^{-1} denotes the inverse electron-positron propagator. In the S-matrix element the electron-positron legs (which are amputated of course) have to be substituted with on-shell $\bar{u}(p_1) = \bar{u}(p_2+q)$ and $u(p_2) = u(p_1-q)$ (or with corresponding v spinors if one considers positrons) spinors, which leads to 0 at the right-hand side. So for both on- and off-shell photons also the left-hand side vanishes as it should according to the above argument. The WTI for arbitrary n-point vertices follows in the same way by taking the corresponding further derivatives of (6.66). This also generates on the left-hand side the sum over all possible insertion points of the particular photon with momentum q in the corresponding diagram and the corresponding appropriate sum on the right-hand side. The WTI thus holds only for these full sums but not for single diagrams.

6.3 Canonical Path Integral

Now we have shown how to quantise QED with a Dirac field for the "matter particles" (so far called electrons and positrons, but applicable also to the electromagnetic sector of other particles, especially the μ- and τ-leptons which are the heavier relatives of the electron) provided the Lagrangian formulation of the path integral is the correct one. So far we have given only the handwaving argument that the interaction Lagrangian (and so the Hamiltonian) contains no

derivative couplings. This chapter shows that this is justified not only for QED with fermionic but also with scalar bosonic matter fields.

For this purpose it is sufficient to take into account only the pure gauge field, i.e., the free photons. Using the Hamiltonian formalism we break Lorentz covariance explicitly. Thus there is no advantage to choose a covariant gauge and we use *axial space-like gauge*, which is more convenient for the Hamiltonian formalism:

$$A_3 = 0. \tag{6.74}$$

We should keep in mind that this fixes the gauge not completely because we can always use a gauge potential χ which is independent on x^3 to define an equivalent gauge field $A'_\mu = A_\mu + \partial_\mu \chi$ and A'_μ still fulfils (6.74) if the original field A_μ does so.

The conjugate momenta are calculated from the gauge-independent Lagrangian

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (\partial_\mu A_\nu) F^{\mu\nu}, \; \Pi_\mu = \frac{\partial \mathscr{L}}{\partial(\partial_t A^\mu)} = F_{0\mu}. \tag{6.75}$$

We find indeed that the Hamiltonian formalism is not applicable directly to the fields A_μ (with $\mu = 0, 1, 2$, A_3 is eliminated using the space-like axial gauge constraint (6.74)). The reason is the gauge invariance with respect to the above mentioned special gauge transformations. It is also well-known from optics that there are only two dynamical field degrees of freedom. The so far obtained result is the formal proof of this from first principles.

The identical vanishing of Π_0 shows that we have to keep only A_1 and A_2 together with the conjugated field momenta Π_1 and Π_2 as the independent dynamical degrees of freedom. But then A_0 has to be a functional of these and we have to find this functional. It is determined by the claim that the Hamiltonian canonical field equations of motion are the same as those from the Lagrangian formalism.

Thus for this purpose we use the equations of motion for A_μ coming from the least action principle:

$$\Box A_\mu - \partial_\mu \partial_\nu A^\nu = 0. \tag{6.76}$$

Especially for $\mu = 3$ we find (making use of the gauge constraint (6.74)):

$$\partial_3 \partial_\nu A^\nu = 0 \Rightarrow \partial_\nu A_\nu = f(x_0, x_1, x_2) \tag{6.77}$$

and from this we find

$$\partial_0 A^0 = -\partial_1 A^1 - \partial_2 A^2 + f(x_0, x_1, x_2) \tag{6.78}$$

and A^0 can be expressed in terms of the other fields

$$A^0 = -\int_{t_0}^{t} \mathrm{d}t (\partial_1 A^1 + \partial_2 A^2) + g(x_0, x_1, x_2). \tag{6.79}$$

g is an arbitrary function of x_0, x_1, x_2 and can be eliminated with help of a special gauge transformation which is consistent with the gauge constraint (6.74).

Taking the formal Legendre transform of the Lagrange density we find the canonical Hamilton density of the system:

$$\mathscr{H} = \Pi_\mu \partial_t A^\mu - \mathscr{L} = \frac{1}{2}\Pi_1^2 + \frac{1}{2}\Pi_2^2 + \frac{1}{2} E_3^2[\Pi_1, \Pi_2] + \frac{1}{2}\vec{B}^2, \tag{6.80}$$

where $\vec{E}$ and $\vec{B}$ are the fields $\vec{E}$ with respect to the now fixed reference frame:

$$\vec{E} = -\partial_0 \vec{A} - \nabla A_0, \ \vec{B} = \nabla \times A. \tag{6.81}$$

At the same time we have used

$$E_1 = -\Pi_1, \ E_2 = -\Pi_2. \tag{6.82}$$

As we have seen above E_3 is a functional of the dynamical degrees of freedom. We shall show now that this functional depends on Π_1 and Π_2 only. One equation of motion for $\vec{E}$ is $\nabla \vec{E} = 0$ and thus

$$E_3[\Pi_1, \Pi_2] = \int_{z_0}^{z} \mathrm{d}\zeta [\partial_1 \Pi_1(x_0, x_1, x_2, \zeta) + \partial_2 \Pi_2(x_0, x_1, x_2, \zeta)]. \tag{6.83}$$

It is also easy to show that with this Hamiltonian the field equations of motion (6.81) follow from the Hamiltonian principle of least action with the Hamilton density given by (6.80).

Now our Hamiltonian action depends only on the two independent field degrees of freedom and their canonical field momenta. Thus we can write down the path integral in the Hamiltonian form, where we have no problem with the gauge dependence because now the gauge is fixed completely by the boundary conditions for the fields:

$$Z[J_\mu] = N \int \mathrm{D}\Pi_1 \mathrm{D}\Pi_2 \int \mathrm{D}A^1 \mathrm{D}A^2 \exp[\mathrm{i}S[A, \Pi] + \mathrm{i} \langle J_\mu A^\mu \rangle]. \tag{6.84}$$

It is also clear that we could quantise the theory with help of the canonical operator formalism. The only disadvantage compared to the Lagrangian path integral is the lost manifest covariance.

We shall use this formalism neither to quantise the theory nor doing calculations of practical interest[5], because the covariant formalism is much more convenient. Nevertheless we have to prove that the path integral (6.84) is equivalent to a Faddeev-Popov path integral of the kind derived in the first two sections. This is important in order to be sure that one obtains a unitary S-matrix. If we can show that our covariant Faddeev-Popov formalism is leading to the same physical S-matrix elements as the covariant formalism. For this purpose we reintroduce the before eliminated spurious degrees of freedom without changing the generating functional (6.84).

As the first step we introduce a factor one in the form

$$1 = \int \mathrm{D}\Pi_3 \delta[\Pi_3 + E_3[\Pi_1, \Pi_2]]. \tag{6.85}$$

Since we have defined the functional E_3 by using the equation of motion $\nabla \vec{E} = \nabla \vec{\pi} = 0$ we can write for the functional δ-distribution

$$\delta[\Pi_3 + E_3[\Pi_1, \Pi_2]] = \delta(\nabla \vec{\Pi}) \det\left(\frac{\delta \nabla \vec{\Pi}}{\delta \Pi_3}\right) = \delta(\nabla \vec{\Pi}) \det(\partial_3). \tag{6.86}$$

With this we can write (6.84) in the form

$$\begin{aligned} Z[J_\mu] = N \int \mathrm{D}\Pi_1 \mathrm{D}\Pi_2 \mathrm{D}\Pi_3 \int \mathrm{D}A_1 \mathrm{D}A_2 \delta(\nabla \vec{\Pi}) \det(\partial_3) \exp[\mathrm{i}S'[A, \Pi] + \mathrm{i} \langle J_\mu A^\mu \rangle] \text{ with} \\ S'[A, \Pi] = -\left\langle \frac{1}{2}\vec{\Pi}^2 + \frac{1}{2}\vec{B}^2 + \vec{\Pi}\partial_t \vec{A} \right\rangle \Big|_{A_3 = 0}. \end{aligned} \tag{6.87}$$

[5] except for the important case of non-relativistic approximations in atomic physics

Next we write the functional δ-distribution as a path integral

$$\delta[\nabla\vec{\Pi}] = \int \mathrm{D}A_0 \exp\left[\mathrm{i}\left\langle A_0 \nabla\vec{\Pi}\right\rangle\right] \tag{6.88}$$

which leads to

$$\begin{aligned} Z[J_\mu] = N \int \mathrm{D}^3\vec{\Pi} \int \mathrm{D}A_0 \cdots \mathrm{D}A_2 \det(\partial_3) \exp[\mathrm{i}S''[A,\vec{\Pi}] + \mathrm{i}\left\langle J_\mu A^\mu\right\rangle] \text{ with} \\ S''[A,\vec{\Pi}] = \left\langle -\frac{1}{2}\vec{\Pi}^2 - \vec{\Pi}\nabla A_0 - \frac{1}{2}\vec{B}^2 + \vec{\Pi}\partial_t\vec{A}\right\rangle\Big|_{A_3=0}. \end{aligned} \tag{6.89}$$

Now we can handle the integrals over the "field momenta" $\vec{\Pi}$.[6] S'' is a quadratic functional in $\vec{\Pi}$ and the coefficient in front of the quadratic part does not depend on the other fields. Thus we have to calculate the stationary point of S'' with respect to $\vec{\Pi}$ at fixed A_0, A_1 and A_2, which leads to

$$\frac{\delta S''}{\delta\vec{\Pi}} = 0 \Rightarrow \vec{\Pi} = \nabla A_0 - \partial_t\vec{A}. \tag{6.90}$$

Now we can also add A_3 as another variable and fix it to zero with help of another functional δ-distribution. Using (6.90) for integrating out the momenta in (6.89) we obtain the desired result

$$Z[J_\mu] = \int \mathrm{D}A_\mu \delta(A_3) \det(\partial_3) \exp[\mathrm{i}S[A] + \mathrm{i}\left\langle J_\mu A^\mu\right\rangle] \tag{6.91}$$

which is in the form (6.17) of the Faddeev-Popov formalism with the choice

$$g[A,x] = A_3(x) \tag{6.92}$$

for the gauge-fixing functional.

We finish this section with the treatment of scalar electrodynamics. The same line of arguments given in section 5.2 for Dirac fermions[7] leads to the Lagrangian

$$\mathscr{L} = (\mathrm{D}_\mu\phi)^*(\mathrm{D}^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \text{ with } \mathrm{D}_\mu = \partial_\mu + \mathrm{i}eA_\mu \text{ and } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.93}$$

This obviously contains derivative couplings between the matter and gauge fields:

$$(\mathrm{D}_\mu\phi)^*(\mathrm{D}^\mu\phi) = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mathrm{i}eA_\mu[\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi] + e^2A_\mu A^\mu\phi^*\phi. \tag{6.94}$$

But in order to integrate out the canonical field momenta in the Hamiltonian path integral we have only to assure that the action is a quadratic functional in all field momenta and that the quadratic part has a field-independent coefficient. In the case of the scalar QED-Lagrangian (6.93) a glance at (6.94) shows that this is the case for choosing the time-like axial gauge, i.e., the gauge-fixing functional

$$g[A,x] = A_0(x). \tag{6.95}$$

So we see that we can use the Lagrangian form for this special gauge and thus with help of the Faddeev-Popov formalism which leads to gauge-invariant S-matrix elements any gauge we like.

[6] One should keep in mind that the reintroduction of the fields Π_3 and A_0 does not mean to add dynamical degrees of freedom but only writing the integration variables for the various path integral manipulations in a suggestive way.

[7] One calls this shortly "gauging" a field theory with a global symmetry. Thus gauging a theory means to extend this global symmetry to a local one. In our case of electrodynamics we gauge the phase invariance of the action for free complex fields which is mathematically the invariance of the action under operations of a representation of the compact Abelian group $U(1)$.

6.4 Invariant Cross Sections

Now we are ready to calculate the lowest order perturbative approximations for the transition amplitudes of QED. We shall use Feynman gauge which leads immediately to covariant amplitudes to calculate physical quantities. In this chapter we shall deal with the simple situation that two particles are scattered again to two particles (which can be different from the incoming). As we have shown in chapter 3 and 4 we have to calculate the connected truncated Green's functions and multiply these with asymptotically free particle amplitudes. From invariance under translations we know that energy and momentum are conserved for our situation. Thus we write the S-matrix-element in the following form

$$S_{fi} = \delta_{fi} + \mathrm{i}(2\pi)^4 \delta^{(4)}(P_f - P_i) T_{fi}. \tag{6.96}$$

Here we have written P_i and P_f for the sum of the four momenta of the particles in the initial and final state respectively. The δ_{fi} takes into account the part of the S-matrix which represents the case that the particles are not scattered at all. In the following we consider only asymptotic free states $|i\rangle$ and $|f\rangle$ which are different. Thus we write

$$S_{fi} = \mathrm{i}(2\pi)^4 \delta^{(4)}(P_f - P_i) T_{fi} \text{ for } |i\rangle \neq |f\rangle\,. \tag{6.97}$$

Then calculating cross sections for non-relativistic potential scattering we have learnt how to deal with the problem of squaring this matrix element in order to calculate the probability distribution out of the transition amplitude: We have to use a finite volume and time where the scattering takes place. Then the δ-distribution is regularised as follows

$$(2\pi)^4 \delta^{(4)}_{\mathrm{reg}}(P_f - P_i) = \int_{-T/2}^{T/2} \mathrm{d}t \int_V \mathrm{d}^3\vec{x} \exp[\mathrm{i}(P_f - P_i)x]. \tag{6.98}$$

Setting $P_f - P_i = \Delta P$ we find

$$\frac{|S_{fi}|^2}{TL^3} = 2^4 \left(\frac{2}{T}\right)\left(\frac{2}{L}\right)^3 \left(\frac{\sin(\Delta P_0 T/2)}{\Delta P_0}\right)^2 \prod_{k=1}^{3} \left(\frac{\sin(\Delta P_k L/2)}{\Delta P_k}\right)^2 |T_{fi}|^2. \tag{6.99}$$

Now we can take the weak limit of this expression for $T \to \infty$ and $L \to \infty$ using the formula

$$\operatorname*{w\text{-}lim}_{y\to\infty} \frac{\sin^2(xy)}{yx^2} = \pi\delta(x). \tag{6.100}$$

This can be proven simply with help of the Fourier transform

$$\int \mathrm{d}x \frac{\sin^2(xy)}{yx^2} \exp(-\mathrm{i}px) = \frac{\pi}{2}\Theta(2y - |p|)\left(2 - \frac{|p|}{y}\right), \tag{6.101}$$

which is shown by using the inverse of this formula. For $y \to \infty$ the right hand side goes to π which shows the above stated weak limit (6.100).

Using this in (6.99) we find

$$\frac{|S_{fi}|^2}{VT} = (2\pi)^4 \delta^{(4)}(P_f - P_i)|T_{fi}|^2. \tag{6.102}$$

Now we like to calculate the cross section for scattering of two particles in the initial state into n particles in the final state. The incoming momenta are $\vec{p}_1$ and $\vec{p}_2$. The corresponding four-momenta are positive time like and obey the on-shell condition $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$. The number of asymptotically free final states in the volume V is given by

$$\prod_{k=1}^{n} \frac{V \mathrm{d}^3 \vec{p}_k'}{(2\pi)^3}. \tag{6.103}$$

Thus the probability to be scattered into this region of the momentum space is given by

$$\mathrm{d}w = (2\pi)^4 \delta^{(4)}(P_f - P_i) |T_{fi}|^2 V \prod_{k=1}^{n} \frac{V \mathrm{d}^3 \vec{p}_k'}{(2\pi)^3}. \tag{6.104}$$

Any external line represents a particle amplitude which is normalised to one particle contained in the volume V. According to appendix C the asymptotically free momentum eigenfunctions contains a normalisation factor $1/\sqrt{2\omega V}$. Leaving out this factor for any of the in- and out-state amplitudes in the calculation of T_{fi}, we define the *Lorentz-invariant matrix element*

$$M_{fi} = T_{fi} \prod_i \sqrt{2\omega_i V} \prod_f \sqrt{2\omega_f V}, \tag{6.105}$$

where the products have to be taken over all incomming and outgoing particles in the initial or final state respectively.

With this we can write (6.104) as

$$\mathrm{d}w = (2\pi)^4 \delta^{(4)}(P_f - P_i) \frac{|M_{fi}|^2}{4\omega_1 \omega_2 V} \prod_{k=1}^{n} \frac{\mathrm{d}^3 \vec{p}_k'}{2\omega_k' (2\pi)^3}. \tag{6.106}$$

Now what is usually measured in scattering experiments is the invariant differential (or total) cross section. This is defined as

$$\mathrm{d}\sigma = \frac{\text{Particles scattered into the final state in the given momentum space region}}{\text{Flow of the projectile particle in the rest frame of the target particle}}, \tag{6.107}$$

i.e., in the frame of reference, where the "target particle 2" in the initial state is at rest ("laboratory frame"). In this frame the three-velocity of the projectile particle is given by

$$\vec{v}_1 = \frac{\vec{p}_1}{\omega_1}. \tag{6.108}$$

Thus the flow in the lab frame is

$$j = \frac{\vec{v}_1}{V} = \frac{\sqrt{m_2^2 \omega_1^2 - (m_1 m_2)^2}}{\omega_1 m_2 V}. \tag{6.109}$$

Since particle 2 is by definition at rest we have $m_2 = \omega_2$, $p_2 = (m_2, 0, 0, 0)$, and thus we have $m_2 \omega_1 = p_1 p_2$. Finally we get

$$j = \frac{\sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}}{\omega_1 \omega_2 V} = \frac{I}{\omega_1 \omega_2 V}. \tag{6.110}$$

Inserting this into (6.107) we find by making use of (6.108)

$$\mathrm{d}\sigma = \frac{\mathrm{d}w}{j} = (2\pi)^4 \delta^{(4)}(P_f - P_i) \frac{|M_{fi}|^2}{4I} \prod_{k=1}^{n} \frac{\mathrm{d}^3 \vec{p}_k'}{2\omega_k' (2\pi)^3}. \tag{6.111}$$

This expression is manifestly covariant because $|M_{fi}|^2$ is a Lorentz scalar and we can write

$$\frac{\mathrm{d}^3 \vec{p}}{2\omega} = \delta(p^2 - m^2)\theta(p_0)\mathrm{d}^4 p. \tag{6.112}$$

This shows again that all four momenta of the outgoing particles are to be taken on the mass shell, while the δ-distribution in (6.111) shows again explicitely the conservation of four-momentum.

Now we look further on (6.111) for the case that we have not only two particles in the initial but also in the final state. The particle sorts in the initial and final state need not be the same[8].

In this case (6.111) reads

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2}{(2\pi)^2 4I} \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \frac{\mathrm{d}^3 \vec{p}_1'}{2\omega_1'} \frac{\mathrm{d}^3 \vec{p}_2'}{2\omega_2'}. \tag{6.113}$$

Using (6.112) for the three-momentum integrals and performing the integration over $\mathrm{d}^4 p_2'$ and $\mathrm{d}p_{10}'$ we find

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2}{64\pi^2 I \omega_1' \omega_2'} \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2') d^3 \vec{p}_1'. \tag{6.114}$$

Now we go to the centre of mass system where

$$\vec{p}_1 = -\vec{p}_2 = \vec{p},\ \vec{p_1}' = -\vec{p}_2' = \vec{p}'. \tag{6.115}$$

To get rid of the last δ-distribution in (6.114) we use polar coordinates for $\vec{p}_1'$ and the fact that

$$\vec{p}'^2 \mathrm{d}|\vec{p}'| = |\vec{p}'| \omega_1' \mathrm{d}\omega_1' = |\vec{p}'| \omega_2' \mathrm{d}\omega_2'. \tag{6.116}$$

Setting $\omega = \omega_1 + \omega_2$ and $\omega' = \omega_1' + \omega_2'$ we can use this to derive

$$\vec{p}'^2 \mathrm{d}|\vec{p}'| = \frac{|\vec{p}'| \omega_1' \omega_2'}{\omega'} \mathrm{d}\omega'. \tag{6.117}$$

Then we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|M_{fi}|^2}{64\pi^2 I} \frac{|\vec{p}'|}{\omega}. \tag{6.118}$$

In the CMS we are calculating this we have

$$I = |\vec{p}|\omega \tag{6.119}$$

and thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|M_{fi}|^2 |\vec{p}'|}{64\pi^2 |\vec{p}| \omega^2}. \tag{6.120}$$

[8] For instance you may think on the process $e\bar{e} \to \mu\bar{\mu}$

This is the form the cross section has in a certain system, namely the centre of mass system and is thus not manifestly covariant. We want to write this result in terms of covariant quantities, which are known as *Mandelstam variables*:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1' + p_2')^2, \\ t &= (p_1 - p_1')^2 = (p_2' - p_1')^2, \\ u &= (p_1 - p_2')^2 = (p_1' - p_2)^2. \end{aligned} \tag{6.121}$$

In the centre of mass system we have

$$t = (p_1 - p_1')^2 = m_1^2 + m_1'^2 - 2\omega_1\omega_1' + 2|\vec{p}||\vec{p}'|\cos\theta. \tag{6.122}$$

For fixed energy we have

$$-\mathrm{d}t = 2|\vec{p}||\vec{p}'|\sin\theta\ \mathrm{d}\theta, \tag{6.123}$$

and with this we have

$$d\sigma = \frac{|M_{fi}|^2}{128\pi^2 I^2}\mathrm{d}(-t)\mathrm{d}\varphi, \tag{6.124}$$

where we have again introduced the invariant flux I defined in (6.110):

$$I^2 = (p_1 p_2)^2 - m_1^2 m_2^2 = \frac{1}{4}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]. \tag{6.125}$$

With help of this invariant form it is simple to calculate the cross section in any reference frame we like.

6.5 Tree level calculations of some physical processes

Starting with this section we calculate the cross sections for some physical processes to lowest order. Restricting to tree-level diagrams we are sure not to run into difficulties with infinities which are subject of the next chapter about renormalisation theory.

6.5.1 Compton Scattering

We calculate the cross section for scattering of a photon with an electron or positron. This process is also known as *Compton scattering*, which was one of the most important experimental results (1923) to convince the physicists that there is a particle structure in electromagnetic waves which where introduced in 1905 by Einstein in his famous explanation of the photo effect. The first calculation of the cross section we shall do now in the framework of Feynman diagrams of QED was undertaken by Klein and Nishina in 1929 using the methods of Dirac's hole-theory.

The Compton scattering is described on the tree-level by the diagrams shown in fig. 6.3.

The right diagram is the same as the left one but with interchanged photon lines. Thus both diagrams go with the same relative sign into the sum for the transition amplitude. In the following we can always use the on-shell conditions for the momenta:

$$p^2 = p'^2 = m^2,\ k^2 = k'^2 = 0 \tag{6.126}$$

and the energy-momentum conservation of the whole process

$$p + k = p' + k'. \tag{6.127}$$

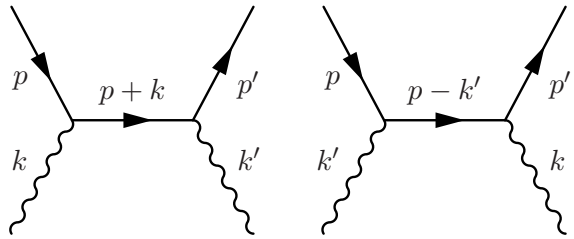


Figure 6.3: *The tree-level diagrams for Compton scattering*

We find for the modified T-matrix element, defined in the previous section, for the given process:

$$M_{fi} = -e^2 \bar{u}(p', \sigma') \gamma^\mu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \gamma^\nu u(p, \sigma) A_\nu(k, \pi) A^*_\mu(k', \pi') + \{(k, \pi) \leftrightarrow (k', \pi')\}. \quad (6.128)$$

Herein σ and σ' are the polarisations of the incoming and outgoing electron while π and π' are those of the incoming and outgoing photon respectively.

The next station of the calculation is to take the modulus squared. To shorten this procedure we define

$$M_{fi} = \bar{u}_{p',\sigma'} A^*_\mu(k', \pi') \hat{Q}^{\mu\nu} u(p, \sigma) A_\nu(k, \pi). \quad (6.129)$$

Now we have

$$(\bar{u}_1 \hat{Q} u_2)^* = u_1^{\dagger *} \gamma^0 \hat{Q}^* u_2^* = u_2^\dagger \hat{Q}^\dagger \gamma^0 u_1 = \bar{u}_2 \bar{\hat{Q}} u_1. \quad (6.130)$$

Since we want only to know the cross section for scattering of unpolarised beams of electrons and photons we have to average over the initial polarisations and to sum over the final polarisations which can be done by substituting the polarisation matrices for unpolarised particles and taking the trace:

$$|M_{fi}|^2 = 4 \,\mathrm{Tr}\left[\rho_0^{(e)}(p) \bar{\hat{Q}}_{\mu\nu} \rho_0^{(e)}(p') \hat{Q}_{\rho\sigma}\right] \rho^{(\gamma)\nu\sigma} \rho^{(\gamma)\mu\rho}, \quad (6.131)$$

where due to appendix C

$$\rho^{(\gamma)\mu\rho} = -\frac{1}{2} g^{\mu\rho}, \; \rho^{(e)}(p) = \frac{1}{2}(\not{p} + m) \quad (6.132)$$

and the factor 4 takes account that we want to sum over the final polarisations rather than to average (two polarisations of the outgoing electron and two polarisations for the outgoing photon).

Further we define the following invariants:

$$s = (p+k)^2, \; t = (p-p')^2, \; u = (p-k')^2, \quad (6.133)$$

which are known as *Mandelstam-variables*.

Now with help of (6.128) and (6.133) we write

$$\hat{Q}^{\mu\nu} = e^2 \left(\gamma^\mu \frac{\not{p} + \not{k} + m}{s - m^2} \gamma^\nu + \gamma^\nu \frac{\not{p} - \not{k}' + m}{u - m^2} \gamma^\mu \right). \quad (6.134)$$

Further it is simple to see that for a product of γ matrices we have

$$\overline{\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_n}} = \gamma^{\mu_n} \gamma^{\mu_{n-1}} \cdots \gamma^{\mu_1}. \quad (6.135)$$

Inserting this into (6.131) we find

$$|M_{fi}|^2 = \frac{e^4}{4}\,\mathrm{tr}\left\{(\not p + m)\left[\frac{\gamma^\nu(\not p + \not k + m)\gamma^\mu}{s-m^2} + \frac{\gamma^\mu(\not p - \not k' + m)\gamma^\nu}{u-m^2}\right]\times \right.$$
$$\left. \times\,(\not p' + m)\left[\frac{\gamma_\mu(\not p + \not k + m)\gamma_\nu}{s-m^2} + \frac{\gamma_\nu(\not p - \not k' + m)\gamma_\mu}{u-m^2}\right]\right\}. \tag{6.136}$$

Thus to evaluate the transition probability we have only to perform the trace over the product of various γ-matrices. The appropriate formulas are given in appendix C.

The first observation is that we have only to calculate two traces, namely

$$\begin{aligned} F_1(s,u) &= \mathrm{tr}[(\not p' + m)\gamma_\mu(\not p + \not k + m)\gamma_\nu(\not p + m)\gamma^\nu(\not p + \not k + m)\gamma^\mu] \\ F_2(s,u) &= \mathrm{tr}[(\not p + m)\gamma^\nu(\not p + \not k + m)\gamma_\mu(\not p' + m)\gamma_\nu(\not p - \not k' + m)\gamma^\mu]. \end{aligned} \tag{6.137}$$

Here we let F_1 and F_2 depend on the Mandelstam variables s and u because the sum of the two diagrams shown in figure 6.3 are symmetric under exchange of these variables. Furthermore it is clear that we can express all invariant quantities with two of the three Mandelstam variables, because with help of the on-shell conditions of the particles one finds

$$s + t + u = 2m^2. \tag{6.138}$$

Applying the formulas in appendix C, using the on shell-conditions (6.126) and the conservation law for four momentum (6.127):

$$F_1(s,u) = 8[2m^4 + 2m^2 s - (s-m^2)(u-m^2)],\ F_2(s,u) = 8m^2(2m^2 + s + u). \tag{6.139}$$

With help of these two functions we can write

$$|M_{fi}|^2 = \frac{e^4}{4}\left[\frac{F_1(s,u)}{(s-m^2)^2} + \frac{F_1(u,s)}{(u-m^2)^2} + \frac{F_2(s,u) + F_2(u,s)}{(s-m^2)(u-m^2)}\right], \tag{6.140}$$

which shows explicitely the already mentioned symmetry in s and u. Using the results from section 5.3 we finally find for the differential cross section:

$$\mathrm{d}\sigma = \frac{e^4}{64\pi^2 I^2}\left[\frac{2m^4 + 2m^2 s - (s-m^2)(u-m^2)}{(s-m^2)^2} + \frac{2m^4 + 2m^2 u - (s-m^2)(u-m^2)}{(u-m^2)^2} + \right.$$
$$\left. + \frac{2m^2(2m^2 + s + u)}{(s-m^2)(u-m^2)}\right]\mathrm{d}(-t)\mathrm{d}\varphi, \tag{6.141}$$

where because of the masslessness of the photon the invariant flow reduces to

$$I^2 = (pk)^2 = \frac{(s-m^2)^2}{4}. \tag{6.142}$$

In order to calculate the total cross section it is more convenient to express this in the laboratory frame which is defined such that the electron in the initial state is at rest. Then the Mandelstam variables are given as

$$s = (p+k)^2 = m^2 + 2m\omega,\ u = (p-k')^2 = m^2 - 2m\omega',\ t = (k'-k)^2 = 2\omega\omega'(1-\cos\theta). \tag{6.143}$$

Herein ω and ω' are the energy of the incoming and the outgoing photon respectively and θ is the scattering angle between the directions of the incoming and the outgoing photon.

Squaring the conservation equation for energy and momentum

$$p' = p + k - k' \Rightarrow m^2 = s - 2k'(p + k) \tag{6.144}$$

we find by solving for ω'

$$\omega' = \frac{m}{\frac{m}{\omega} + 1 - \cos\theta}. \tag{6.145}$$

Using (6.143) and (6.145) we have for fixed energy of the incoming photon ω:

$$t = -2\omega(1 - \cos\theta)\frac{m}{\frac{m}{\omega} + 1 - \cos\theta}, \; -\mathrm{d}t = 2\omega'^2\mathrm{d}(\cos\theta) = -2m\mathrm{d}\omega'. \tag{6.146}$$

Introducing

$$x = \frac{s - m^2}{m^2} = \frac{2\omega}{m} \text{ and } r_e = \frac{e^2}{4\pi m} \tag{6.147}$$

and integrating (6.141) over θ and φ we find for the total cross section

$$\sigma = \frac{\pi r_e^2[x(16 + 32x + 18x^2 + x^3) + 2(1 + x)^2(x^2 - 4x - 8)\ln(1 + x)]}{x^3(1 + x)^2}. \tag{6.148}$$

This result was obtained by Klein and Nishina in 1929 using Dirac's hole theory and therefore known as *Klein-Nishina cross section*.

Especially for small x, i.e. in the low energy region, we obtain the well-known result for photon scattering from classical electro-dynamics, namely the *Thomson cross section*:

$$\sigma \underset{x\to 0}{\cong} \frac{8\pi}{3} r_e^2(1 - x) = \sigma_{\mathrm{Thom}}(1 - x). \tag{6.149}$$

The cross section (6.148) is shown in fig. 6.4.

6.5.2 Annihilation of an e^-e^+-pair

With the very same calculation we can find the cross section of the process that an electron and a positron are annihilated to two photons. The diagram is the same as in figure 6.3. The only difference to Compton scattering is that we have to use a positron wave function $\bar{v}(-p_2, \sigma_2)$ for the incoming positron instead of the wave function $\bar{u}(p', \sigma')$ of an outgoing electron in (6.128) and the corresponding polarisation matrix for an ensemble of unpolarised positrons in (6.136):

$$\rho^{(e^+)}(p) = \frac{1}{2}(\not{p} - m). \tag{6.150}$$

Then it turns out that we can use the very same matrix element as for Compton scattering with the qualification that now the Mandelstam variables have to be defined as follows:

$$\begin{aligned} s &= (p_1 - k_1)^2 = (k_2 - p_2)^2 \\ t &= (p_1 + p_2)^2 = (k_1 + k_2)^2 \\ u &= (p_1 - k_2)^2 = (p_2 - k_1)^2. \end{aligned} \tag{6.151}$$

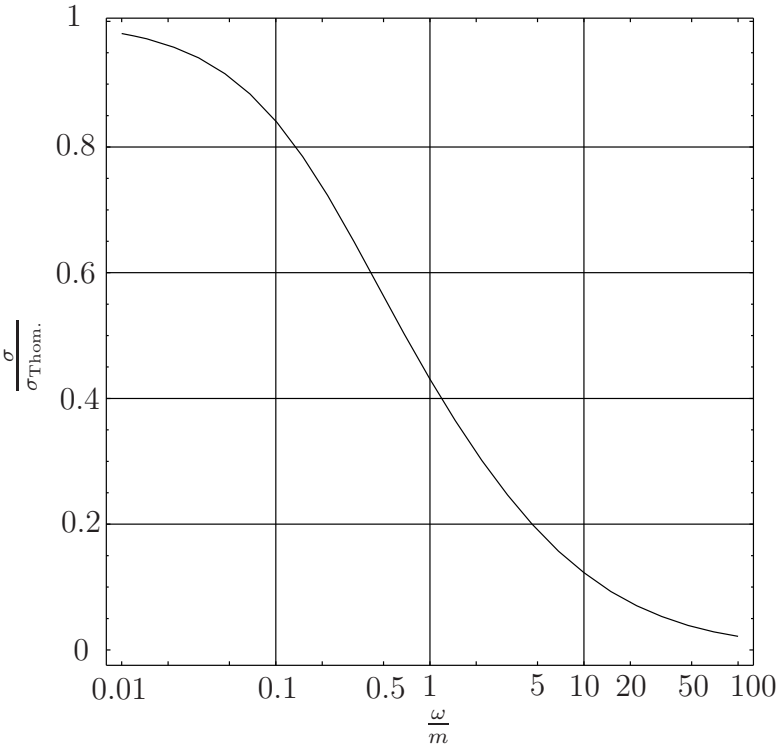


Figure 6.4: *The Klein-Nishina total cross section for Compton scattering*

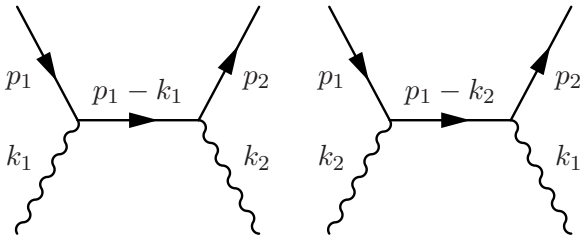


Figure 6.5: *The tree-level diagrams for e^-e^+-annihilation*

The invariant current (6.125) reads

$$I^2 = (p_1 p_2)^2 - m^4 = \frac{t(t-4m^2)}{4}. \tag{6.152}$$

Instead of $\mathrm{d}(-t)$ in (6.124) we have to write $\mathrm{d}(-s)$ according to our changed definition of the Mandelstam variables cf. (6.151). Thus the final result reads

$$\mathrm{d}\sigma = \frac{e^4}{64\pi^2 I^2}\left[\frac{2m^4+2m^2s-(s-m^2)(u-m^2)}{(s-m^2)^2} + \frac{2m^4+2m^2u-(s-m^2)(u-m^2)}{(u-m^2)^2} + \right. \\ \left. + \frac{2m^2(2m^2+s+u)}{(s-m^2)(u-m^2)}\right]\mathrm{d}(-s)\mathrm{d}\varphi, \tag{6.153}$$

To find the total cross section we have to give the physical range of s-values at a given t (which is now the square of the centre of mass energy). Together with $s+t+u=2m^2$ at the end we

find

$$m^2 - \frac{-t}{2} - I \leq s \leq m^2 - \frac{t}{2} + I \tag{6.154}$$

and the total cross section for e^-e^+-annihilation becomes

$$\sigma = \frac{2\pi r_e^2}{\tau^2(\tau-4)}\left[(\tau^2+4\tau-8)\ln\left(\frac{\sqrt{\tau}+\sqrt{\tau-4}}{\sqrt{\tau}-\sqrt{\tau-4}}\right) - (\tau+4)\sqrt{\tau(\tau-4)}\right] \quad \text{with } r_e = \frac{e^2}{4\pi m},\ \tau = \frac{t}{m}. \tag{6.155}$$

This result was found first by Dirac in 1930 within his hole theoretic formulation of QED.

6.6 The Background Field Method

So far our definitions of the generating functionals are not gauge invariant. Especially the effective action is not invariant under gauge transformations of its field arguments. This is due to their construction by the Faddeev-Popov gauge fixing formalism. It is clear that nevertheless in principle for all physical purposes they are very useful since we have shown that the generating functional for the S-matrix elements is gauge invariant as it must be, because it contains the physical information.

Nevertheless it is very customary to have a manifestly gauge invariant formalism, especially an invariant effective action. For instance if we try to prove the renormalisability of the theory we must show that all counter terms are consistent with the gauge invariance of the classical action and with the gauge fixing procedure of the Faddeev Popov formalism. Fortunately there exists a manifest gauge invariant formalism which uses a special choice of gauge fixing within the Faddeev-Popov formalism. It is known as the *background field method*, and we follow the line of arguments in Abbott's papers [Abb81, Abb82, AGS83]. Originally the method was invented by DeWitt and 't Hooft (for references see Abbott's papers).

6.6.1 The background field method for non-gauge theories

In order to explain the method we go again back to our most simple model of ϕ^4-theory. Here the background field method does not give any differences or advantages to the conventional functional approach given in chapter 4, but it shows clearly the idea of the method without the complications of gauge symmetries.

As is well known to the reader now the Lagrangian is given by

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{6.156}$$

Then we introduce the background field generating functional as follows

$$\tilde{Z}[J,\tilde{\varphi}] = N\int \mathrm{D}\phi\,\exp[\mathrm{i}S[\phi+\tilde{\varphi}] + \mathrm{i}\,\{J_1\phi_1\}_1]. \tag{6.157}$$

We see that the only difference to the usual definition is that in addition to the usual classical source J we introduced also a classical *background field* $\tilde{\varphi}$ as an external quantity, and the background field generating functional becomes additionally dependent on the background field.

The relation to the usual generating functional is very simple. This can be seen by substituting $\phi' = \phi + \tilde{\varphi}$ in the path integral:

$$\tilde{Z}[J,\tilde{\varphi}] = N\int \mathrm{D}\phi' \exp[\mathrm{i}S[\phi'] + \mathrm{i}\left\{J_1(\phi'_1 - \tilde{\varphi}_1)\right\}_1] = Z[J]\exp[-\mathrm{i}\left\{J_1\tilde{\varphi}_1\right\}_1]. \tag{6.158}$$

From this we find immediately after a functional Legendre transformation

$$\tilde{\Gamma}[\varphi,\tilde{\varphi}] = \Gamma[\varphi+\tilde{\varphi}] \text{ with } \tilde{W}[J,\tilde{\varphi}] = -\mathrm{i}\ln Z[J,\tilde{\varphi}],\ \varphi_1 = \frac{\delta\tilde{W}[J,\tilde{\varphi}]}{\delta J_1}. \tag{6.159}$$

Setting $\varphi = 0$ we find the background field representation of the usual effective action:

$$\Gamma[\tilde{\varphi}] = \tilde{\Gamma}[\varphi = 0, \tilde{\varphi}]. \tag{6.160}$$

The background field method does not give much simplification in this case, but as we shall show now with the introduction of the background field for gauge fields we can find a special gauge fixing such that the effective action is manifestly gauge invariant.

6.6.2 Gauge theories and background fields

Now we shall use the background field method for QED. In later chapters we shall also treat more general gauge field theories with non-abelian gauge groups and also there this method is of great advantage.

We start with the Faddeev Popov path integral (6.23) with a background field for the gauge field added. For the gauge invariant classical action we chose fermionic electrodynamics given by (6.49):

$$\begin{aligned}\tilde{Z}[J,\eta^*,\eta;a] = \int \mathrm{D}A\mathrm{D}\bar{\psi}\mathrm{D}\psi\ \mathrm{Det}\left(\frac{\delta\tilde{g}[A^\chi,a]}{\delta\chi}\right)\exp\Bigg[\mathrm{i}S_{\mathrm{QED}}[A+a,\bar{\psi},\psi]- \\ -\frac{\mathrm{i}}{2\xi}\left\{\tilde{g}_1^2[A]\right\}_1 + \mathrm{i}\left\{j_{1\mu}A_1^\mu + \mathrm{i}\bar{\eta}_1\psi_1 + \mathrm{i}\bar{\psi}_1\eta_1\right\}_1\Bigg].\end{aligned} \tag{6.161}$$

It is clear that we can do the same manipulations with scalar electrodynamics. Note that we did not add background fields for the matter fields $\bar{\psi}$ and ψ.

Now the trick is to chose the covariant Landau-Feynman gauge condition only for the "quantum field" A and not for the sum $A + a$ of both the quantum and the background field, which latter choice would result if we simply shifted the fields in the gauge fixed action (6.51):

$$\tilde{g}[A] = \partial_\mu A^\mu. \tag{6.162}$$

This special choice of gauge leads to an effective Lagrangian

$$\begin{aligned}\mathscr{L}_{\mathrm{eff}} =& \bar{\psi}\left[\mathrm{i}\not{\partial} + e(\not{A} + \not{a}) - m\right]\psi - \frac{1}{4}(F_{\mu\nu} + f_{\mu\nu})(F^{\mu\nu} + f^{\mu\nu}) - \\ &- \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu) + j_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta\end{aligned} \tag{6.163}$$

which is invariant under the local transformation

$$\delta A^\mu = 0,\ \delta a^\mu = \partial_\mu\delta\chi,\ \delta\psi = \mathrm{i}e\delta\chi\psi,\ \delta\bar{\psi} = -\mathrm{i}e\delta\chi\bar{\psi},\ \delta\eta = \mathrm{i}e\delta\chi\eta,\ \delta\bar{\eta} = -\mathrm{i}e\delta\chi\bar{\eta}. \tag{6.164}$$

Note that only the background field is transformed, not the quantum field. Since only the quantum field is gauge fixed the generating functional is invariant under this gauge transformation.

Now the functional matrix in the Faddeev-Popov determinant reads

$$\frac{\delta \tilde{g}_1[A^\chi]}{\delta \chi_2} = \Box_1 \delta_{12} \tag{6.165}$$

and is thus independent of all fields and sources. Thus it can be taken out of the path integral as another indefinite factor. This feature is unique to abelian gauge theories, in non-abelian ones they will give rise to the introduction of the famous Faddeev-Popov ghost terms.

Now we perform the infinitesimal background field gauge transformations (6.164) for the sources and the background field a^μ accompanied by the change of functional integration variables A^μ, ψ and $\bar{\psi}$ given in the same equation. This yields the *Ward-Takahashi-identity* for the generating functional:

$$\left\{ \mathrm{i}e\delta\chi_1 \left[\bar{\eta}_1 \left(\frac{\delta \tilde{Z}}{\delta \bar{\eta}_1} \right)_L - \left(\frac{\delta \tilde{Z}}{\delta \eta_1} \right)_R \eta_1 \right] - \partial_\mu \frac{\delta \tilde{Z}}{\delta a_\mu} \delta\chi_1 \right\}_1 = 0 \tag{6.166}$$

Since $\delta\chi_1$ is arbitrary we find the *local* identity

$$\mathrm{i}e \left[\bar{\eta}_1 \left(\frac{\delta \tilde{Z}}{\delta \bar{\eta}_1} \right)_L - \left(\frac{\delta \tilde{Z}}{\delta \eta_1} \right)_R \eta_1 \right] - \partial_\mu \frac{\delta \tilde{Z}}{\delta a_\mu} = 0. \tag{6.167}$$

Now we define the generating functional for connected Green's functions as usual by

$$\tilde{W} = -\mathrm{i} \ln \tilde{Z} \tag{6.168}$$

and also the effective action, i.e., the generating functional for proper one-particle irreducible vertex functions by

$$\tilde{\Gamma}[\langle \bar{\psi} \rangle, \langle \psi \rangle, \langle A_\mu \rangle; a_\mu] = \tilde{W}[\bar{\eta}, \eta, j_\mu] - \left\{ \bar{\eta}_1 \langle \psi_1 \rangle + \langle \bar{\psi}_1 \rangle \eta_1 + \langle A_{\mu 1} \rangle j_1^\mu \right\} \tag{6.169}$$

with

$$\langle \psi_1 \rangle = \left(\frac{\delta \tilde{W}}{\delta \bar{\eta}_1} \right)_L, \ \langle \psi_1 \rangle = \left(\frac{\delta \tilde{W}}{\delta \eta_1} \right)_R, \ \langle A_{\mu 1} \rangle = \frac{\delta \tilde{W}}{\delta j_1^\mu}. \tag{6.170}$$

Since from (6.169) we get

$$\eta_1 = -\left(\frac{\delta \tilde{\Gamma}}{\delta \langle \bar{\psi}_1 \rangle} \right)_L, \ \bar{\eta}_1 = -\left(\frac{\delta \tilde{\Gamma}}{\delta \langle \psi_1 \rangle} \right)_R, \ j_1^\mu = -\frac{\delta \tilde{\Gamma}}{\delta \langle A_{\mu 1} \rangle}, \ \frac{\delta \tilde{W}}{\delta a_{\mu 1}} = \frac{\delta \tilde{\Gamma}}{\delta a_{\mu 1}} \tag{6.171}$$

we find from (6.167) which holds true equally well for $\tilde{W}$:

$$\partial_\mu \frac{\delta \tilde{\Gamma}}{\delta a_{\mu 1}} - \mathrm{i}e \left[\left(\frac{\delta \tilde{\Gamma}}{\delta \langle \psi_1 \rangle} \right)_R \langle \psi_1 \rangle - \langle \bar{\psi}_1 \rangle \left(\frac{\delta \tilde{\Gamma}}{\delta \langle \bar{\psi}_1 \rangle} \right)_L \right] = 0. \tag{6.172}$$

This means nothing else than that $\tilde{\Gamma}$ is invariant under a special local gauge transformation of its arguments

$$\delta \langle \bar{\psi}_1 \rangle = -\mathrm{i}e\delta\chi_1 \langle \bar{\psi}_1 \rangle, \ \delta \langle \psi_1 \rangle = \mathrm{i}e\delta\chi_1 \langle \psi_1 \rangle, \ \delta a_{\mu 1} = \partial_\mu \delta\chi_1, \ \delta \langle A_{\mu 1} \rangle = 0. \tag{6.173}$$

The usual generating functional to calculate one-particle irreducible vertex functions can now be defined by setting $\langle A_{\mu 1}\rangle = 0$:

$$\Gamma_{\text{bfg}}[\langle\bar{\psi}\rangle, \langle\psi\rangle, a_\mu] = \tilde{\Gamma}[\langle\bar{\psi}\rangle, \langle\psi\rangle, \langle A_\mu\rangle = 0, a_\mu]. \tag{6.174}$$

This corresponds to a calculation of the usual Faddeev-Popov action making use of the special gauge fixing condition

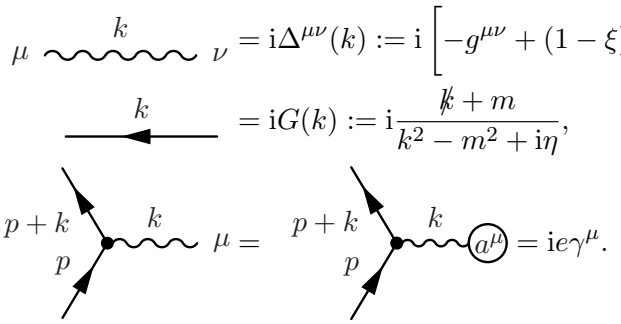
$$\partial_\mu(A^\mu - a^\mu) = 0. \tag{6.175}$$

and setting $\langle A^\mu\rangle = a^\mu$ at the end of the calculation.

From this we conclude that in the background field gauge we can calculate the proper vertex functions perturbatively from the explicitly gauge invariant action (6.174) where the internal lines correspond to propagators with respect to the quantum field A_μ (and of course the usual fermion propagators), derived from the term quadratic in A_μ (and ψ) in the effective Lagrangian (6.163). The price we have to pay is the introduction of one more vertex except the usual one with $\bar{\psi}$, ψ, and A_μ-lines: namely one with $\bar{\psi}$, ψ and an external field a_μ. In vertex functions the external photon lines are always arising from the vertices of the latter case. So far the gauge is only fixed for the *quantum vector fields* A_μ determining the propagator for inner lines, while the fields a_μ are completely arbitrary.

To calculate (connected) S-matrix elements now we have to fix the gauge for the *background field* a_μ which determines the propagator used to connect the gauge invariant 1PI-vertices as well as the asymptotically free gauge fields connected with the external legs.

In our choice of (6.162) the Feynman rules thus read as follows:



(6.176)

The external lines for the calculation of S-matrix elements are the same as given in figure 6.2 with the qualification that now any gauge fixing for the external gauge fields and its *propagator* can be chosen, independent from the previous choice for the quantum field (6.162).

The propagator of the background field is used to connect the proper vertex functions, which propagator we call the *external propagator* to distinguish it from the propagator of the quantum field which is used to calculate the proper vertex functions perturbatively. It is clear that only the on-shell polarisation vectors $e^\mu(\vec{p}, \lambda)$ for the external legs in the S-matrix Feynman rules have to be chosen according to the same gauge-fixing condition as the *external propagator*. Thus for the quantum field we can use 't Hooft's class of covariant renormalisable gauges ("R_ξ-gauges") cf. 6.162 to calculate the proper vertex functions which due to the background field method fullfil the naive Ward-Takahashi identities (WTIs) of the tree-level Lagrangian which are simpler than the WTIs for the analogous choice of gauge fixing in the usual method described before. It is clear that nevertheless the proper vertex functions calculated with this method depend on the choice of gauge for the quantum fields but as shown above for any choice of gauge within the Faddeev Popov method the *on-shell S-Matrix elements* are gauge independent and only those are used to calculate physical quantities such as life times of unstable states or cross sections.

6.6.3 Renormalisability of the effective action in background field gauge

Now we like to show that the effective action in background field gauge is renormalisable in the sense that we can render all proper vertex diagrams finite with local counter terms and that these are gauge invariant as the bare effective action in background field gauge. It is clear that this depends on the gauge choice for the *quantum fields* (6.162). In our case we have chosen a so called renormalisable gauge which leads to a photon propagator of momentum power -2 as for scalar fields. As we shall see below this is crucial for the proof of renormalisability.

We have to start with power counting. Each internal e^+e^--line has a momentum power (superficial degree of divergence) -1, each γ-line one of -2 (in our choice of the gauge-fixing condition!), each vertex counts with a power of 0 since there are no derivative couplings in fermionic QED. Now we define

$$\begin{aligned} E_\gamma &: \text{Number of external photon legs} \\ E_e &: \text{Number of external electron/positron legs} \\ I_\gamma &: \text{Number of internal photon legs} \\ I_e &: \text{Number of internal electron/positron legs} \\ L &: \text{Number of loops} \\ V &: \text{Number of vertices} \end{aligned} \tag{6.177}$$

Since each vertex has one internal or external photon leg and two electron/positron legs we have:

$$2V = 2I_e + E_e = 2(2I_\gamma + E_\gamma), \tag{6.178}$$

where we have taken into account that each internal line belongs to two legs of one ("tadpole diagrams") or two vertices. Further the number of loops is given by

$$L = I_e + I_\gamma - V + 1, \tag{6.179}$$

because we have $(I_e + I_\gamma)$ internal momenta, which are restricted by momentum conservation at each vertex but overall momentum conservation is always implied in the definition of the proper vertex functions. The superficial degree of divergence is

$$\delta = 4L - I_e - 2I_\gamma \tag{6.180}$$

Using (6.178) and (6.179) to eliminate L, I_γ and I_e from this equation we find

$$\delta = 4 - E_\gamma - \frac{3}{2}E_e. \tag{6.181}$$

Now we use the fact that we can define γ^μ-matrices in any space time dimension and thus have no problem to regularise quantum electrodynamics with help of dimensional regularisation. This means that we can be sure that the regularised bare vertex functions obey the background field gauge Ward-Takahashi identities which can be derived from (6.172) by taking further functional derivatives. Now we shall give a typical inductive argument: In zeroth loop order, i.e., for the tree diagrams of vertex functions by definition all is finite when using the renormalised normalisations for the wave functions, the renormalised electron, and the renormalised coupling constant. We remark that we only think about the UV-divergence problem in this section. The IR-divergences which are due to the masslessness of the photon are regularised by dimensional regularisation but

can only be cancelled for physical quantities by defining realistic inclusive cross sections where in addition to a process with a fixed number of electrons, positrons and photons in the final state all processes with an arbitrary number of additional soft photons in the final state is taken into account which cannot be resolved due to the finite energy resolution of the detector. This was first shown in the famous paper [BN37].

Now suppose we have renormalised all proper vertex functions up to L loops with *gauge symmetric* counter terms of the same form as in the original QED-Lagrangian. Thus we can renormalise each subdivergence of a proper vertex diagram Γ with such gauge symmetric counter terms and we know that the remainder $\bar{R}_\Gamma$ has only overall counter terms which are polynomials in the external momenta. Further we know that $\bar{R}_\Gamma$ fulfils its Ward Takahashi identity (WTI) contained in (6.172). To finish the proof of renormalisability we only have to show that all monomials to the effective action Lagrangian which fulfil these WTIs (and the additional symmetries of the effective Lagrangian like C- and P-invariance) and are renormalisation parts, i.e., which have a superficial degree of divergence $\delta(\Gamma) \geq 0$ are of the same form as $\mathscr{L}_{\text{QED}}$ cf. (6.49).

To this end we simply have to investigate the finite set of vertex functions which have $\delta(\gamma) \geq 0$ according to (6.181).

We start with the diagrams with $E_e = 0$, i.e., all pure photon vertices. First we show that in QED for $E_e = 0$ all vertices with an odd number of photon lines vanish due to *charge conjugation invariance*. Indeed: Since the charge conjugation for the ψ-field (see section 4.4.1) changes the sign of the current $\bar{\psi}\gamma^\mu\psi$ while the kinetic term for the fermions remains invariant we have to set $(A^C)^\mu = -A^\mu$, so that the full QED-Lagrangian is invariant under charge conjugation[9].

Thus we have for $E_e = 0$ the following renormalisation parts in the set of proper vertices: $E_\gamma = 2$ and $E_\gamma = 4$. The first case is the inverse photon propagator. According to (6.181) it is of dimension 2. It depends on only one momentum p due to momentum conservation and is a symmetric tensor field. Since the overall divergence is a polynomial of the momentum with dimension 2 it must be of the form

$$\overline{(D_\gamma^{-1})}_{\mu\nu}(p) = (Ap^2 + C)g_{\mu\nu} + Bp_\mu p_\nu + \text{finite} \tag{6.182}$$

with A and B constants of mass dimension 0 and C of mass dimension 2 containing the divergent parts at $L+1$-loop order $\propto 1/\epsilon^{L+1}$. Here and in the following the bar over the symbol means that we have subtracted all subdivergences at loop orders $\leq L$.

Now the infinite part must fulfil the WTI for D_γ^{-1} for itself since the $1/\epsilon$-expansion of dimensional regularisation is consistent with the gauge symmetry. From (6.172) we find by taking a derivative with respect to $a_{2\nu}$ and setting $\langle\psi\rangle = \langle\bar{\psi}\rangle = a_\mu = 0$ (where we use the fact that the vacuum is Lorentz invariant and thus admits no expectation values of non-scalar fields):

$$\begin{aligned} \partial_{\mu 1}\overline{(D_\gamma^{-1})}_{12}^{\mu\nu} &= 0 \\ p_\mu\overline{(D_\gamma^{-1})}(p)^{\mu\nu} &= 0. \end{aligned} \tag{6.183}$$

For the infinite part this means

$$p_\nu[(A+B)p^2 + C] = 0, \tag{6.184}$$

[9] This is also heuristically clear from classical electromagnetics: Charge conjugation means to change the sign of all charges and thus also of all currents. The electromagnetic field produced by the charge conjugated matter is thus opposite of the field which is produced by the original matter state.

and comparing the coefficients on the left and the right hand side of this equation yields the *Ward identities*

$$A = -B,\ C = 0 \Rightarrow \overline{(D_\gamma^{-1})}_{\mu\nu}(p) = A(p^2 g_{\mu\nu} - p_{\mu\nu}) + \text{finite}. \tag{6.185}$$

This means that there is no mass-counter term for the photon, which remains massless to any order as it should be due to gauge invariance, and we need to add only a *logarithmically divergent wave-function normalisation counterterm*:

$$\frac{1}{4}\delta Z_\gamma f_{\mu\nu} f^{\mu\nu} \tag{6.186}$$

to the effective Lagrangian which is of the same gauge invariant form as the bare Lagrangian we started with.

The other renormalisation part with $E_e = 0$ is the four-photon vertex *which is absent in the original QED-Lagrangian* since there does not exist a *gauge invariant* four-photon term of dimension ≤ 4 which is necessary for superficial renormalisability. We have to show that we do not need a counter term for the four-photon vertex which is superficially logarithmically divergent. Since its overall counter term is of momentum order 0 and a totally symmetric tensor with respect to its four Lorentz indices the four-photon-vertex must be of the form

$$\overline{(\Gamma^{(4\gamma)})}_{\mu\nu\rho\sigma}(p_1, p_2, p_3) = D(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\rho\nu}) + \text{finite}, \tag{6.187}$$

where the momenta p_1, p_2, p_3 are the independent four-momenta and D is a divergent constant of mass dimension 0. Deriving (6.172) three times with respect to the gauge field a with the appropriate space-time arguments and transforming to momentum space we find the simple condition

$$p_1^\mu \overline{(\Gamma^{(4\gamma)})}_{\mu\nu\rho\sigma}(p_1, p_2, p_3) = 0 \to D(p_\nu g_{\rho\sigma} + p_\rho g_{\nu\sigma} + p_\sigma g_{\rho\nu}) = 0 \tag{6.188}$$

which by comparison of coefficients leads to $D = 0$. Due to the WTI for the four-photon vertex it thus is finite and *in accordance with gauge invariance no four-photon counterterm is needed*! This does not mean that the four-photon vertex vanishes completely, but that the scattering of photons at photons (*Delbrück scattering*) is a pure quantum effect and absent at tree level (i.e., classical electrodynamics) due to the lack of a superficially renormalisable *local* gauge invariant four-photon interaction term.

We now come to the case $E_e = 2$. The inverse electron-positron propagator has no restriction for itself due to gauge invariance, which also formally can be seen by looking at (6.172). It is clear from (6.181) that it is of divergence degree 1. Further the QED-Lagrangian is invariant under spatial reflections (parity conservation). Thus we have

$$\overline{(G_e^{-1})}(p) = E\not{p} + Fm + \text{finite} \tag{6.189}$$

where E and F both are constants of mass dimension 0. While E is the electron wave function counter term F is a contribution to the mass renormalisation factor.

Now there remains only one divergence left, namely this with $E_e = 2$ and $E_\gamma = 1$, i.e., the counter term of the only local vertex already present in the original QED-Lagrangian. Our description of the gauge field as an covariant derivative (6.45) in order to enforce not only global but *local* gauge symmetry, yields the conclusion that E must be related to the counter term for the vertex:

$$\overline{(\Gamma^\mu_{(\gamma e\bar{e})})}(k, p) = eG\gamma^\mu + \text{finite}, \tag{6.190}$$

where the kinematics of the momenta is chosen as in (6.176).

Deriving (6.172) with a left sided derivative with respect to $\langle \bar{\psi}_2 \rangle$ and a right sided derivative with respect to $\langle \psi_3 \rangle$ after some algebra with the Fourier transform we find the WTI for the vertex which connects it to the inverse electron-positron propagator:

$$k_\mu(\overline{\Gamma^\mu_{(\gamma e\bar{e})}}) = e[(\overline{G_e^{-1}})(k+p) - (\overline{G_e^{-1}})(p)] \tag{6.191}$$

For the infinite part this means

$$eG\not{k} = eE\not{k} \Rightarrow E = G, \tag{6.192}$$

which again tells us that indeed also the counter terms necessary to renormalise the $\gamma e\bar{e}$-vertex and the electron-wave function connected to each other such that gauge invariance is guaranteed. This finishes the proof that QED in background field gauge is renormalisable with local gauge invariant counter terms, i.e., the renormalised Lagrangian is gauge invariant to any order and of the same form as the original Lagrangian (6.49).

6.7 One-loop corrections in QED

Now we can look at some of the one-loop radiative corrections in QED. We shall restrict ourselves to investigate the parts that have to be renormalized, i.e., the photon polarization tensor (or photon self-energy), the electron-positron self-energy, and the electron-positron-γ vertex. We shall use dimensional regularisation and discuss both the on-shell and the minimal-substraction renormalization renormalization schemes.

The usual definition of the multiplicative renormalisation for the Lagrangian reads

$$\mathscr{L} = -\frac{Z_3}{4} F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi} \mathrm{i} \not{\partial} \psi - Z_m m \bar{\psi} \psi + Z_1 e \mu^\epsilon \bar{\psi} \not{A} \psi, \quad d = 4 - 2\epsilon, \tag{6.193}$$

which implies the relation between the bare and the renormalized quantities

$$A_0^\mu = \sqrt{Z_3} A^\mu, \quad \psi_0 = \sqrt{Z_2}\psi, \quad m_0 = \frac{Z_m}{Z_2} m, \quad e_0 = \frac{Z_1}{Z_2\sqrt{Z_3}} e\mu^\epsilon. \tag{6.194}$$

For the following perturbative calculations, it is more convenient to write the counter-term Lagrangian in the form

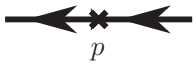
$$\delta\mathscr{L} = -\frac{\delta Z_3}{2} F_{\mu\nu} F^{\mu\nu} + \delta Z_2 \bar{\psi} \mathrm{i} \not{\partial} \psi - \delta m \bar{\psi} \psi + e\mu^\epsilon \delta Z_1 \bar{\psi} \not{A} \psi. \tag{6.195}$$

This implies

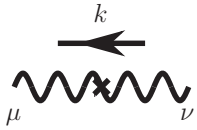
$$Z_1 = 1 + \delta Z_2, \quad Z_2 = 1 + \delta Z_2, \quad Z_3 = 1 + \delta Z_3, \quad Z_m = 1 + \frac{\delta m}{m}. \tag{6.196}$$

The counter-terms are evaluated as a power series in $\hbar$ or $\alpha = e^2/(4\pi)$.

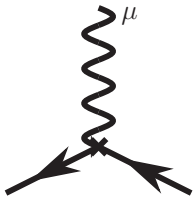
The Feynman rules for the counter-terms read



$$= \mathrm{i}(\not{p}\delta Z_2 - \delta m), \tag{6.197}$$



$$= -\mathrm{i}k^2 \delta Z_3 \Theta^{\mu\nu}(k), \tag{6.198}$$



$$= \mathrm{i}e\mu^\epsilon \delta Z_1 \gamma^\mu \tag{6.199}$$

with the projection matrix to four-transverse vector components

$$\Theta^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}. \tag{6.200}$$

We note that the Ward identity (6.192) implies in this notation

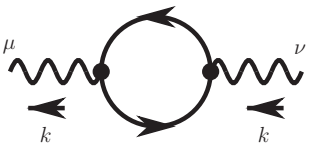
$$Z_1 = Z_2, \quad e_0 = e\mu^\epsilon Z_3^{-1/2}. \tag{6.201}$$

We define the photon-polarization tensor such that the one-particle irreducible diagrams with two amputated photon lines reads $+\mathrm{i}\Pi^{\mu\nu}(k) = +\mathrm{i}k^2\Pi(k)\Theta^{\mu\nu}(k)$, which implies that the Dyson equation for the transverse part of the full photon propagator reads[10]

$$D_\perp^{\mu\nu}(k) = -\frac{\Theta^{\mu\nu}(k)}{k^2[1-\Pi(k)]}. \tag{6.202}$$

In one-loop approximation for the regularized photon polarization we have to evaluate only one diagram,

$$\mathrm{i}\Pi_{\mathrm{reg}}^{\mu\nu}(k) =$$



(6.203)

Applying the Feynman rules, including the factor (-1) for the fermion loop leads to

$$\Pi_{\mathrm{reg}}^{\mu\nu}(k) = -e^2\mu^{2\epsilon} \int \frac{\mathrm{d}^{2\omega} l}{(2\pi)^{2\omega}} \operatorname{tr} \frac{\gamma^\mu[(\not{l}+\not{k})+m]\gamma^\nu(\not{l}+m)}{[m^2-(l+k)^2](m^2-l^2)}. \tag{6.204}$$

Of course, we could simplify our calculation by using the Ward-Takahashi identity 6.183 for the photon-polarization tensor, which tells us that it must be transverse, but we want to explicitly confirm this in our calculation.

[10] Note that in the background-field gauge the Lagrangian in the effective action is gauge invariant and thus the free as well as the full inverse photon propagator is transverse. Due to the Ward-Takahashi identities for proper vertex functions we never need to fix the gauge for the background-gauge field to connect the 1PI diagrams and provide external legs to express the (connected) Green's functions in terms of proper vertex functions and propagators.

The Dirac trace in the numerator gives

$$\Pi^{\mu\nu}_{\mathrm{reg}}(k) = -4e^2\mu^{2\epsilon}\int \frac{\mathrm{d}^{2\omega}l}{(2\pi)^{2\omega}} \frac{k^\mu l^\nu + k^\nu l^\mu + 2l^\mu l^\mu + [m^2 - l\cdot(l+k)]g^{\mu\nu}}{[m^2-(l+k)^2](m^2-l^2)}. \tag{6.205}$$

Now we use the first Feynman-parametrization formula (C.16) to write

$$\mathrm{i}\Pi^{\mu\nu}_{\mathrm{reg}}(k) = -4e^2\mu^{2\epsilon}\int \frac{\mathrm{d}^{2\omega}l}{(2\pi)^{2\omega}} \int_0^1 \mathrm{d}x \frac{k^\mu l^\nu + k^\nu l^\mu + 2l^\mu l^\mu + [m^2 - l\cdot(l+k)]g^{\mu\nu}}{[M^2-(l+xk)^2]^2} \tag{6.206}$$

with

$$M^2 = m^2 - x(1-x)k^2. \tag{6.207}$$

Now we shift the momentum-integration variable to $q = l + xk$ and interchange the q and x integration, which is allowed because the integral converges in the sense of dimensional regularization. Since the denominator is an even function in the new integration variable q we can omit all expressions linear to q in the numerator, because the dimensionally regularized integration over an odd function vanishes. This leads to

$$\begin{aligned}\mathrm{i}\Pi^{\mu\nu}_{\mathrm{reg}}(k) = &-4e^2\mu^{2\epsilon}\int_0^1 \mathrm{d}x \\ &\times \int \frac{\mathrm{d}^{2\omega}q}{(2\pi)^{2\omega}} \frac{2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu + g^{\mu\nu}[m^2 - q^2 + x(1-x)k^2]}{(M^2-q^2)^2}.\end{aligned} \tag{6.208}$$

Using (C.8), (C.10) and (C.14) and (C.15) to do the Laurent expansion around $\epsilon = (4-d)/2 =$, we get after some algebra

$$\Pi^{\mu\nu}_{\mathrm{reg}}(k) = -\frac{e^2}{4\pi^2}k^2\Theta^{\mu\nu}(k)\int_0^1 \mathrm{d}x x(1-x)\left(\frac{1}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{M^2}\right) = k^2\Theta^{\mu\nu}\Pi_{\mathrm{reg}}(k). \tag{6.209}$$

As we see, the photon-polarization tensor is indeed transverse as it should be according to the Ward-Takahashi identity. The x integral can be done in closed form leading to

$$\begin{aligned}\Pi_{\mathrm{reg}}(k) = \frac{e^2}{36\pi^2k^2}\Bigg[&-\frac{3k^2}{\epsilon} + 3\gamma k^2 - 12m^2 - 5k^2 + 3k^2\ln\left(\frac{m^2}{4\pi\mu^2}\right) \\ &+ 6\sqrt{1-\frac{4m^2}{s}}(s+2m^2)\,\mathrm{arcoth}\left(\sqrt{1-\frac{4m^2}{s}}\right)\Bigg].\end{aligned} \tag{6.210}$$

The tradiational renormalization scheme is the **on-shell scheme**. According to (6.202) the photon propagator has a pole at $k^2 = 0$, i.e., it describes a massless quantum. The on-shell renormalization scheme is defined such that the particle propagators should have a residuum of 1 at their pole mass. This means that the renormalization condition for this scheme reads

$$\Pi_{\mathrm{OS}}(k)|_{k^2=0} = 0, \tag{6.211}$$

which implies according to the Feynman rule for the corresponding counter term (6.200)

$$\delta Z_3^{(\mathrm{OS})} = \lim_{k^2\to 0}\Pi_{\mathrm{reg}}(k) = \frac{e^2}{12\pi^2}\left[\gamma - \frac{1}{\epsilon} + \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right], \tag{6.212}$$

and thus

$$\Pi_{\mathrm{OS}}(k) = \frac{e^2}{36\pi^2 s}\left[-12m^2 - 5k^2 + 6\sqrt{1-\frac{4m^2}{s}}(s+2m^2)\,\mathrm{arcoth}\left(\sqrt{1-\frac{4m^2}{s}}\right)\right]. \tag{6.213}$$

The disadvantage of this renormalization scheme is that the photon-wave-function counterterm becomes mass dependent and thus it is not so convenient in investigating the running of the renormalized parameters with the renormalization-group equation. For this purpose the **minimal-subtraction renormalization scheme** is more convenient, because it is a mass-independent renormalization scheme and thus the renormalization-group parameters only depend on the dimensionless coupling e. Here the choice is to just subtract the divergent part when $\epsilon \to 0$, i.e.,

$$\delta Z_3^{(\mathrm{MS})} = -\frac{e^2}{12\pi^2\epsilon}, \tag{6.214}$$

i.e.,

$$\begin{aligned}\Pi_{\mathrm{MS}}(k) = \frac{e^2}{36\pi^2 k^2}\Bigg[&3\gamma k^2 - 12m^2 - 5k^2 + 3k^2\ln\left(\frac{m^2}{4\pi\mu^2}\right)\\ &+ 6\sqrt{1-\frac{4m^2}{s}}(s+2m^2)\,\mathrm{arcoth}\left(\sqrt{1-\frac{4m^2}{s}}\right)\Bigg].\end{aligned} \tag{6.215}$$

Chapter 7

Nonabelian Gauge fields

In this chapter we come to the heart of modern particle physics, namely the notion of general local gauge symmetry based on compact semi-simple Lie groups as local symmetry groups. The paradigm of gauge symmetry is taken from the example of QED which is contained in the general case with the abelian gauge group U(1).

We shall start with the description of some toy models which aim to show the reader the main mathematical contents on hand of the most simple examples. Together this mathematics gives the ingredients for the standard model: local gauge invariance of the classical action, the Faddeev-Popov quantisation procedure, BRST-symmetry of the quantised theory and the Anderson-Higgs mechanism for massive gauge vector particles without breaking gauge symmetry and so keeping the theories renormalisable and physically consistent.

We shall also treat the question of *anomalies* which we have neglected so far in these notes.

With this mathematical fundament we shall describe the standard model of elementary particles which is based on the gauge group SU(3)×SU(2)×U(1) and a choice of the representation and realization of this group which is given by the collected *empirical knowledge* about all the particles produced and observed so far in accelerators.

7.1 The principle of local gauge invariance

In this section we shall introduce the concept of local gauge invariance at the level of the classical field theory. For sake of convenience we shall work with a multiplet of fermions. Its free Lagrangian

$$\mathscr{L}_{0f} = \sum_{i=1}^{N} \bar{\psi}^i (\mathrm{i}\partial\!\!\!/ - m)\psi^i \tag{7.1}$$

is invariant under the action of the fundamental representation of the group SU(N):

$$\psi'(x) = U\psi(x),\ \bar{\psi}' = \overline{U\psi(x)} = \bar{\psi}U^\dagger \quad \text{with} \quad U \in \mathrm{SU}(N). \tag{7.2}$$

The idea of local gauge theories was invented by Yang and Mills in 1954. They argued that the transformation U in (7.2) has to be constant in space and time. Because of the derivative in (7.1) indeed a space-time dependent U would not be a symmetry transformation of the action defined from $\mathscr{L}_{0f}$.

On the other hand, they argued, observers should be allowed to choose the basis of the multiplet independently on each place and at each time. The invariant object is

$$\underline{\psi}(x) = \underline{b}_i(x)\psi^i(x). \tag{7.3}$$

The product in the multiplet space is a sesquilinear form and thus all observers will choose an orthonormalised basis on each space-time point. Due to the symmetry group $\mathrm{SU}(N)$ underlying the symmetry the change of the basis at space-time point x compared to one at an arbitrary reference point (we shall take $x = 0$ for it) is given by an arbitrary smooth function $U : \mathbb{R}^4 \to \mathrm{SU}(N)$:

$$\underline{b}_i(x) = \underline{b}_j(0)U^j{}_i(x). \tag{7.4}$$

The derivative of the field components with respect to the local basis is obtained from this by simple calculation:

$$\begin{aligned} \partial_\mu\underline{\psi}(x) &= (\partial_\mu\psi^i)\underline{b_i} + \psi^i(x)\underline{b}_j(0)\partial_\mu U^j{}_i(x) = \\ &= (\partial_\mu\psi^i)\underline{b_i} + \psi^i(x)[U^{-1}(x)]_j{}^k[\partial_\mu U^j{}_i(x)]\underline{b}_j. \end{aligned} \tag{7.5}$$

We define

$$\mathrm{i}g\mathcal{A}_\mu(x) = U^{-1}(x)\partial_\mu U(x) = \mathrm{i}gA^a_\mu T^a. \tag{7.6}$$

The vector field $\mathcal{A}_\mu$ is called the *gauge field* and is $\mathcal{L}\mathrm{SU}(N) = \mathrm{su}(N)$-valued. Since $\mathrm{SU}(N)$ is a semi-simple Lie group there exists a basis T^a of the Lie-algebra such that

$$\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}. \tag{7.7}$$

In the appendix we give a proof for this theorem by E. Cartan. It holds true for any semi-simple Lie algebra. It is also shown there that in this basis the structure constants f^{abc}, defined by

$$\left[T^a, T^b\right] = \mathrm{i}f^{abc}T^c, \tag{7.8}$$

are totally anti-symmetric with respect to reordering of its indices.

Now we define the *covariant derivative*

$$\mathcal{D}_\mu = \partial_\mu + \mathrm{i}g\mathcal{A}_\mu. \tag{7.9}$$

If we substitute the partial derivative in the Lagrangian (7.1) by this covariant one by construction the Lagrangian becomes *invariant under general local gauge transformations*:

$$\mathscr{L}_f = \bar{\psi}(\mathrm{i}\not{\mathcal{D}} - m)\psi. \tag{7.10}$$

Here and in the following ψ denotes the column vector with the spinor-valued components ψ^i while $\bar{\psi}$ is the corresponding vector with components $\bar{\psi}^i$.

To show that (7.10) is indeed gauge invariant we have to find the transformation law for the gauge field (7.6). A gauge transformation is given by a change of the choice of the basis (7.4) in 7.4 by means of an $\mathrm{SU}(N)$ *space-time dependent* transformation $V(x)$:

$$\underline{b}_i(x) = \underline{b}'_j(x)V^j{}_i(x). \tag{7.11}$$

It is easy to see that for the matrix U in (7.4) this means

$$U'(x) = V(0)U(x)V^{-1}(x). \tag{7.12}$$

It is important to keep in mind that $U(x)$ does *not* transform with the local adjoint representation of the group since the matrices $V(0)$ and $V(x)$ are different!

Using the definition (7.6) of the gauge field yields together with (7.12):

$$\mathrm{i}g\mathcal{A}'_\mu(x) := U'(x)\partial_\mu U'^{-1}(x) = \mathrm{i}gV(x)\mathcal{A}_\mu V^{-1}(x) + V(x)\partial_\mu V^{-1}(x)]. \tag{7.13}$$

From (7.11) we derive immediately that the components of the fermion fields transform with the matrix $V(x)$:

$$\psi'(x) = V(x)\psi(x). \tag{7.14}$$

With (7.13) and (7.15) we see

$$(\mathcal{D}_\mu\psi)' = \mathcal{D}'_\mu\psi' = V\mathcal{D}_\mu\psi \Rightarrow \mathcal{D}'_\mu = V\mathcal{D}_\mu V^{-1}, \tag{7.15}$$

and from this it is clear that the Lagrangian (7.10) is invariant not only under global but also under local gauge transformations. The substitution of the partial derivative in (7.1) by the covariant one in (7.10) is known as the principle of *minimal substitution.*

For later use we specialise the transformation rules (7.13) and (7.14) for *infinitesimal gauge transformations*:

$$V(x) = 1 - \mathrm{i}g\delta\chi_a(x)T^a. \tag{7.16}$$

Inserting this in (7.13) and omitting all terms in higher than linear order of $\delta\chi_a$ we find

$$\delta\mathcal{A}_\mu = -\mathrm{i}g\delta\chi_a\,[T^a, \mathcal{A}_\mu] + \partial_\mu\chi_a T^a. \tag{7.17}$$

Using the definition of the structure constants (7.8) we obtain for the transformation law of the gauge-field components

$$\delta A^c_\mu = \partial_\mu\delta\chi^c + gf^{cab}\delta\chi^a A^b_\mu. \tag{7.18}$$

From a differential-geometrical point of view the gauge field $\mathrm{i}g\mathcal{A}_\mu$ is a local connection in the SU(N) bundle: At each point of space-time a "charge space" is located which is in our example realised by the multiplet of matter-field components ψ.

It is clear from (7.15) that $\mathcal{D}_\mu$ is an SU(N)-vector operator. Although $\mathcal{A}_\mu$ is not a vector under the adjoint representation the construct

$$\mathcal{F}_{\mu\nu} = \frac{1}{\mathrm{i}g}\,[\mathcal{D}_\mu, \mathcal{D}_\nu] = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\right)T^a \tag{7.19}$$

is a vector under the adjoint representation. Indeed it is easy to prove the behaviour of $\mathcal{F}_{\mu\nu}$ under a gauge transformation. Using (7.15) we obtain immediately

$$\mathcal{F}'_{\mu\nu} = V\mathcal{F}_{\mu\nu}V^{-1}, \tag{7.20}$$

i.e., $\mathcal{F}$ transforms under the *adjoint representation* of the underlying gauge group SU(N).

To make the gauge field to a dynamically interacting field we have to add a kinetic term to the Lagrangian (7.10) which must be gauge invariant. The most simple possibility, already very successful in the case of QED, is

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{tr}\,(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{7.21}$$

the so called *Yang-Mills action*. While mathematically the construction we used is the same in both cases, QED and nonabelian Yang-Mills gauge theory: The fields $\mathcal{F}_{\mu\nu}$ define the *local curvature* from the affine connection $\mathcal{A}_\mu$.

Physically there are important differences: first of all due to (7.19) for nonabelian gauge groups the Lagrangian (7.21) describes *interacting* fields since if not all of the f^{abc} vanish (7.21) contains expressions with three and four gauge fields. Thus the nonabelian gauge fields are always interacting, i.e., they carry charges determined by the adjoint representation of the gauge group.

From the point of view of model building it is important to realise that due to (7.19) for nonabelian gauge groups $\mathcal{F}_{\mu\nu}$ contains the gauge coupling g which determines the strength of interaction. This means that the coupling has to be the same for *all* matter fields. Each matter field must be grouped in a multiplet on which a representation of the gauge group operates. The only possibility to change the coupling strength from one multiplet to the other is to change the representation. But this does not mean too much freedom, because the choice of the gauge group determines uniquely the possible multiplets and the charge patterns in each of them; they are fixed by the Lie-algebra matrices T^a in the chosen representation which are normalised due to (7.7), but one is not allowed to change the gauge coupling g from one multiplet to another[1], because otherwise the gauge-symmetry is explicitely broken and, as we shall see in the next section, then the physical content of the quantised theory is completely lost. This feature of the nonabelian gauge theories is known as *universality of the gauge coupling*.

In the abelian case of the group U(1), the gauge group of QED, the gauge coupling g (which in this case is $-e$ for the electron-positron matter field) is not necessarily universal. It is one of the big remaining enigmas within the standard model why all observable particles come in integer multiples of $e/3$.

Remark: Not e is the "elementary electromagnetic charge" but $e/3$ since the quarks which constitute the observed hadrons carry electric charges of $-e/3$ and $+2e/3$, while their anti-particles of course carry the opposite sign of charges. It is due to the remarkable feature of nonabelian gauge theories, in this case of quantum chromodynamics which is a Yang-Mills theory with the gauge group SU(3) to be "confining". The charges of SU(3) are called the *colour*. The quarks are grouped in the fundamental representation of this colour SU(3) gauge group and thus for them there exist three colours (red, green, blue). Confinement means that only "colour-less objects" are observable in nature, i.e., only bound states of such a configuration of quarks which are found in the physical spectrum, which are colour-charge neutral. If one tries to split these bound objects such strong forces have to be used that immediately a bunch of new particles, among them a lot of quarks and the QCD-gauge fields, which quanta are called "*gluons*", such that those can again be bound to colourless objects. Indeed up to now no free quarks have been observed, but from *deep inelastic scattering* experiments we know that the *hadrons* are bound states of a quark anti-quark pair (the *mesons*) or of three quarks of different colour (the *baryons*). These experiments show also that the strong interaction becomes weakly coupled in the regime of high scattering energies, i.e., the "running coupling constant" is small if the renormalisation scale is chosen high enough. This means that for QCD perturbation theory is applicable for high-energy processes. We shall show this property of *asymptotic freedom* of nonabelian gauge theories when we come to the issue of quantisation and renormalisation of gauge theories.

Before we come to the important question, how to describe massive gauge vector fields, in the next section we shall treat the quantisation of gauge field theories which can be formulated in

[1] it is clear that one is allowed change the sign of the gauge coupling

the same way using the *Faddeev-Popov formalism* already described for QED in chapter 6.

7.2 Quantisation of nonabelian gauge field theories

The naive path integral expression for the expectation value of a time ordered product of local gauge invariant operators reads

$$\langle T\mathbf{A}(x_1)\mathbf{B}(x_2)\ldots\rangle = N\int \mathrm{D}\Phi[A(x_1)B(x_1)\ldots]\exp(\mathrm{i}S[\Phi]), \tag{7.22}$$

where we have abbreviated the fields ψ, $\bar{\psi}$ and $\mathcal{A}$ with Φ.

This path integral is not well defined due to the fact that it contains an additional infinite factor caused by the gauge copies of the fields, i.e., one integrates over all field configurations although only those are physically distinct which are not connected by a local gauge transformation.

From the perturbative point of view this can be observed by the fact that there does not exist a free propagator for the gauge fields because in momentum space its putative "inverse" reads

$$(\Delta^{-1})^{ab}_{\mu\nu}(k) = \delta^{ab}(k_\mu k_\nu - k^2 g_{\mu\nu}), \tag{7.23}$$

which has no inverse because it maps the vector k to 0, i.e., it has a vanishing eigenvalue.

The way out of this is simple: Since the integration over the gauge orbits yields only an field independent constant it is irrelevant for the calculation of expectation values of the type (7.22). We have only to integrate over the gauge group selecting an arbitrary gauge fixing. This idea is due to *Feynman, Faddeev and Popov*. For that purpose we introduce the *gauge fixing condition*

$$g^a[x;\Phi] = c^a(x), \tag{7.24}$$

where c^a is a set of arbitrary functions. For now the local functional g has only to be chosen such that it really fixes the gauge (at least in the here developed perturbative sense), i.e., the condition (7.24) on the fields Φ should uniquely determine the member of each gauge orbit.

Now we define the functional

$$\Delta_g^{-1}[\Phi] = \int \mathrm{D}U\delta[g^a[x,\Phi^U] - c^a], \tag{7.25}$$

where $\mathrm{D}U$ is the *Haar measure* of the gauge group G, taken at each space-time point x. This measure is by definition invariant under left- and right operations of the group, which feature makes it gauge invariant:

$$\Delta_g^{-1}[\Phi^{U'}] = \int \mathrm{D}U\delta[g^a[x,(\Phi^{U'})^U] - c^a] \overset{U''=UU'}{=} \int \mathrm{D}U''\delta[g^a[x;\Phi^{U''}] - c^a] := \Delta_g^{-1}[\Phi]. \tag{7.26}$$

We can parameterise the gauge group elements with the χ^a of Eq. (7.16), which reads, e.g., for the matter field ψ:

$$U(\chi) = \exp(-\mathrm{i}g\chi^a T^a) \tag{7.27}$$

In this parameterisation the Haar measure is simply defined by the usual integral measure $\mathrm{d}^n\chi$ since then an infinitesimal transformation reads $\chi^a \to \chi^a + \delta\chi^a$. Further we can by definition choose the fields Φ such that the gauge condition (7.24) is fulfilled. Thus we have

$$\Delta_g^{-1}[\Phi] = \int \mathrm{D}g^a \mathrm{Det}\left(\frac{\delta\chi^b}{\delta g_a}\right)\delta[g^a[\Phi^\chi] - c^a] = \mathrm{Det}\left(\frac{\delta\chi^b}{\delta g^a}\right)\Big|_{\chi^a=0} \tag{7.28}$$

due to our choice of gauge for the fields Φ within the gauge orbits.
Thus we have

$$\Delta_g[\Phi] = \mathrm{Det}\left(\frac{\delta g^a}{\delta \chi^b}\right). \tag{7.29}$$

With this preparation we can introduce a clever factor 1 into the naive path integral (7.22):

$$\langle T\mathbf{A}(x_1)\mathbf{B}(x_2)\ldots\rangle = N\int \mathrm{D}\Phi\mathrm{D}U\Delta_g[\Phi]\delta[g^a[\Phi^U]-c^a](A(x_1)B(x_2)\ldots)\exp(\mathrm{i}S[\Phi]) \tag{7.30}$$

Substituting herein Φ by $\Phi^{U^{-1}}$ yields

$$\langle T\mathbf{A}(x_1)\mathbf{B}(x_2)\ldots\rangle = N\int \mathrm{D}\Phi\mathrm{D}U(A(x_1)B(x_2)\ldots)\Delta_g[\Phi]\delta[g^i[\Phi]-c^i]\exp(\mathrm{i}S[\Phi]), \tag{7.31}$$

where we have used the invariance of the path-integral measure, the action functional $S[\Phi]$ and Δ_g (see (7.26)) as well as the local operators $\mathbf{A},\mathbf{B},\ldots$. We have thus to prove the invariance of the path-integral measure under local gauge transformations. It is sufficient that this holds true for infinitesimal transformations:

$$\begin{aligned}\mathrm{Det}\left[\frac{\delta}{\delta\Phi_2}(\Phi_1+\delta\Phi_1)\right] &= \mathrm{Det}\Big[\delta^{2\omega}(x_1-x_2)(\delta^{ab}+gf^{abc}\delta\chi^b)\\ &\quad\times(\delta^i_k-\mathrm{i}g\delta\chi^a(T^a)^i{}_k)(\delta^i_k+\mathrm{i}g\delta\chi^a(T^a)^i{}_k)\Big]\\ &= 1+O(\delta\chi^2)\end{aligned} \tag{7.32}$$

Here we have made use of the behaviour of the various fields under infinitesimal local gauge transformations cf. (7.16) for the ψ and (7.18) for the $\mathcal{A}_\mu$-fields.

Now in (7.31) the infinite factor coming from the integration over the gauge group is explicit and can be lumped into the overall normalisation constant N. Further by construction the expression is independent of the arbitrary function c^i, so that we can functionally integrate over it with an arbitrary weight giving rise for another field-independent factor which again can be absorbed into N. For the weight factor we use an exponential, which for linear gauge fixing functions g^i is a Gaussian:

$$\begin{aligned}\langle T\mathbf{A}(x_1)\mathbf{B}(x_2)\ldots\rangle = N'\int \mathrm{D}\Phi\Delta_g[\Phi][A(x_1)B(x_2)\ldots]\exp(\mathrm{i}S[\Phi])\times\\ \times\exp\left(-\frac{\mathrm{i}}{2\xi}\left\{g^i_1[\Phi]g^i_1[\Phi]\right\}_1\right).\end{aligned} \tag{7.33}$$

To obtain perturbatively calculable expressions we have to express the Faddeev-Popov determinant Δ_g (7.29) with help of scalar *Grassmann* fields, the so called *Faddeev-Popov ghosts*. They have to be Grassmannian since the determinant appears in the numerator of the path integral.

$$\begin{aligned}\langle T\mathbf{A}(x_1)\mathbf{B}(x_2)\ldots\rangle = N'\int \mathrm{D}\Phi\mathrm{D}\eta^*\mathrm{D}\eta[A(x_1)B(x_2)\ldots]\exp(\mathrm{i}S[\Phi])\times\\ \times\exp\left(-\frac{\mathrm{i}}{2\xi}\left\{g^i_1[\Phi]g^i_1[\Phi]\right\}_1-\mathrm{i}\left\{\eta^{*i}_1\left.\frac{\delta g^i[\Phi^\chi]}{\delta\chi^a_2}\right|_{\chi=0}\eta^a_2\right\}_{12}\right).\end{aligned} \tag{7.34}$$

It is clear that the expression with the Faddeev-Popov ghosts is local since g^i is local by assumption, and thus the functional derivative is $\propto\delta^{(2\omega)}(x_1-x_2)$.

(7.34) tells us that we can evaluate expectation values of time ordered products of *gauge independent local operators* with the usual Feynman-Kac formula when we use the *effective* action

$$\begin{aligned} S_{\text{eff}}[\Phi,\eta^*,\eta] &= S[\Phi] - \frac{1}{2\xi}\left\{g_1^i[\Phi]g_1^i[\Phi]\right\}_1 - \left\{\eta_1^{*i}\mathcal{M}_{12}^{ia}\eta_2^a\right\}_{12} \\ \text{with}\quad \mathcal{M}_{12}^{ia} &= \left.\frac{\delta g_1^i[\Phi^\chi]}{\delta\chi_2^a}\right|_{\chi=0} \end{aligned} \tag{7.35}$$

As an example we choose the covariant 't Hooft gauge

$$g^i = \partial_\mu A^{i\mu}. \tag{7.36}$$

Then we have

$$\begin{aligned} \mathcal{M}_{12}^{ia} &= \frac{\delta g^i[x_1,A^\chi]}{\delta\chi^a(x_2)} = \int \mathrm{d}^4x_3 \frac{\delta g^i[x_1,A]}{\delta A_\nu^b(x_3)}\frac{\delta A_\nu^{\chi b}(x_3)}{\delta\chi^a(x_2)} \\ &= \int \mathrm{d}^4x_3 \partial_{x_1,\nu}\delta^{(4)}(x_1-x_3)\delta^{ib} \\ &\quad\times\left[\partial_{x_3}^\nu\delta^{(4)}(x_3-x_2)\delta^{ba} + gf^{bca}A^{c\nu}(x_3)\delta^{(4)}(x_3-x_2)\right] \\ &= \Box_1\delta^{(4)}(x_1-x_2) + gf^{cab}A^{c\nu}(x_2)\partial_{1\nu}\delta(x_1-x_2). \end{aligned} \tag{7.37}$$

Thus we have

$$S_{\text{gh}} = \left\{-\eta_1^{*i}\mathcal{M}_{12}^{ia}\eta_2^a\right\}_{12} = \left\{(\partial_\mu\eta_1^*)\mathrm{D}_{\mathrm{A}}^\mu\eta_1\right\}_1, \tag{7.38}$$

where

$$(\mathrm{D}_{\mathrm{A}\mu}\eta)^a = \partial_\mu\eta^a + \mathrm{i}gT_{\mathrm{A}}^{ca}{}_b A^{c\mu}\eta^b = \partial_\mu\eta^a + gf^{cab}A^{c\mu}\eta^b \tag{7.39}$$

is the covariant derivative, which applies to a matter field that transforms under the adjoint representation of the local gauge group.

7.2.1 BRST-Invariance

Although in principle it is possible to use local gauge invariance of the original action $S[\Phi]$ to derive the Ward-Takahashi-identities for nonabelian gauge theories as we have done in QED it is much more convenient to use a symmetry of the effective gauge-fixed action $S_{\text{eff}}[\Phi,\eta^*,\eta]$, which is named after *Becchi, Rouet, Stora and Tyutin* the *BRST-symmetry*. To simplify the analysis we use the fact that we can write

$$\exp[\mathrm{i}\left\{g_1^i g_1^i\right\}_1] = N\int \mathrm{D}h \exp\left[\frac{\mathrm{i}\xi}{2}\left\{h_1^i h_1^i\right\}_1 + \mathrm{i}\left\{g_1^i h_1^i\right\}\right]. \tag{7.40}$$

Herein the h_1^i are auxiliary fields, often named after their inventors *Nakanishi-Lautrup fields*. With help of these fields we write

$$S'_{\text{eff}}[\Phi,\eta^*,\eta,h] = S[\Phi] + \frac{\xi}{2}\left\{h_1^i h_1^i\right\}_1 + \left\{g_1^i h_1^i\right\} - \left\{\eta_1^{*i}\left.\frac{\delta g^i[\Phi^\chi]}{\delta\chi_2^a}\right|_{\chi=0}\eta_2^a\right\}_{12} \tag{7.41}$$

For the following we note that the Nakanishi-Lautrup fields h^i are usual commuting c-number fields. The BRST transformations read

$$\begin{aligned}
\delta_\Theta \psi(x) &= -\mathrm{i}g\Theta\eta^a(x)T^a\psi(x),\\
\delta_\Theta A^a_\mu(x) &= \Theta[\partial_\mu\eta^a(x) + gf^{abc}A^c(x)\eta^b(x)] = \mathrm{D}_\mu\eta^a\\
\delta_\Theta \eta^a(x) &= \frac{g}{2}\Theta f^{abc}\eta^b(x)\eta^c(x)\\
\delta_\Theta \eta^{*a}(x) &= \Theta h^a\\
\delta_\Theta h^a &= 0.
\end{aligned} \tag{7.42}$$

The last two lines show that the introduction of the Nakanishi-Lautrup fields is customary, because they make the BRST transformation as a whole homogeneous. We also used the fact that η^a is a field transforming under the adjoint representation of the gauge group:

$$\mathrm{D}_\mu\eta^a = \partial_\mu\eta^a + gf^{cab}A^{c\mu}\eta^b, \quad (T_{\mathrm{ad}})^{ca}{}_b = -\mathrm{i}f^{cab}. \tag{7.43}$$

Since η^a are Grassmann-valued fields the parameter Θ must also be Grassmannian in order to keep the bosonic or fermionic nature of the fields consistent with the BRST-transformed fields. In the following we define for any local expression $F_1[\Xi]$ of the fields $\Xi \in \{\psi, A^a_\mu, \eta^a\eta^{*i}, h^i\}$:

$$\delta_\Theta F_1[\Xi] = \Theta\delta_{\mathrm{BRST}}F_1[\Xi]. \tag{7.44}$$

The most important feature of BRST-transformations is their *nil-potency*, i.e. for any expression $F_1[\Xi]$ we have

$$\delta^2_{\mathrm{BRST}}F_1[\Xi] = 0. \tag{7.45}$$

To prove this we start with the elementary fields Ξ themselves. From (7.42) we find

$$\delta_\Theta\delta_{\mathrm{BRST}}\psi = -\mathrm{i}g\eta^aT^a(-\mathrm{i}g)\Theta\eta^bT^b\psi + \frac{g}{2}\Theta f^{abc}\eta^b(-\mathrm{i}g)\eta^cT^a\psi. \tag{7.46}$$

Since Θ is a Grassmannian and the same is true for the Spin-1/2-Dirac fields ψ we have a change of sign by bringing $\delta\Theta$ to the left in the first expression. In the first expression we can also substitute T^aT^b by $1/2\left[T^a, T^b\right] = \mathrm{i}/2f^{abc}T^c$ because of the Grassmannian nature of the ghost fields η^a. Thus we have

$$\delta_\Theta\delta_{\mathrm{BRST}}\psi = \frac{\mathrm{i}g^2}{2}f^{abc}\Theta(\eta^a\eta^bT^c - \eta^b\eta^cT^a)\psi = 0. \tag{7.47}$$

The last operation was to interchange the summation indices a and c in the last expression and using the total antisymmetry of the structure constants f^{abc}. This proves (7.45) for $F_1 = \psi_1$.

For the gauge fields we find

$$\begin{aligned}
\delta_\Theta\delta_{\mathrm{BRST}}A^a_\mu =& \frac{g}{2}\Theta f^{abc}\partial_\mu(\eta^b\eta^c) +\\
&+ gf^{abc}\Theta(\partial_\mu\eta^c + gf^{cb'c'}A^{c'}\eta^{b'})\eta^b +\\
&+ gf^{abc}A^c\Theta\frac{g}{2}f^{bc'a'}\eta^{c'}\eta^{a'}.
\end{aligned} \tag{7.48}$$

Since the fields A^a_μ are bosonic Θ commutes with them and we can bring Θ without changes of signs to the left of the whole expression. Further we sort with respect to g and g^2:

$$\begin{aligned}
\delta_\Theta\delta_{\mathrm{BRST}}A^a_\mu = \Theta\frac{g}{2}f^{abc}&\left[(\partial_\mu\eta^b)\eta^c + \eta^b\partial_\mu\eta^c + 2(\partial_\mu\eta^c)\eta^b\right] +\\
&+ \Theta\frac{g^2}{2}f^{abc}\left[f^{cb'c'}A^{c'}\eta^{b'}\eta^b + \frac{1}{2}f^{bc'a'}A^c\eta^{c'}\eta^{a'}\right]
\end{aligned} \tag{7.49}$$

The term $\propto g$ vanishes identically because we have for the second term

$$f^{abc}\eta^b\partial_\mu\eta^c = -f^{abc}(\partial_\mu\eta^c)\eta^b = +f^{abc}(\partial_\mu\eta^b)\eta^c, \tag{7.50}$$

where we have used that the η^a are anti-commuting fields and the total anti-symmetry of the f^{abc}. For the third term we use also the anti-symmetry of the structure constants to obtain 0 for the first line in (7.49). The second line $\propto g^2$ is a little bit more involved, but using again the anti-symmetry of the structure constants the anti-commuting of the ghost fields we find by renaming the summation indices:

$$\text{second line of (7.49)} = -\Theta\frac{g^2}{2}A^{c'}\eta^{b'}\eta^b(f^{cb'c'}f^{bca} + f^{cc'b}f^{b'ca} + f^{cbb'}f^{c'ca}) = 0, \tag{7.51}$$

which latter equation follows from the definition of the structure constants (7.8) and the *Jacobi identity*

$$\Big[\Big[T^a,T^b\Big],T^c\Big] + \Big[\Big[T^b,T^c\Big],T^a\Big] + \Big[[T^c,T^a],T^b\Big] = 0, \tag{7.52}$$

which holds true for any three operators T^a, T^b and T^c:

$$f^{cb'c'}f^{bca} + f^{cc'b}f^{b'ca} + f^{cbb'}f^{c'ca} = 0. \tag{7.53}$$

The next identity is for the ghost field itself:

$$\begin{aligned}\delta_\Theta\delta_{\mathrm{BRST}}\eta^a &= \frac{g^2}{4}f^{abc}(\Theta f^{bb'c'}\eta^{b'}\eta^{c'}\eta^c + f^{cb'c'}\eta^b\Theta\eta^{b'}\eta^{c'} = \\ &= \Theta\frac{g^2}{2}f^{abc}f^{bb'c'}(\eta^{b'}\eta^{c'}\eta^c - \eta^{c'}\eta^c\eta^{b'}) = 0,\end{aligned} \tag{7.54}$$

where we have used again the Grassmannian nature of the ghost fields and the anti-symmetry of the structure constants as well as the fact that also the parameter Θ is anti-commuting with the ghost fields.

From (7.42) we immediately find

$$\begin{aligned}\delta_\Theta\delta_{\mathrm{BRST}}\eta^{*a} &= \delta_\Theta h^a = 0,\\ \delta_\Theta\delta_{\mathrm{BRST}}h^i &= \delta_\Theta 0 = 0.\end{aligned} \tag{7.55}$$

Thus (7.45) holds true for all linear forms of fields. Since we can write derivatives as limits of fields it also holds true if the linear forms contain derivatives of fields.

Now the rest of the proof follows by induction: Suppose we have proven (7.45) for a monomial $F_1^{(n)}[\Xi]$ with n fields. Then for a monomial with $(n+1)$-fields we can write

$$F_1^{(n+1)}[\Xi] = \Xi_1^j F_1^{(n)}[\Xi]. \tag{7.56}$$

For this we have

$$\begin{aligned}\delta_\Theta F_1^{(n+1)}[\Xi] &= (\Theta\delta_{\mathrm{BRST}}\Xi_1^j)F_1^{(n)}[\Xi] + \Xi_1^j\Theta\delta_{\mathrm{BRST}}F_1^{(n)}[\Xi] = \\ &= \Theta\{(\delta_{\mathrm{BRST}}\Xi_1^k)F_1^{(n)}[\Xi] + \sigma(\Xi_1^k)\Xi_1^k\delta_{\mathrm{BRST}}F_1^{(n)}[\Xi]\},\end{aligned} \tag{7.57}$$

where we have introduced the sign of the field Ξ_1^k which is 1 for commuting and -1 for anti-commuting (i.e., Grassmannian valued) fields. From this we find

$$\begin{aligned}\delta_\Theta\delta_{\mathrm{BRST}}F_1^{(n+1)}[\Xi] &= \delta_{\mathrm{BRST}}\Xi_1^k\cdot\Theta\delta_{\mathrm{BRST}}F_1^{(n)}[\Xi] + \sigma(\Xi_1^k)\Theta\delta_{\mathrm{BRST}}\Xi_1^k\cdot\delta_{\mathrm{BRST}}F_1^{(n)}[\Xi] = \\ &= \Theta\delta_{\mathrm{BRST}}\Xi_1^k\cdot\delta_{\mathrm{BRST}}F_1^{(n)}[\Xi][-\sigma(\Xi_1^k) + \sigma(\Xi_1^k)] = 0.\end{aligned} \tag{7.58}$$

In the step to the last line we have used the fact that

$$\sigma(\delta_{\mathrm{BRST}}\Xi_1^k) = -\sigma(\Xi_1^k). \tag{7.59}$$

This proves for all polynomials of fields and their derivatives that indeed the BRST-transformation is nil-potent, i.e., Eq. (7.45).

To show that the action (7.41) is invariant under BRST transformations first we realise that it is a usual local gauge transformation with infinitesimal parameters

$$\delta\chi^a(x) = \Theta\eta^a(x) \tag{7.60}$$

for the fields $\Phi \in \{A_\mu^a, \psi\}$, so that the classical action $S[\Phi]$ is indeed BRST-invariant.

To prove that also the rest of the action, i.e., $S_{\mathrm{gh}} + S_{\mathrm{NL}}$ with

$$\begin{aligned} S_{\mathrm{gh}} &= -\left\{\eta_1^{*i}\left.\frac{\delta g_1^i[\Phi^\chi]}{\delta\chi_2^a}\right|_{\chi^a=0}\eta_2^a\right\}_{12}, \\ S_{\mathrm{NL}} &= \frac{\xi}{2}\left\{h_1^i h_1^i\right\}_1 + \left\{h_1^i g_1^i[\Phi]\right\}_1 \end{aligned} \tag{7.61}$$

is gauge invariant, we show that it can be written as $\delta_{\mathrm{BRST}}\Psi[\Xi]$.

To this end we calculate

$$\delta_\Theta g_1^i[\Phi] = \left\{\left.\frac{\delta g_1^i[\Phi^\chi]}{\delta\chi_2^a}\right|_{\chi=0}\Theta\eta_2^a\right\}_2 = \Theta\left\{\mathcal{M}_{12}^{ia}\eta_2^a\right\}_2, \tag{7.62}$$

where we firstly have used that for the original fields Φ the BRST transformation is a gauge transformation with local infinitesimal gauge parameters (7.60) and secondly that g_1^i and χ_2^a are both c-number valued, so that there is no change of sign when bringing Θ to the left. Comparing (7.62) with the upper line of (7.61) we find

$$S_{\mathrm{gh}} = -\left\{\eta_1^{*i}\delta_{\mathrm{BRST}}g_1^i[\Phi]\right\}_1 = +\delta_{\mathrm{BRST}}\left\{\eta_1^{*i}g_1^i[\Phi]\right\}_1 - \left\{h_1^i g_1^i[\Phi]\right\}_1 \tag{7.63}$$

and thus

$$S_{\mathrm{gh}} + S_{\mathrm{NL}} = \delta_{\mathrm{BRST}}\left\{\eta_1^{*i}\left(g_1^i(\Phi) + \frac{\xi}{2}h_1^i\right)\right\}_1 \tag{7.64}$$

Thus from the above proven nil-potency of the BRST-transformation we find also

$$\delta_\Theta(S_{\mathrm{gh}} + S_{\mathrm{NL}}) = 0, \tag{7.65}$$

i.e., the invariance of the modified action (7.41).

7.2.2 Gauge independence of the S-matrix

Now we want to show that we can use the above elaborated formalism to calculate S-matrix elements in the usual way, i.e., by defining the generating functional for Green's functions by[2]

$$Z[J_k] = N\int \mathrm{D}\Xi\exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi] + \mathrm{i}\left\{J_1^k\Xi_1^k\right\}_1\right] \tag{7.66}$$

[2]From now on we understand always functional derivatives to the left (right) with respect to the auxiliary sources (the mean fields $\xi_1^k = \langle\boldsymbol{\Xi}_1^k\rangle$).

and then making use of the LSZ reduction formalism. It is important that we have introduced sources for all fields, including the Faddeev-Popov ghost fields η^{*i} and η^a and the Nakanishi-Lautrup fields h^i.

To show that our formalism leads to a physically sensible definition of an S-matrix, at least in the sense of perturbative quantum field theory, we have to show three features

1. The independence of the regularised S-matrix from the choice of gauge, i.e., its independence from the choice of a gauge fixing functional $g^i[\Phi]$.
2. The unitarity of the regularised S-matrix.
3. Existence of a gauge for which the theory is renormalisable in the sense of the BPHZ-formalism.

All points together ensure that the unitary S-matrix is renormalisable independently of the choice of gauge and the renormalised physical quantities are thus also independent from the choice of gauge.

To prove the gauge-independence of the S matrix we need a special Ward-Takahashi identity which was firstly derived by Slavnov and Taylor. Since η^{*i} are Grassmann fields we have

$$\int \mathrm{D}\Xi \eta_1^{*i} \exp[\mathrm{i}S_{\mathrm{eff}'}[\Xi] + \mathrm{i}\left\{J_{1k}\Xi_1^k\right\}_1] = 0. \tag{7.67}$$

To see this we expand the part of the exponential containing η^{*j}. Then all terms with η_1^{*i} are cancelled by the anticommuting character of this field and because for Grassmann numbers the integration rule

$$\int \mathrm{d}c_j c_k = \delta_{jk} \tag{7.68}$$

holds true by definition.

Now we substitute the BRST-transformed fields in (7.67) which does not change the value of the integral. Using the BRST-invariance of $S_{\mathrm{eff}'}$ and the BRST transformation for η^* we find

$$\int \mathrm{D}\Xi \left(h_1^i - \mathrm{i}\left\{J_2^k \eta_1^{*i} \delta_{\mathrm{BRST}} \Xi_2^k\right\}_2\right) \exp[\mathrm{i}S_{\mathrm{eff}'}[\Xi] + \mathrm{i}\left\{J_1^k \Xi_1^k\right\}_1] = 0. \tag{7.69}$$

Again we have made use of the fact that δ_{BRST} is anti-commuting and that $\sigma(J^k) = \sigma(\Xi^k)$.

Now we look what happens if we infinitesimally distort the gauge fixing functionals $g_1^i[\Phi]$ by $\Delta g_1^i[\Phi]$. We look now for the effect of this change of gauge on the generating functional (7.66). First we use (7.64) to obtain

$$\Delta S'_{\mathrm{eff}}[\Xi] = \Delta(S_{\mathrm{gh}} + S_{\mathrm{NL}}) = \delta_{BRST}\left\{\eta_1^{*i} \Delta g_1^i[\Xi]\right\}_1, \tag{7.70}$$

which leads to first order in Δg to

$$\begin{aligned} \Delta Z &= \mathrm{i}N \int \mathrm{D}\Xi \delta_{\mathrm{BRST}} \left\{\eta_1^{*i} \Delta g_1^i[\Xi]\right\}_1 \exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi] + \mathrm{i}\left\{J_2^k \Xi_2^k\right\}_2\right] = \\ &= \mathrm{i}N \int \mathrm{D}\Xi \left\{h_1^i \Delta g_1^i - \eta_1^{*i}\left\{\frac{\delta \Delta g_1^i[\xi]}{\delta \Xi_2^k} \delta_{\mathrm{BRST}} \Xi_2^k\right\}_2\right\}_1 \exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi] + \mathrm{i}\left\{J_2^k \Xi_2^k\right\}_2\right] \end{aligned} \tag{7.71}$$

The second term can be written with help of the Slavnov-Taylor identity (7.69):

$$\begin{aligned}\int \mathrm{D}\Xi \left\{\eta_1^{*i}\left\{\frac{\delta\Delta g_1^i[\xi]}{\delta\Xi_2^k}\delta_{\mathrm{BRST}}\Xi_2^k\right\}_2\right\}_1 \exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi]+\mathrm{i}\left\{J_2^k\Xi_2^k\right\}_2\right] = \\ = \Delta g_1^i\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]\int \mathrm{D}\Xi h_1^i \exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi]+\mathrm{i}\left\{J_2^k\Xi_2^k\right\}_2\right] = \\ = \mathrm{i}\Delta g_1^i\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]\int \mathrm{D}\Xi\left\{J_2^k\eta_1^{*i}\delta_{\mathrm{BRST}\Xi_2^k}\right\}_2 \exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi]+\mathrm{i}\left\{J_2^k\Xi_2^k\right\}_2\right]\end{aligned} \tag{7.72}$$

Further we note that since Δg_1^i is by assumption a local functional

$$\Delta g_1^i\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right](\mathrm{i}J_2)\exp\left[\mathrm{i}\left\{J_2^k\Xi_2^k\right\}_2\right] = \delta^{2\omega}(x_2-x_1)\frac{\partial g_1^i[\Xi]}{\partial\Xi^k}\exp\left[\mathrm{i}\left\{J_2^k\Xi_2^k\right\}_2\right] \tag{7.73}$$

Plugging (7.72) and (7.73) into (7.71) yields

$$\Delta Z = \mathrm{i}N\int \mathrm{D}\Xi\left\{\mathrm{i}J_1^k\Delta g_1^i[\Phi]\eta_1^{*i}\delta_{\mathrm{BRST}}\Xi_1^k\right\}\exp\left[\mathrm{i}S'_{\mathrm{eff}}[\Xi]+\mathrm{i}\left\{J_2^k\Xi_2^k\right\}_2\right] \tag{7.74}$$

From this we find due to the infinitesimal character of Δg:

$$Z_{g+\Delta g} = N\int \mathrm{D}\Xi\exp\left\{\mathrm{i}S'_{\mathrm{eff}}[\Xi]+\mathrm{i}\left\{J_1^k\left(\Xi_1^k+\mathrm{i}\Delta g_1^i[\Phi]\eta_1^{*i}\delta_{\mathrm{BRST}}\Xi_1^k\right)\right\}_1\right\} \tag{7.75}$$

The equivalence theorem, proved in section 4.6.2, shows that although the Green's functions are of course gauge dependent the S-matrix calculated with help of the functional (7.66) is gauge-independent.

The proof for the unitarity of the regularised S-matrix can be taken from the treatment of QED in section 6.3. There are only a few modifications due to the nonabelian group structure. So we can come directly to the renormalisability proof.

7.3 Renormalisability of nonabelian gauge theories in BFG

In this section we shall prove the renormalisability of nonabelian gauge theories for the background field gauge. This proof goes exactly along the lines of arguments used for abelian gauge theories (here given in Sect. 6.6.3).

7.3.1 The symmetry properties in the background field gauge

We look at a gauge theory which is defined by the classical Lagrangian

$$\mathscr{L}_{\mathrm{cl}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\bar\psi(\mathrm{i}\not{D}-m)\psi. \tag{7.76}$$

All the ingredients are defined as above by

$$D_\mu\psi = (\partial_\mu+\mathrm{i}gT^aA^a_\mu)\psi,\quad F^a_{\mu\nu} = \partial_\mu A^a_\nu-\partial_\nu A^a_\mu-gf^{abc}A^b_\mu A^c_\nu. \tag{7.77}$$

The Lagrangian $\mathscr{L}_{\mathrm{cl}}$ was constructed such that it is invariant under a local gauge transformation

$$\delta\psi = -\mathrm{i}g\delta\chi^aT^a\psi,\quad \delta\bar\psi = \mathrm{i}g\delta\chi^a\bar\psi T^a,\quad \delta A^a_\mu = (\mathcal{D}_\mu\delta\chi)^a = \partial_\mu\delta\chi^a+gf^{abc}A^c_\mu\delta\chi^b. \tag{7.78}$$

The application of a covariant derivative to an infinitesimal gauge transformation parameter $\delta\chi$ is meant to belong to the adjoint representation. Note that the generators for the adjoint representation are given by

$$(T_A^a)^b{}_c = -\mathrm{i}f^{abc} \tag{7.79}$$

As in the abelian case we now choose the background-field gauge for quantisation and look at the classical action shifted by a background gauge field, i.e., $S[A_\mu^a + a_\mu^a, \bar{\psi}, \psi]$, where a_μ^a is the background field and A_μ^a is the gauge field, over which we will integrate in the path integral, i.e., the "quantum gauge field".

Before we can integrate over the quantum gauge field we need to fix the gauge in the way explained above. Here the trick is to use

$$g^a[A,x] = (\tilde{\mathcal{D}}_\mu \mathcal{A}^\mu)^a := \partial_\mu A^{a\mu} + g f^{abc} a^{c\mu} A_\mu^b \tag{7.80}$$

as the gauge fixing functional. Note that $\tilde{\mathcal{D}}_\mu$ denotes the background-field covariant derivative, here acting on the quantum gauge field, corresponding to the adjoint representation. Then we have to use (7.35) to calculate the effective action which can be plugged into the path integral with the naive measure. To calculate $\mathcal{M}$ we have to apply the "true gauge transformation", keeping the action $S[A_\mu^a + a_\mu^a, \psi, \bar{\psi}]$ invariant, which are given by (7.78), but with the last equation substituted by

$$\delta A_\mu^a = \partial_\mu \delta\chi^a + g f^{abc}(A_\mu^c + a_\mu^c)\delta\chi^b = [\tilde{\mathcal{D}}_\mu \delta\chi]^a + g f^{abc} A_\mu^c \delta\chi^b. \tag{7.81}$$

Due to (7.61) we find from (7.81)

$$\int \mathrm{d}^4x \mathscr{L}_{\mathrm{gh}} = -\left\{\eta_1^{*a} \mathcal{M}_{12}^{ab} \eta_2^b\right\}_{12} = \left\{(\tilde{\mathcal{D}}_\mu \eta^*)_1^a \left[(\tilde{\mathcal{D}}_\mu \eta)_1^a + g f^{abc} A_{1\mu}^c \eta_1^b\right]\right\}_1. \tag{7.82}$$

Here we have used the fact that we can handle the $\tilde{\mathcal{D}}_\mu$, which acts on the Faddeev-Popov ghost fields in the same way as on $\mathcal{A}_\mu$, c.f. (7.80), like a partial derivative with respect to partial integrations, i.e., for instance

$$\left\{\eta_1^{*a} (\tilde{\mathcal{D}}_\mu \tilde{\mathcal{D}}^\mu)^{ab} \eta_1^b\right\}_1 = -\left\{(\tilde{\mathcal{D}}_\mu \eta^*)_1^a (\tilde{\mathcal{D}}_\mu \eta)_1^a\right\}_1. \tag{7.83}$$

The great feature of the choice (7.80) for the gauge fixing functional is that the effective background field gauge (BFG) classical action, given due to (7.35) by

$$S_{\mathrm{bfg}}[A_\mu^a, \psi, \bar{\psi}, \eta, \eta^*; a_\mu^a] = S_{\mathrm{cl}}[A_\mu^a + a_\mu^a, \psi, \bar{\psi}] - \frac{1}{2\xi}\left\{g^a[A,x] g^a[A,x]\right\}_x + S_{\mathrm{gh}}[A_\mu^a, \eta, \eta^*; a_\mu^a], \tag{7.84}$$

is invariant under the following *background field gauge transformation*:

$$\begin{aligned}
\delta\psi &= -\mathrm{i}g\delta\chi^a T^a \psi, \quad \delta\bar{\psi} = \mathrm{i}g\delta\chi^a \bar{\psi} T^a, \\
\delta a_\mu^a &= [\tilde{\mathcal{D}}_\mu \delta\chi]^a = \partial_\mu \delta\chi^a + g f^{abc} a_\mu^c \delta\chi^b, \\
\delta A_\mu^a &= g f^{abc} A_\mu^c \delta\chi^b, \\
\delta\eta^{*a} &= g f^{abc} \eta^{*c} \delta\chi^b, \\
\delta\eta^a &= g f^{abc} \eta^c \delta\chi^b.
\end{aligned} \tag{7.85}$$

While the background field transforms like a local gauge field, the quantum gauge field transforms just under the *adjoint representation*. The same holds true for the Faddeev-Popov ghost fields.

The invariance of the action (7.84) is easy to see: the classical action is invariant, since there the gauge field always appears as $A_\mu^a + a_\mu^a$, and this sum transforms due to a normal local gauge transformation. Indeed, (7.85) tells us that

$$\delta(A_\mu^a + a_\mu^a) = \partial_\mu \delta\chi^a + g f^{abc}(A_\mu^a + a_\mu^a)\delta\chi^b. \tag{7.86}$$

Since also the matter field transforms under its local fundamental representation, thus S_{cl} is invariant under (7.85). Further, since the gauge fixing functional (7.80) is the background-field covariant derivative of the quantum gauge field, transforming according to the homogeneous adjoint representation in (7.85), it transforms according to the adjoint representation. Thus $g^a g^a$ is an invariant under (7.85), since the background field appearing in $\tilde{\mathcal{D}}_\mu$ transforms as a local gauge field. The same holds true for the ghost term. If one likes to check this explicitely, one has to use the Jacobi identity (7.53).

Now the generating functional for *quantum field* expectation values is given by

$$Z[J_k; a_\mu^a] = \int \mathrm{D}\Xi^k \exp\left[\mathrm{i} S_{\mathrm{bfg}}[\Xi^k; a_\mu^a] + \mathrm{i}\left\{J_{1k}\Xi_1^k\right\}_1\right]. \tag{7.87}$$

Since the action is *invariant* under the transformations (7.85) we can show in the usual manner that Z is invariant under the contragredient transformations for the currents, i.e.,

$$\delta J_\psi = \mathrm{i} g \delta\chi^a T^a J_\psi, \quad \delta J_{\psi^*} = -\mathrm{i} g \delta\chi^a T^a J_{\psi^*}, \quad \delta J_{A,\eta^*,\eta} = g f^{abc} J^c_{A,\eta^*,\eta}\delta\chi^b \tag{7.88}$$

and the local gauge transformation for the background field, already given in (7.85). Of course, introducing the transformed background field a_μ^a and the transformed currents into (7.87) and using a substitution of the quantum fields to the transformed values given by (7.88) shows, that Z is indeed an invariant:

$$Z[J_k + \delta J_k; a_\mu^a + \delta a_\mu^a] = Z[J_k; a_\mu^a]. \tag{7.89}$$

Of course, the same holds true for

$$W[J_k; a_\mu^a] = -\mathrm{i} \ln Z[J_k; a_\mu^a]. \tag{7.90}$$

Finally we obtain the BFG generating functional

$$\Gamma_{\mathrm{bfg}}[\left\langle \Xi^k \right\rangle; a_\mu^a] = W[J_k; a_\mu^a] - \left\{J_{1k}\left\langle \Xi_1^k \right\rangle\right\}_1 \text{ with } \left\langle \Xi_1^k \right\rangle = \frac{\delta W[J_k; a_\mu^a]}{\delta J_{1k}}. \tag{7.91}$$

From the latter definition of the mean quantum fields $\langle \Xi^k \rangle$, it follows that they transform contragrediently to the external currents J_k, i.e., like to the quantum fields given by (7.85). Finally, as in Sect. 6.6 we can show that the usual effective action, generating the 1PI truncated vertex functions, is given by

$$\Gamma[a_\mu^a, \psi, \bar{\psi}, \eta, \eta^*] = \Gamma_{\mathrm{bfg}}[\left\langle \Xi^k \right\rangle; a_\mu^a]\Big|_{\langle A_\mu^a\rangle = 0}. \tag{7.92}$$

Since Γ_{bfg} is invariant under the transformation (7.85) and especially the $\langle A_\mu^a \rangle$ transforms homogeneously, $\Gamma[a_\mu^a, \psi, \bar{\psi}, \eta, \eta^*]$ is *invariant under* ***local*** *gauge transformations*, where the matter fields ψ and $\bar{\psi}$ transform under the fundamental representation, the Faddeev-Popov ghosts like matter fields under the adjoint representation, and the gauge field a_μ^a as a local gauge field.

As we shall see in a moment, the background field gauge is particularly nice to prove manifest renormalisability since Γ obeys the naive Ward-Takahashi identities (WTIs) like the abelian gauge field theories. The WTIs follow directly from the local gauge invariance of the functional (7.92). Now we shall be more specific with the source term, concerning the left- and right derivatives for fermions. We write the source term as

$$J_k \Xi^k = j_\mu^a A^{a\mu} + \bar{k}\psi_1^a + \bar{\psi}_1^a k_a + l_a^* \eta^a + \eta^{*a} l_a. \tag{7.93}$$

Now we write the invariance of (7.92) under the infinitesimal local gauge transformation (7.85) explicitly in the following form:

$$\begin{aligned} \Bigg\{ & \frac{\delta\Gamma}{\delta a_\mu^b(x)} (\tilde{\mathcal{D}}_\mu \delta\chi(x))^b - \mathrm{i}g \left(\frac{\delta\Gamma}{\delta\psi(x)}\right)_R T^b \psi_1(x) \delta\chi^b(x) + \mathrm{i}g\bar{\psi}(x) T^b \left(\frac{\delta\Gamma}{\delta\bar{\psi}(x)}\right)_L \delta\chi(x)^b \\ & + g f^{abc} \left(\frac{\delta\Gamma}{\delta\eta^a(x)}\right)_R \eta^c(x) \delta\chi^b(x) + g f^{abc} \eta^{*c}(x) \left(\frac{\delta\Gamma}{\delta\eta^{*a}(x)}\right)_L \delta\chi^b(x) \Bigg\}_x = 0. \end{aligned} \tag{7.94}$$

Since this holds true for any $\delta\chi^a(x)$, after an integration by parts, we find the local version of the WTI:

$$\begin{aligned} & \left(\tilde{\mathcal{D}}_\mu \frac{\delta\Gamma}{\delta a_\mu(x)}\right)^b + \mathrm{i}g \left[\left(\frac{\delta\Gamma}{\delta\psi(x)}\right)_R T^b \psi(x) - \bar{\psi}(x) T^b \left(\frac{\delta\Gamma}{\delta\psi(x)}\right)_R \right] \\ & \qquad - g f^{abc} \left[\left(\frac{\delta\Gamma}{\delta\eta^a(x)}\right)_R \eta^c(x) + \eta^{*c}(x) \left(\frac{\delta\Gamma}{\delta\eta^{*a}(x)}\right)_L \right] = 0. \end{aligned} \tag{7.95}$$

These are simple, QED-like WTIs for the nonabelian gauge theory! The only difference to the abelian case is that here the Faddeev Popov ghosts must be treated like a usual matter field, which transforms under the adjoint representation of the gauge group, because it is an interacting field. As we shall see, it is important to compensate for contributions of unphysical parts of the quantum gauge fields within loops, leading to vertex functions fulfilling all WTIs, derivable from (7.95) order by order in $\hbar$.

7.3.2 The BFG Feynman rules

The effective Lagrangian in the background field gauge is given by (7.84). We get practicable Feynman rules for calculating the 1PI vertex corrections by putting all pieces containing the background gauge field a_μ^a to the vertices. We can omit all contributions with only one quantum-field leg, because these vertices can never appear in 1PI diagrams.

The free propagator for the gauge field is thus given by that of the quantum field A_μ^a. Of course, only this one exists, because we fixed the gauge only for the quantum field. It is identical to the Landau-'t Hooft gauge propagator[3]:

$$\Delta_{\mu\nu}^{ab}(p) = -\delta^{ab} \frac{1}{p^2} \left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} (1-\xi) \right]. \tag{7.96}$$

The ghost propagator is that of a set of massless scalar fields, diagonal in the adjoint representation of the gauge group:

$$\tilde{\Delta}^{ab}(p) = \frac{\delta^{ab}}{p^2}. \tag{7.97}$$

[3]The appropriate $\mathrm{i}0^+$ expression in the denominators is implicitly understood.

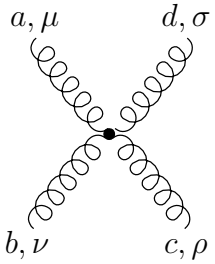
The propagator for the Dirac matter field is also the usual one:

$$G^{ab}(p) = \delta^{ab} \frac{\not{p} + m}{p^2 - m^2}. \tag{7.98}$$

The vertices must be distinguished for those legs standing for a quantum gauge field and a background field. Thus it is more convenient to give them directly in a diagrammatic form.

We start with the pure quantum gauge-field vertices. These are the same as in the usual Landau-'t Hooft gauge:

The four-gluon vertex is given by[4]

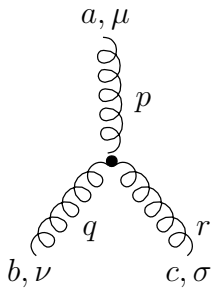


$$= -\frac{\mathrm{i}g^2}{4!}[\quad f^{xab} f^{xcd} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \tag{7.99}$$

$$+ f^{xad} f^{xcb} (g^{\nu\sigma} g^{\rho\mu} - g^{\nu\mu} g^{\rho\sigma}) \tag{7.100}$$

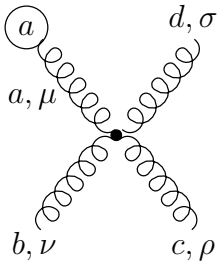
$$+ f^{xdb} f^{xca} (g^{\sigma\rho} g^{\mu\nu} - g^{\mu\sigma} g^{\nu\rho})] \tag{7.101}$$

The three-gluon vertex reads



$$= \frac{g}{3!} f^{abc} [(p_\nu - r_\nu) g_{\mu\rho} + (q_\rho - p_\rho) g_{\nu\mu} + (r_\mu - q_\mu) g_{\rho\nu}] \tag{7.102}$$

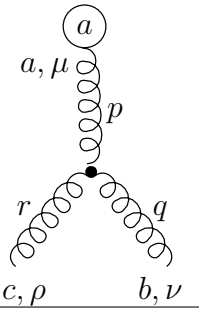
The vertices containing a background gauge field are



$$= -\frac{\mathrm{i}g^2}{3!}[\quad f^{xab} f^{xcd} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \tag{7.103}$$

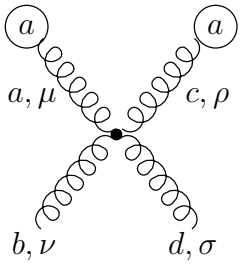
$$+ f^{xad} f^{xcb} (g^{\nu\sigma} g^{\rho\mu} - g^{\nu\mu} g^{\rho\sigma}) \tag{7.104}$$

$$+ f^{xdb} f^{xca} (g^{\sigma\rho} g^{\mu\nu} - g^{\mu\sigma} g^{\nu\rho})] \tag{7.105}$$



$$= \frac{g}{2!} f^{abc} [(p_\nu - r_\nu - q_\nu/\xi) g_{\mu\rho} + (r_\mu - q_\mu) g_{\nu\rho} + (q_\rho - p_\rho + r_\rho/\xi) g_{\mu\nu}] \tag{7.106}$$

[4] In the Feynman diagrams all gluon momenta are oriented to flow *into* the vertex.



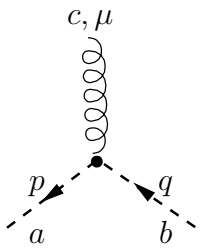
$$= -\frac{\mathrm{i}g^2}{4}[\quad f^{xab}f^{xcd}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \tag{7.107}$$

$$+f^{xad}f^{xcb}(g^{\nu\sigma}g^{\rho\mu} - g^{\nu\mu}g^{\rho\sigma}) \tag{7.108}$$

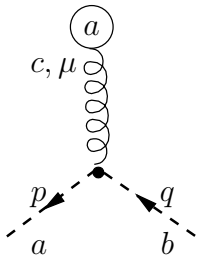
$$+f^{xdb}f^{xcd}(g^{\sigma\rho}g^{\mu\nu} - g^{\mu\sigma}g^{\nu\rho})] \tag{7.109}$$

$$+f^{xab}f^{xcd}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})/\xi \tag{7.110}$$

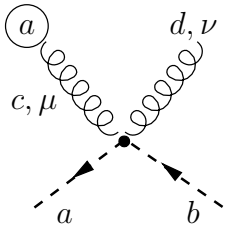
The vertices including ghosts are



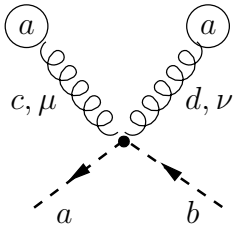
$$= gf^{abc}p_\mu \tag{7.111}$$



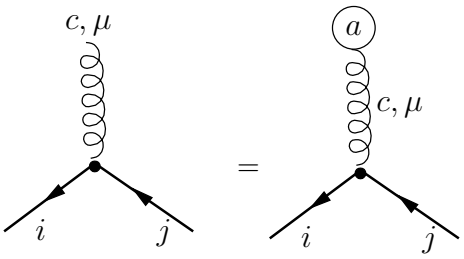
$$= gf^{abc}(p_\mu + q_\mu) \tag{7.112}$$



$$= -\mathrm{i}g^2 f^{xac}f^{xdb}g^{\mu\nu} \tag{7.113}$$



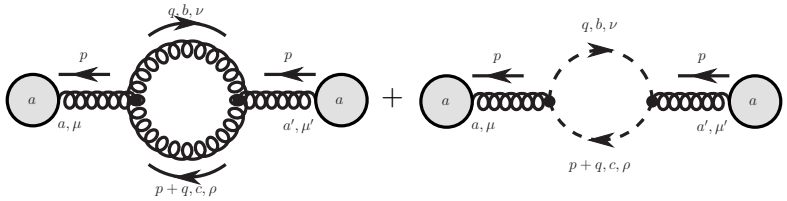
$$= -\mathrm{i}g^2(f^{xac}f^{xdb} + f^{xad}f^{xcb})g^{\mu\nu} \tag{7.114}$$



$$= -\mathrm{i}gT^c_{ij}\gamma_\mu \tag{7.115}$$

As an example we calculate the one-loop approximation for the gauge-boson self-energy. We neglect the matter fields, i.e., doing pure Yang-Mills theory. We use dimensional regularisation. In this regularisation all tadpole diagrams with massless propagators in the loop are vanishing. Thus in the gauge-field and ghost sector of the model we are left with only two diagrams:

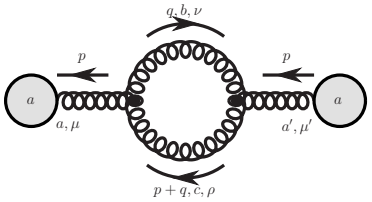
$\mathrm{i}\Pi^{aa'}_{\mu\mu'}(p) =$



(7.116)

The calculation is most easy in the Feynman gauge, $\xi = 1$. Then the first graph leads to the contribution to the gauge-field polarisation tensor

$\mathrm{i}\Pi^{(1)aa'}_{\mu\nu'}(p) =$



$$\begin{aligned} &= -\frac{g^2}{2}\mu^{2\epsilon}\int\frac{\mathrm{d}^dq}{(2\pi)^d}[-2p_\nu g_{\mu\rho} + (2q+p)_\mu g_{\nu\rho} + 2p_\rho g_{\mu\nu}] \\ &\quad\times[2p^\nu\delta^\rho_{\mu'} - (2q+p)_{\mu'}g^{\nu\rho} - 2p_\rho\delta^\nu_{\mu'}]\frac{C_A\delta^{aa'}}{p^2(p+q)^2}. \end{aligned} \tag{7.117}$$

Here we have used

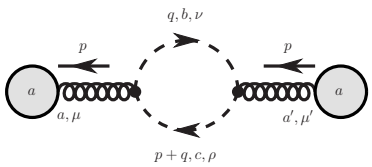
$$f^{abc}f^{a'bc} = C_A\delta^{aa'}, \quad C_A = N \quad \text{for} \quad \mathrm{SU}(N). \tag{7.118}$$

Multiplying out the numerator and, after Feynman parameterising, using the integrals in Appendix C one finds

$$\Pi^{(1)aa'}_{\mu\mu'}(p) = \frac{10C_Ag^2}{48\pi^2\epsilon}(p^2g^{\mu\mu'} - p^\mu p^{\mu'}) + \text{finite}. \tag{7.119}$$

The ghost-loop diagram in (7.116) evaluates to

$\mathrm{i}\Pi^{(2)aa'}_{\mu\mu'}(p) =$



$$= -g^2C_A\delta^{aa'}\mu^{2\epsilon}\int\frac{\mathrm{d}^dq}{(2\pi)}\frac{(2q+p)_\mu(2q+p)_\nu}{q^2(q+p)^2} \tag{7.120}$$

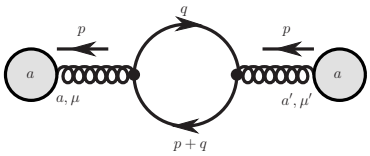
with the final result

$$\Pi^{(2)aa'}_{\mu\mu'}(p) = \frac{C_A g^2}{48\pi^2\epsilon}(p^2 g^{\mu\mu'} - p^\mu p^{\mu'}) + \text{finite}. \tag{7.121}$$

Note that (7.120) contains an additional $-$ sign from the "fermion-loop rule", which applies here for the "scalar" Faddeev-Popov ghost loop, because the corresponding fields in the path integral are Grassmann-number valued.

Last but not least there is a contribution from the fermions. For each fermion we find

$$\begin{aligned} \mathrm{i}\Pi^{(3)aa'}_{\mu\mu'}(p) = & \\ & = -\frac{g^2\delta^{aa'}}{2}\mu^{2\epsilon}\int \frac{\mathrm{d}^d}{(2\pi)^d}\,\mathrm{tr}\,\frac{\gamma^\mu(\not{q}+\not{p}+m)\gamma^{\mu'}(\not{q}+m)}{(q^2-m^2)[(q+p)^2-m^2]}, \end{aligned} \tag{7.122}$$



which leads after the standard evaluation techniques to

$$\Pi^{(3)aa'}_{\mu\mu'}(p) = -\frac{2g^2}{48\pi^2\epsilon}(p^2 g_{\mu\mu'} - p_\mu p_{\mu'}) + \text{finite}. \tag{7.123}$$

Since kinetic part of the original gauge-invariant action for the background-gauge field reads

$$S_{\text{bfg,kin}} = -\int \mathrm{d}^4x \frac{1}{2}(\partial_\mu A^a_\nu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + \int \mathrm{d}^4x \frac{1}{2} A^a_\nu(\Box A^a_\nu - \partial^\nu\partial_\mu A^{a\mu}) \tag{7.124}$$

and thus the operator between the field binary form in the momentum space translates to $-(p^2 g_{\mu\nu} - p_\mu p_\nu)$ the infinite part of the gauge-field polarisation contributions (7.119), (7.121) and (7.123) leads in the *minimal-subtraction renormalization scheme* to the back-ground field renormalization factor

$$Z_A^{(\text{MS})} = 1 + \frac{g^2\hbar}{48\pi^2\epsilon}(11C_A - 2N_\text{F}) = 1 + \frac{\alpha\hbar}{\epsilon}, \tag{7.125}$$

where we have reintroduced the factor $\hbar$ to count the number of loops, which need in the next section to calculate the β function of the renormalization group.

7.4 Asymptotic freedom

Now we use the techniques of the renormalization group as detailed for the simpler ϕ^4 theory in Subsection 5.11.2 to study the change of the renormalized coupling constant with the change of the renormalization scale μ of dimensional regularization. The great simplifying feature of the background-field gauge fixing worked out in the previous section is the *local* gauge invariance of the effective action, which leads to a very simple relation between the background-gauge field renormalization factor (7.125) and the coupling-constant renormalization factor, i.e., a *Ward identity* as in the case of the Abelian gauge symmetry as discussed for QED in Section 6.6.3. While in QED we have only a relation between the fermion-wave-function renormalization factor and the coupling-constant renormalization factor, for the non-Abelian case we have also such a relation between the gauge-field and the coupling-constant renormalization factors, because here the gauge field is self-interacting. Since the background-field-gauge action is invariant under

local gauge transformations the local part must be built with the bare field-strength tensor, which reads

$$(F^a_{\mu\nu})_0 = \sqrt{Z_A}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \sqrt{Z_A} Z_g g f^{abc} A^b_\mu A^c_\nu]. \tag{7.126}$$

This transforms correctly under the adjoint representation of the local gauge symmetry, if

$$\sqrt{Z_A} Z_g = 1 \;\Rightarrow\; Z_g = Z_A^{-1/2}. \tag{7.127}$$

Now we can evaluate the β function which determines the running of the coupling in the MS-renormalization scheme. The bare coupling is related to the renormalized coupling by

$$g_0 = Z_g \mu^\epsilon g, \tag{7.128}$$

where we have introduced the renormalization scale of dimensional regularization. Because of $d = 4 - 2\epsilon$ the factor μ^ϵ keeps g dimensionless for any space-time dimension d. Now we have

$$\mu \frac{\partial}{\partial \mu} g_0 = 0 = \mu^\epsilon Z_g \left(\epsilon g + g \mu \frac{\partial}{\partial \mu} \ln Z_g + \mu \frac{\partial}{\partial \mu} g \right). \tag{7.129}$$

Now

$$\beta = \mu \frac{\partial g}{\partial \mu} = -\epsilon g - g \mu \frac{\partial}{\partial \mu} \ln Z_g. \tag{7.130}$$

Further we have

$$\mu \frac{\partial}{\partial \mu} \ln Z_g = \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} \ln Z_g = \beta \frac{\partial}{\partial g} \ln Z_g. \tag{7.131}$$

Plugging this into (7.130) we find

$$\beta = -\epsilon g - g \beta \frac{\partial}{\partial g} \ln Z_g = -\epsilon g + \frac{g}{2} \beta \frac{\partial}{\partial g} \ln Z_A, \tag{7.132}$$

where in the last step we have used the background-field gauge Ward identity (7.127). Plugging in (7.125), solving for β we find to $\mathcal{O}(\hbar)$ in the physical limit $\epsilon \to 0$

$$\mu \frac{\partial g}{\partial \mu} = \beta = -\alpha g^3. \tag{7.133}$$

This differential equation is easily solved by separation of variables

$$g^2 = \frac{1}{\alpha \ln(\mu^2/\Lambda^2)} = \frac{1}{\frac{11 C_A - 2 N_F}{48 \pi^2} \ln\left(\frac{\mu^2}{\Lambda^2}\right)}. \tag{7.134}$$

Here, Λ is an integration constant, defining the scale, where the renormalized coupling diverges. The remarkable feature of this result is that g decreases with increasing renormalization scale, if N_F becomes not too large, so that $11 C_A - 2 N_F > 0$. This finding is known as *asymptotic freedom*, i.e., the interaction becomes small at large momentum transfers in interactions [Pol73, GW73] (Physics Nobel Prize 2004 to Gross, Wilczek and Politzer).

One should note that the running dimensionless coupling is now given as a function of the renormalization scale μ which as the dimension of energy. This feature is also known as "*dimensional transmutation*". One should note that the result (7.134) is independent of the fermion mass. Thus the renormalization-scale dependence occurs also in the limit of massless fermions, i.e.,

a scale-free theory after quantization becomes dependent on a dimensionful energy-momentum scale.

These findings find their application in the Standard Model of elementary particles in quantum chromodynamics (QCD), the theory describing the strong interaction. In this context (7.134) is given in terms of the strong fine-structure constant

$$\alpha_s = \frac{g^2}{4\pi} = \frac{1}{\frac{11C_A - 2N_F}{12\pi} \ln\left(\frac{\mu^2}{\Lambda^2}\right)}. \tag{7.135}$$

The simplification of this calculation due to the choice of the background-field gauge comes to its full advantage also at higher loop orders [Abb81, Abb82, AGS83]. For a review of the most recent results, also see [O+14].

7.5 Renormalisability of nonabelian gauge theories (BRST)

To show the renormalisability of nonabelian gauge theories in more general renormalisable gauges, which must be chosen such that the gauge fixing term does not spoil the condition of superficial renormalisability, i.e., it must be of mass dimension 4 or less, we have to use the Ward-Takahashi identities due to BRST invariance. This is much more complicated than the simple argument of the previous chapter, but the renormalisability of nonabelian gauge field theories is so important that we show it twice.

7.5.1 The Ward-Takahashi identities

For sake of completeness we also admit scalar matter fields ϕ^i in addition to the fermionic matter fields treated so far. Of course the ϕ^i build a representation of the gauge group. Then the most general gauge fixing functions are given by

$$g^a = \partial^\mu A^a_\mu + f^a_i \phi^i + \frac{\xi}{2} h^a + c^a, \tag{7.136}$$

where c^a are arbitrary external fields not contained in the set of matter, gauge and ghost fields Ξ^k. The various parts of the Lagrangian now read

$$\begin{aligned} \mathscr{L}_{\text{gauge}} &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mathscr{L}_{\text{matter}}(\phi, \psi, D_\mu \phi, \not{D}\psi), \\ \mathscr{L}_{\text{gf}} &= h^a(\partial^\mu A^a_\mu + f^a_i \phi^i + w^a) + \frac{\xi}{2} h^a h^a, \\ \mathscr{L}_{\text{gh}} &= \eta^{*a}[-\partial^\mu (\mathrm{D}_\mu)\eta^a - \mathrm{i} g f^a_i (T^b)^i{}_j \phi^j \eta^b]. \end{aligned} \tag{7.137}$$

The matter part is given by substituting ∂_μ in the free Lagrangian by D_μ, i.e.,

$$\begin{aligned} (\mathrm{D}_\mu \phi)^j &= \partial_\mu \phi^j + \mathrm{i} g A^a_\mu (T^a)^j{}_i \phi^i \text{ and analoguous for the } \psi^j, \\ \mathscr{L}_{\text{matter}} &= (\mathrm{D}_\mu \phi)^\dagger (\mathrm{D}^\mu \phi) - \frac{m_1^2}{2} \phi^\dagger \phi - \frac{\lambda}{8} (\phi^\dagger \phi)^2 + \bar{\psi}(\mathrm{i}\not{D} - m_2)\psi. \end{aligned} \tag{7.138}$$

Note that the T^a for the scalar and the fermionic fields may build different representations of the gauge Lie algebra and that for the bosonic fields we need a gauge-invariant quartic self-interaction due to renormalisability. Without restriction of generality we have assumed a semi-simple nonabelian gauge group so that there occurs only one gauge coupling g.

In the following we use a short-hand notation for the fields, namely $(\varphi_I) = (A^a_\mu, \phi^i, \psi^j)$ and $f_I = (\partial_\mu, f^a_i, 0)$.

Now we introduce not only external sources for the fields themselves but also for the composite operators $\delta_{\mathrm{BRST}}\Xi$ which are given by (7.42). The source term Lagrangian thus reads

$$\mathscr{L}_{\mathrm{source}} = J\Xi + K\delta_{\mathrm{BRST}}\Xi. \tag{7.139}$$

Note that each J has the same Grassmann sign as its corresponding field while K has the opposite sign: $\sigma(J_\xi) = \sigma(\xi)$, $\sigma(K_\xi) = -\sigma(\xi)$. Since $\delta_{\mathrm{BRST}} h^a \equiv 0$ we set $K^a_h \equiv 0$. Then the new generating functional reads

$$Z[J,K] = \int \mathrm{D}\Xi \exp(\mathrm{i}S_{\mathrm{gauge}} + \mathrm{i}S_{\mathrm{gf}} + \mathrm{i}S_{\mathrm{gh}} + \mathrm{i}S_{\mathrm{source}}) \tag{7.140}$$

As we have shown in the last section all $\delta_{\mathrm{BRST}}\Xi$ are BRST-invariant. Thus when doing a substitution $\Xi \to \Xi + \theta\delta_{\mathrm{BRST}}\Xi$ only the $J\Xi$-term gives a contribution. Using the invariance of the path-integral measure $\mathrm{D}\Xi$ under this transformation we obtain

$$\left\{ \sum_{\xi\in\Xi} \sigma(\xi) J_\xi \frac{\delta Z}{\delta K_{\xi 1}} \right\}_1 = 0. \tag{7.141}$$

Here and in the following all derivatives with respect to Grassmann fields or currents are to be read as left derivatives. Writing $W[J,K] = -\mathrm{i}\ln(Z[J,K])$ and using the Legendre transformation

$$\Gamma[\Xi,K] = W[J,K] - \{J_1\Xi_1\}_1 \text{ with } \xi_1 = \frac{\delta W}{\delta J_{\xi 1}}. \tag{7.142}$$

It follows immediately

$$\frac{\delta\Gamma}{\delta\xi_1} = -\sigma(\xi)J_\xi, \quad \frac{\delta W}{\delta K_\xi} = \frac{\delta\Gamma}{\delta K_\xi}. \tag{7.143}$$

Since in (7.141) only first derivatives appear, which is the main advantage to introduce the additional sources K, (7.141) holds true for W. Using (7.143) one finds the generalised *Ward-Takahashi*-identities

$$\sum_\xi \left\{ \frac{\delta\Gamma}{\delta\xi_1}\frac{\delta\Gamma}{\delta K_{\xi 1}} \right\}_1 = 0. \tag{7.144}$$

A little bit more explicit this reads

$$\left\{ \frac{\delta\Gamma}{\delta\varphi_I}\frac{\delta\Gamma}{\delta K_{\varphi 1 I}} + \frac{\delta\Gamma}{\delta\eta^a_1}\frac{\delta\Gamma}{\delta K^a_{\eta 1}} + \frac{\delta\Gamma}{\delta\eta^{*a}_1}h^a_1 \right\}_1 = 0 \tag{7.145}$$

Now we use the fact that the Lagrangian (7.137) is quadratic in h^a and linear in η^* which leads to a almost trivial dependence of the quantum action Γ on these fields. Using the invariance of the path-integral measure with respect general shifts of the fields h and η^* we get taking advantage of (7.143)

$$\frac{\delta\Gamma}{\delta h^a} = \partial_\mu A^{a\mu} + f^a_i\varphi^i + w^a + \xi h^a \tag{7.146}$$

$$\frac{\delta\Gamma}{\delta\eta^*} = \partial_\mu \frac{\delta\Gamma}{\delta K^a_{A\mu}} + f^a_i \frac{\delta\Gamma}{\delta K_i}. \tag{7.147}$$

From (7.146) we find

$$\Gamma[\varphi,\eta^*,\eta,h;K]=\tilde{\Gamma}[\varphi,\eta^*,\eta;K]+\left\{h_1^a\partial_\mu A_1^{a\mu}+h_1^a(f_i^a\varphi_1^i+w^a)+\frac{\xi}{2}h_1^a h_1^a\right\}_1, \tag{7.148}$$

where $\tilde{\Gamma}$ is independent of h. It is clear that (7.147) holds true for $\tilde{\Gamma}$. Now we define

$$\tilde{K}_{A\mu}^a=K_{A\mu}^a-\eta^{*a}\partial_\mu,\quad \tilde{K}_i=K_i-\eta^{*a}f_i^a. \tag{7.149}$$

Then we find, using the chain rule for functional derivatives

$$\left(\frac{\delta\tilde{\Gamma}}{\delta\eta^{*a}}\right)_{\tilde{K}=\text{const}}=\left\{\frac{\delta K_{I1'}}{\delta\eta_1^{*a}}\frac{\delta\tilde{\Gamma}}{\delta K_{I1'}}\right\}_1+\frac{\delta\tilde{\Gamma}}{\delta\eta_1^{*a}}=-\partial_\mu\frac{\delta\tilde{\Gamma}}{\delta K_{A\mu 1}^a}-f_i^a\frac{\delta\tilde{\Gamma}}{\delta K_{i1}}+\frac{\delta\tilde{\Gamma}}{\delta\eta_1^{*a}}=0. \tag{7.150}$$

From now on we shall regard $\tilde{\Gamma}$ as a function of the fields and $\tilde{K}$. For sake of brevity we write $\tilde{K}_\eta=K_\eta$ Then (7.150) tells us that $\tilde{\Gamma}$ is a functional of the fields φ_I, η and the external sources $\tilde{K}_I$ and $\tilde{K}_\eta$, i.e., that it is independent of η^*. (7.145) thus reads for $\tilde{\Gamma}$ in these new variables

$$\left\{\frac{\delta\tilde{\Gamma}}{\delta\varphi_{I1}}\frac{\delta\tilde{\Gamma}}{\delta\tilde{K}_{I1}}+\frac{\delta\tilde{\Gamma}}{\delta\eta_1^a}\frac{\delta\tilde{\Gamma}}{\delta\tilde{K}_1^a}\right\}_1=0. \tag{7.151}$$

Now we define generalised "coordinates" and "momenta" due to

$$Q_A=(A_\mu^a,\phi^i,K_{\psi^i},K_\eta^a),\quad P_A=(\tilde{K}_A^{a\mu},\tilde{K}_{\phi i},\psi_i,\eta^a) \tag{7.152}$$

and the *antibracket*

$$F*G=\left\{\frac{\delta F}{\delta Q_{A1}}\frac{\delta G}{\delta P_{A1}}+(-1)^{[F]}\frac{\delta F}{\delta P_{A1}}\frac{\delta G}{\delta Q_{A1}}\right\}_1, \tag{7.153}$$

where F and G can be arbitrary Grassmann even or Grassmann odd functionals of the fields and sources. Since the Q_A are even and P_A are odd we have

$$G*F=-(-1)^{\overline{[F]}\,\overline{[G]}}F*G \tag{7.154}$$

$$F*(G*H)+(-1)^{\overline{[F]}(\overline{[G]}+\overline{[H]})}G*(H*F)+(-1)^{\overline{[H]}(\overline{[F]}+\overline{[G]})}H*(F*G)=0. \tag{7.155}$$

where we define $\overline{[F]}$ as 0 for odd and 1 for even functionals F.
From (7.154) we read off that we can write (7.151) as

$$\tilde{\Gamma}*\tilde{\Gamma}=0. \tag{7.156}$$

7.6 Anomalies

So far we have investigated only symmetries which are not only valid in the classical field theory but also "survive" quantisation, i.e., the quantum effective action is also symmetric under the corresponding symmetry transformation on the fields, implying that the Noether theorem holds true also in the quantum field theory, which implies the Ward-Takahashi identities (WTIs) for both global and local symmetries. One formal way to derive this result is to use the path-integral analysis of the generating functionals for Green's functions $Z[J]$, connected Green's functions $W[J]$ and proper vertex functions $\Gamma[\varphi]$, where J are some external sources and φ the mean fields

at presence of these sources. To derive the WTIs, including Noether's theorem for the expectation values of the conserved charge corresponding to the Noether current associated with the theory, we used the invariance of the path-integral measure under the typical symmetry transformations acting on the fields. We did not pay too much attention to this point, but as it will become clear in this Chapter, there are cases, where the path-integral measure is not invariant under a transformation that is a symmetry of the classical field theory. Then the WTIs will contain additional terms, leading to the conclusion that the Noether theorem does not hold, i.e., the corresponding Noether charge is not conserved in this case and the symmetry is explictly broken [Fuj80, Fuj79].

Originally anomalies have been discovered in perturbation theory [Adl69, BJ69] in the context of a puzzle concerning the decay of neutral pions to two photons $\pi^0 \to \gamma\gamma$ in the context of the otherwise very successful hypothesis of the "partially conserved axial vector current" (PCAC), according to which the pions are approximately the Goldstone bosons of a chiral symmetry. The prediction of the decay width $\Gamma_{\pi^0\to\gamma\gamma}$ from a naive QED model, not making use of the PCAC hypothesis, lead to a very good agreement with the experimental value [Ste49]. Including the PCAC hypothesis, however leads to a decay width that is several orders of magnitude lower. The resolution was the discovery of anomalies by Adler, Bell and Jackiw [Adl69, BJ69].

In this case the anomaly of the chiral $\mathrm{U}_A(1)$ symmetry, which we will explain in a moment, comes to the rescue of a physically important approximate symmetry, namely the chiral $\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}$ symmetry of the strong interactions, which is very important to build models for hadrons at low energies ("chiral perturbation theory"). Nowadays it is well understood as an "accidental symmetry" of Quantum Chromodynamics, which describes the interaction between quarks and gluons. The approximate chiral symmetry is based on the fact that the light quarks (up, down and to lesser accuracy also strange) have masses of the order of some MeV, which is much smaller than typical hadronic scales of order 1 GeV.

On the other hand, we have seen the vital importance of local gauge invariance for the consistency of models with vector bosons. Thus, it is important to make sure that local gauge symmetries are not broken by anomalies.

7.6.1 Motivation: The decay of neutral pions to photons ($\pi^0 \to \gamma\gamma$)

We start with the "naive" evaluation of the decay width of the neutral pion decaying to two photons. The most simple model is QED with protons as charged particles with the Lagrangian

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\not{D} - M)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \mathrm{D}_\mu = \partial_\mu + \mathrm{i}eA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.157}$$

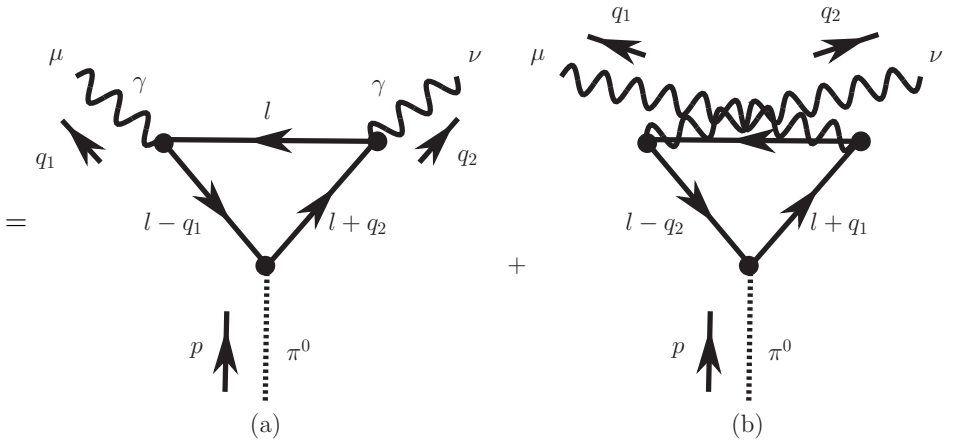
This is an effective model, treating the proton as an elementary particle, which can be considered a valid approximation for not too large reaction energies.

To guess a reliable pion coupling, it is sufficient to know that a pion is a pseudoscalar particle, and thus the neutral pion, also treated as an elementary particle can be represented by a real pseudoscalar field π. Then the most simple coupling, which is even renormalisable, of this field to a Dirac spinor is of the Yukawa type to the pseudoscalar bilinear form,

$$\mathscr{L}_{\pi N} = \mathrm{i}g\pi\bar{\psi}\gamma^5\psi. \tag{7.158}$$

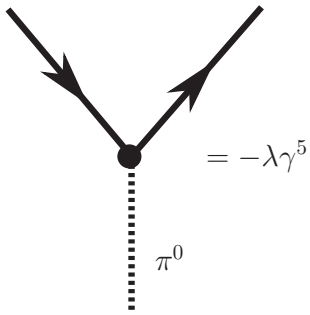
To lowest order the matrix element for the decay $\pi \to \gamma\gamma$ is given by the two triangle diagrams

$$\mathrm{i}\mathcal{M}_{\pi\gamma\gamma} = -(-\lambda)(-\mathrm{i}e)^2 \epsilon_\mu(q_1,\lambda_1)\epsilon_\nu(q_2,\lambda_2)\mathrm{i}\mathcal{M}^{\mu\nu}$$



(7.159)

The Feynman rule for the interaction (7.158) of the pion with protons obviously reads



(7.160)

For diagram (a) in (7.159) we can write

$$\begin{aligned}\mathrm{i}\mathcal{M}_a^{\mu\nu} &= \mathrm{i}^3 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma^\mu(\not{l}+M)\gamma^\nu(\not{l}+\not{q}_2+M)\gamma^5(\not{l}-\not{q}_q+M)}{(l^2-M^2)[(l+q_2)^2-M^2][(l-q_1)^2-M^2]}\right] \\ &= 4M q_{1\rho} q_{2\sigma}\epsilon^{\mu\nu\rho\sigma}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{(l-M^2)[(l+q_2)^2-M^2][(l-q_1)^2-M^2]}.\end{aligned} \tag{7.161}$$

The other diagram $\mathrm{i}\mathcal{M}_b^{\mu\nu}$ follows by interchanging μ and ν as well as q_1 and q_2, which finally leads to the same result as (7.161). The integral is obviously convergent, and we can use the standard formulae in Appendix C.2 directly for $d = 2\omega = 4$, leading to

$$\mathcal{M}^{\mu\nu} = \frac{\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma}}{2\pi^2 M^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{M^2 - m_\pi^2 xy} \underset{m_\pi^2 \ll M^2}{\cong} \frac{\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma}}{4\pi^2 M^2}. \tag{7.162}$$

Since obviously $q_{1\mu}\mathcal{M}^{\mu\nu} = q_{2\nu}\mathcal{M}^{\mu\nu} = 0$, as dictated by the WTIs of electromagnetic gauge invariance and also expected from the conservation of the electric current, we can apply the usual rules for the polarisation vectors $\epsilon_\mu(q,\lambda)$ of the electromagnetic field, i.e., $\sum_{\lambda=1,2}\epsilon_\mu^*(q,\lambda)\epsilon_\nu(q,\lambda) \to -g^{\mu\nu}$ in (7.159). Thus the squared matrix element, summed over the polarisation states of the final photons, finally results in

$$\overline{\mathcal{M}^2_{\pi^0\to\gamma\gamma}} = \frac{g^2 e^4 m_\pi^4}{32\pi^4 M^2} = \frac{g^2 \alpha_{\mathrm{em}}^2 m_\pi^4}{2\pi^2 M^2}. \tag{7.163}$$

The total decay width, including the appropriate symmetry factor 1/2 for the identical photons in the final state is finally given by

$$\Gamma = \frac{1}{2}\int \mathrm{d}\Omega \frac{\mathcal{M}^2}{32\pi^2}\frac{|\vec{q}_1|}{m_\pi^2} = \frac{g^2\alpha_{\mathrm{em}}^2 m_\pi^3}{64\pi^3 M^2}. \tag{7.164}$$

Here we have used that for the decay of the pion at rest $|\vec{q}_1| = E_1 = m_\pi/2$.

On the other hand the pion is the Goldstone boson of the spontaneously broken chiral $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ symmetry of the strong interaction. Keeping the hadronic effective-field theory version of this symmetry, the SU(2) groups are related to the isospin symmetry, which is the unbroken subgroup of the chiral group which. The most simple realization of such a model the proton and neutron form an isospin doublet.

The most simple realization of chiral symmetry is the socalled linear σ model, which we have already discussed in Sect. 5.10. For our purpose here, we formulate it in the following way. In the mesonic sector we introduce a scalar field $\sigma \in \mathbb{R}$ in addition to the three pseudo-scalar fields $\vec{\pi} \in \mathbb{R}^3$, representing the charged pions, $\pi^\pm = (\pi_1 \pm \mathrm{i}\pi_2)/\sqrt{2}$, and the neutral pion $\pi^0 = \pi_3$. The chiral group is represented by SO(4) matrices acting on the fields $\underline{\phi} = (\vec{\pi}, \sigma)^\mathrm{T} \in \mathbb{R}^4$. To make contact with the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ representation, we introduce the $\mathbb{C}^{2\times 2}$-valued vector $\hat{\Sigma} = (\mathbb{1}_2, \hat{\vec{\sigma}})$, where $\vec{\sigma}$ are the usual Pauli matrices. Then we write $\hat{\Phi} = \sigma\mathbb{1} + \mathrm{i}\hat{\vec{\sigma}}\cdot\vec{\pi}$. Because of $\mathrm{Tr}(\hat{\Sigma}_j\hat{\Sigma}_k) = 2\delta_{jk}$ the Lagrangian (5.235) of the mesonic part of the linear σ model can be written in the form

$$\mathscr{L}_\mathrm{mes} = \frac{1}{4}\mathrm{Tr}[(\partial_\mu\hat{\Phi}^\dagger)(\partial^\mu\hat{\Phi}) + \tilde{m}^2\hat{\Phi}^\dagger\hat{\Phi}] - \frac{\lambda}{32}[\mathrm{Tr}\,\hat{\Phi}^\dagger\hat{\Phi}]^2. \tag{7.165}$$

The SO(4) transformations can now be represented with two independent transformations $g_\mathrm{L}, g_\mathrm{R} \in \mathrm{SU}(2)$,

$$\hat{\Phi} \to g_\mathrm{L}\hat{\Phi}g_\mathrm{R}^\dagger. \tag{7.166}$$

The minimum of the potential is given by $\sigma^2 = v^2 = 2\tilde{m}^2/\lambda =: F_\pi^2$, cf. (5.238), and to derive Feynman rules, one has to shift the σ field by this vacuum expectation value, as detailed in Sect. 5.10. For our purposes it is more convenient to stay with the manifestly chirally symmetric form.

To describe the protons and neutrons we introduce the isospin doublet $\Psi = (\psi_\mathrm{p}, \psi_\mathrm{n})$ with the Dirac-spinor fields for proton and neutron. We can split any Dirac spinor in two pieces, which are the eigenvectors of the γ^5 matrix with eigenvalues ± 1 (defining the quantum number chirality), called the right- and left-handed parts,

$$\Psi_{R/L} = \frac{1}{2}(\mathbb{1} \pm \gamma_5)\Psi, \tag{7.167}$$

where the unit matrix and γ_5 act in spinor space. Now we have

$$\bar{\Psi}\gamma^\mu\Psi = \bar{\Psi}_\mathrm{L}\gamma^\mu\Psi_\mathrm{L} + \bar{\Psi}_\mathrm{R}\gamma^\mu\Psi_\mathrm{R}, \quad \bar{\Psi}\Psi = \bar{\Psi}_\mathrm{L}\Psi_\mathrm{R} + \bar{\Psi}_\mathrm{R}\Psi_\mathrm{L}, \tag{7.168}$$

which tells us that the kinetic term $\bar{\Psi}\mathrm{i}\partial\!\!\!/\Psi$ is invariant under the chiral transformation

$$\Psi_\mathrm{L} \to g_\mathrm{L}\Psi_L, \quad \Psi_R \to g_R\Psi_R \tag{7.169}$$

with independent SU(2) matrices g_L and g_R acting in flavour space. The mass term $\bar{\Psi}M\Psi$, breaks this symmetry explicitly. This term stays symmetric only under the SU(2) transformation with $g_L = g_R = g_V$. However, we can provide a mass to the nuclons via the spontaneous breaking

of the symmetry, i.e., the presence of the vacuum expectation value F_Π of the σ field by writing the Lagrangian for the nucleons as

$$\mathscr{L}_{\text{nucl}} = \frac{M}{F_\pi}(\bar{\psi}_L \hat{\Phi} \psi_R + \bar{\psi}_R \hat{\Phi}^\dagger \psi_L). \tag{7.170}$$

Indeed, because of (7.168), by shifting the σ field via $\sigma = F_\pi + \tilde{\sigma}$, we find that (7.170) reads

$$\mathscr{L}_{\text{nucl}} = \frac{M}{F_\pi}[\bar{\Psi}(F_\pi + \tilde{\sigma})\Psi + \mathrm{i}\bar{\Psi}\vec{\pi}\cdot\hat{\vec{\sigma}}\gamma_5\Psi]. \tag{7.171}$$

For our purposes, we can introduce the explicit symmetry breaking of chiral symmetry, which is necessary to provide the pions with their small but finite masses by simply introducing an additional mass term

$$\mathscr{L}_{\chi\text{SB}} = -\frac{m_\pi^2}{2}\vec{\pi}^2. \tag{7.172}$$

This is invariant under the unbroken "vector part" of the chiral symmetry, i.e., for $g_L = g_R = g_V$, which just translates into an SO(3) rotation of the pion fields $\vec{\pi}$, but it obviously violates full chiral symmetry since for $g_L \neq g_R$ the σ and pion fields get mixed under the chiral transformation. We should, however, note that this say to introduce the explicit breaking of chiral symmetry is not the way it is realized in nature, as a more detailed analysis of the chiral-symmetry pattern of the underlying fundamental theory of the strong interaction, QCD, reveals. For details, see the excellent review on chiral symmetry [Koc97].

The only point of this detour into chiral symmetry is that we can relate the coupling constant g in (7.158) to measurable physical parameters, the pion-decay constant F_π and the nucleon mass M,

$$g = \frac{M}{F_\pi}. \tag{7.173}$$

This follows from the identification of π in (7.158) with the component π_3 of the pion field (which represents the neutral pion) in (7.171) The constant F_π can (at least in principle) measured by comparing to the pion-nucleon cross section as following from the chiral model. The value of the pion-decay constant is $F_\pi \simeq 92.21\,\text{MeV}$. With $\alpha_{\text{em}} \simeq 1/137.036$, $M \sim 938\,\text{MeV}$ and $m_{\pi_0} \simeq 135\,\text{MeV}$ we find from (7.164) with (7.173)

$$\Gamma_{\pi^0\to\gamma\gamma} = 7.76\,\text{eV}, \tag{7.174}$$

which is in excellent agreement with the measured value of $(7.73 \pm 0.16)\,\text{eV}$ [O$^+$14].

7.6.2 The PCAC hypothesis and trouble with $\pi_0 \to \gamma\gamma$

To understand the problem concerning the decay $\pi_0 \to \gamma\gamma$ in connection with modern hadron theory, we consider Quantum Chromodynamics (QCD) with two flavors, i.e., up- and down-quarks. Writing $\psi = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}$, where each of the Dirac fields also carries an $\text{SU}(3)_{\text{color}}$ index, to which the non-Abelian gauge-boson field (gluon field) couples,

$$\mathscr{L}_{\text{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \overline{\psi}\mathrm{i}\not{\mathcal{D}}\psi - \overline{\psi}\hat{M}\psi, \tag{7.175}$$

where we write G^a_μ for the gluon gauge fields and $G^a_{\mu\nu}$ for the non-Abelian field-strength tensor, and $\mathcal{D}_\mu = \partial_\mu + \mathrm{i}gG^a_\mu T^a$ for the covariant derivative which are defined as in Sect. 7.1. The matrix

$\hat{M} = \mathrm{diag}(m_u, m_d)$ acts in flavor space. The light-quark masses have values of a few MeV and thus are small compared to typical masses of hadrons of $\mathcal{O}(1\ \mathrm{GeV})$ and thus can be considered as a perturbation. In the limit of massless quarks, the QCD Lagrangian shows a chiral symmetry with the symmetry group $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R} \times \mathrm{U(1)_V} \times \mathrm{U(1)_A}$, i.e., the left- and right-handed parts of the quark fields can be independently transformed as defined in (7.159) and via

$$\begin{aligned} \mathrm{U(1)_V}: &\quad \psi' = \exp(\mathrm{i}\alpha)\psi, \quad \overline{\psi}' = \overline{\psi}\exp(-\mathrm{i}\alpha) \\ \mathrm{U(1)_A}: &\quad \psi' = \exp(\mathrm{i}\beta\gamma_5)\psi, \quad \overline{\psi}' = \overline{\psi}\exp(+\mathrm{i}\beta\gamma_5). \end{aligned} \tag{7.176}$$

As we shall see, the crucial difference between the vector and axialvector transformation is that in the latter case both ψ and $\overline{\psi}$ have to transform with the same sign in the exponential. This is because only then $\overline{\psi}\partial\!\!\!/\psi$ is invariant under the $\mathrm{U(1)_A}$ transformation, because the γ_5 matrix *anti*commutes with $\partial\!\!\!/ = \partial_\mu\gamma^\mu$. So the $\mathrm{U(1)_A}$ is *not* a unitary but a "pseudo-unitary" transformation in Dirac-spinor space.

Since the Lagrangian (7.175) in the limit $\hat{M} \to 0$ is invariant under these transformations the corresponding Noether currents (the isoscalar-vector and isoscalar-vector currents) are conserved in this limit. With the classical equations of motion indeed one finds that the currents

$$j^\mu_{\mathrm{V}} = \overline{\psi}\gamma^\mu\psi, \quad j^\mu_{\mathrm{A}} = \overline{\psi}\gamma^\mu\gamma_5\psi \tag{7.177}$$

fulfill

$$\partial_\mu j^\mu_{\mathrm{V}} = 0, \quad \partial_\mu j^\mu_{\mathrm{A}} = 2\overline{\psi}\gamma^\mu\gamma_5\hat{M}\psi \quad \text{(classical)}. \tag{7.178}$$

The latter relation is known as the "partial conservation of the axial-vector current" (PCAC).

The chiral $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ also leads to the isovector Noether vector and axialvector currents

$$J^{a\mu}_{\mathrm{V}} = \overline{\psi}\gamma^\mu\tau^a\psi, \quad J^{a\mu}_{\mathrm{A}} = \overline{\psi}\gamma^\mu\tau^a\gamma_5\psi \tag{7.179}$$

and a PCAC relation for the isovector-axialvector current

$$\partial_\mu J^{a\mu}_{\mathrm{A}} = 2\overline{\psi}\gamma^\mu\gamma_5\tau^a\hat{M}\psi. \tag{7.180}$$

Here $\tau^a = \sigma^a/2$ denote the SU(2) matrices acting in flavor space. The strong force, however, is attractive in the quark-antiquark channel, and thus the chiral symmetry is spontaneously broken due to the formation of a quark condensate, i.e.,

$$\left\langle \Omega \left| \overline{\psi}\psi \right| \Omega \right\rangle \neq 0. \tag{7.181}$$

The symmetry of the vacuum thus reduces from $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ to $\mathrm{SU(2)_V}$ even in the chiral limit. According to the Goldstone theorem, there must be three pseudoscalar massless Nambu-Goldstone bosons in the chiral limit, which are identified with the pions. Their finite mass is due to the small explicit symmetry breaking due to the light-quark masses.

Now the modern way to define the pion decay constant is by the PCAC hypothesis

$$\left\langle \Omega \left| J^{a\mu}_{\mathrm{A}}(x) \right| \pi^b(\vec{p}) \right\rangle = \mathrm{i}\exp(\mathrm{i}p\cdot x)p^\mu F_\pi\delta^{ab}, \quad p^0 = \sqrt{m_\pi^2 + \vec{p}^2}. \tag{7.182}$$

Then via the PCAC relation (7.181) we have

$$p_\mu \left\langle \Omega \left| J^{a\mu}_{\mathrm{A}}(x) \right| \pi^b(\vec{p}) \right\rangle = \mathrm{i}\exp(\mathrm{i}p\cdot x)m_\pi^2 F_\pi\delta^{ab}. \tag{7.183}$$

So we can identify the (real) pion fields to be $\pi^a = \frac{1}{F_\pi}\partial_\mu J_{\mathrm{A}}^{a\mu}(x)$.
But then according to our evaluation of the triangle diagrams (7.159) we'd have to assume

$$\left\langle \overline{\psi}_q \gamma_5 \psi_q \right\rangle_{A^\mu} = \frac{\mathrm{i}e^2}{32\pi^2}\frac{1}{m_q}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{7.184}$$

and on the other hand through the PCAC relation for the isoscalar-axialvector current (7.178)

$$\partial_\mu \left\langle \overline{\psi}_q \gamma^\mu \gamma_5 \psi_q \right\rangle = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{7.185}$$

which is independent of the quark mass, but this in turn violates the naive PCAC relation (7.178) in the chiral limit, because then the Noether theorem seems to imply that the isoscalar axial current is conserved for both massless QCD and QED.

The only conclusion can be that the naive PCAC relation is violated in the quantized theory. As we will see in the next two sections this is indeed the case. First we will look at the violation of the PCAC relation in the perturbative approach with massless quarks. There we have to calculate the matrix element $\left\langle T_c j_{\mathrm{A}}^\mu j_{\mathrm{V}}^\rho j_{\mathrm{V}}^\sigma \right\rangle$, because through contraction with the pion momentum p_μ we should get the pion field due to the PCAC hypothesis (7.183) for the matrix element $\mathcal{M}_{\pi^0\to\gamma\gamma}$, and the vector currents are the electric currents (up to the appropriate quark charges $q_q e$ with $q_u = 2/3$ and $q_d = -1/3$). The above expectation value in leading-order perturbation theory is linearly divergent, which brings up the question of how to regularize this divergence properly.

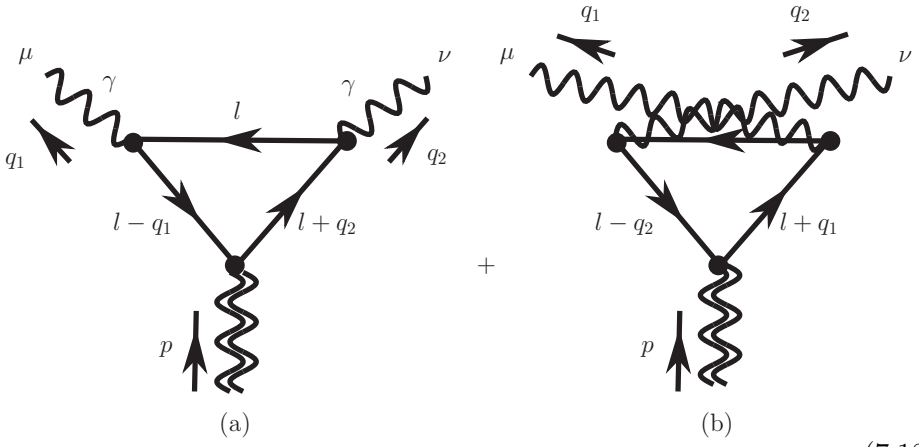
7.6.3 The $\mathrm{U}(1)_{\mathrm{A}}$ anomaly: perturbative analysis

If now one tries to verify the conservation of the Ward-Takahashi identities for the above written triple-current correlation function it turns out that it is impossible to find a regularization prescription which keeps both the isoscalar vector and axialvector transformations a symmetry. E.g., in dimensional regularization it is not clear, how to define the generically four-dimensional γ_5 matrix in space-time dimensions different from 4.

Trying to use BPHZ renormalization, i.e., subtracting the linear divergences yields to a result which depends on the choice of the loop momenta in the two triangle diagrams. As we shall see, contracting with the momenta referring to the different currents in the correlation function, leads to a finite result without any subtractions, which however depends on the choice of the loop momenta. Now it is clear that the conservation of the vector current must be fulfilled at any cost, because it is the current the electromagnetic field couples to, and violating current conservation of a local gauge symmetry leads to an explicit breaking of the gauge invariance and renders the theory physically inconsistent, and the usual interpretation of the S-matrix elements breaks down, because the unphysical photon-field degrees of freedom would become interacting. So the right choice of the loop momenta and ofthe renormalization prescription must be such as to preserve the vector-current conservation Ward-Takahashi identity, and this necessarily violates the naive Ward-Takashi identity for the axialvector current, i.e., within this description the $\mathrm{U}(1)_{\mathrm{V}}$ symmetry stays intact, as forced by the local electromagnetic gauge invariance, while the $\mathrm{U}(1)_{\mathrm{A}}$ symmetry is broken in the quantum theory in precisely the way given by (7.185), which renders the PCAC description of the pion as a (pseudo-) Nambu-Goldstone boson of the spontaneously broken (approximate) chiral $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ compatible with the successful evaluation of the $\pi^0 \to \gamma\gamma$ decay rate.

To see, how (7.185) comes about we calculate the perturbative one-loop contribution to the triple-current correlation function

$\mathrm{i}\mathcal{M}^{\alpha\mu\nu}_{AVV}(q_1,q_2) =$



(7.186)

The first diagram translates into

$$\mathrm{i}\mathcal{M}^{\alpha\mu\nu}_{aAVV}(q_1,q_2) = -\mathrm{i}^3 \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{\gamma^\mu \not{l}\gamma^\nu(\not{l}+\not{q}_2)\gamma^\alpha\gamma_5(\not{l}-\not{q}_1)}{l^2(l+q_2)^2(l-q_1)^2}\right], \tag{7.187}$$

where we work in the chiral limit, i.e., with massless fermions. The second diagram is then given by interchanging $q_1 \leftrightarrow q_2$ and $\mu \leftrightarrow \nu$. The integral (7.187) is linearly divergent, but we do not regularize it, because it is not clear how to do this in a way which keeps both the $\mathrm{U(1)_V}$ and $\mathrm{U(1)_A}$ symmetries intact. We are also not interested in (7.187) itself, but in the contractions $p_\alpha\mathcal{M}^{\alpha\mu\nu}_{AVV}$ and $q_{1\mu}\mathcal{M}^{\alpha\mu\nu}_{AVV}$ which probe the conservation of the axialvector and the vector current within the correlation function. As we shall see, this leads to finite results, which however are not unique, because they depend on the arbitrary choice of the loop momentum, of which (7.187) realizes only one special simple choice.

To evaluate $p_\alpha\mathcal{M}^{\alpha\mu\nu}_{AVV}$ we use the identity

$$\not{p}\gamma_5 = (\not{q}_1+\not{q}_2)\gamma_5 = (\not{q}_1-\not{l})\gamma_5 + (\not{l}+\not{q}_2)\gamma_5 = \gamma_5(\not{l}-\not{q}_1) - (\not{l}+\not{q}_2)\gamma_5 \tag{7.188}$$

in the numerator, leading to

$$p_\alpha\mathcal{M}^{\alpha\mu\nu}_{AVV} = \int \frac{\mathrm{d}^4 l}{(2\pi)4}\,\mathrm{Tr}\left[\frac{\gamma^\mu \not{l}\gamma^\nu(\not{l}+\not{q}_2)\gamma_5}{l^2(l+q_2)^2} + \frac{\gamma^\mu \not{l}\gamma^\nu\gamma^5(\not{l}-\not{q}_1)}{l^2(l-q_2)^2} + \mathrm{exch}\right], \tag{7.189}$$

where "exch" means to add the same expression with the interchanges $q_1 \leftrightarrow q_2$ and $\mu \leftrightarrow \nu$. Performing the Dirac traces, one finds

$$p_\alpha\mathcal{M}^{\alpha\mu\nu}_{AVV} = 4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left[\frac{l_\rho q_{1\sigma}}{l^2(l-q_1)^2} - \frac{l_\rho q_{1\sigma}}{l^2(l+q_1)^2} + \frac{l_\rho q_{2\sigma}}{l^2(l+q_2)^2} - \frac{l_\rho q_{2\sigma}}{l^2(l-q_2)^2}\right]. \tag{7.190}$$

Now both integrals are of the form

$$I(a) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4}[f_\rho(l+a) - f_\rho(l)], \tag{7.191}$$

where for the first two terms we have

$$f_\rho(l) = 4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\frac{l_\rho q_{1\sigma}}{l^2(l+q_1)^2}, \quad a = q_1. \tag{7.192}$$

Now, if each of the two terms in the integral (7.191) were convergent, we could shift the integration variable to $l' = l + a$, leading to $I(a) = 0$. However, this part of the integral is linearly divergent. To analyse (7.191) further we first Wick rotate to Euclidean momentum space,

$$I(a) = \mathrm{i}\int \frac{\mathrm{d}^4\tilde{l}}{(2\pi)^4}[f_\rho(\tilde{l}+\tilde{a}) - f_\rho(\tilde{l})]. \tag{7.193}$$

For a rational function of l linear divergence of the integral over f_ρ means it must have the asymptotic behavior

$$f(\tilde{l}) \underset{\tilde{l}^2\to\infty}{\cong} A\frac{l^\rho}{l^4}. \tag{7.194}$$

Now we can regularize the integral by integrating over a sphere with radius Λ and afterwards letting the regulator $\Lambda \to \infty$. Now Taylor expansion of the integrand leads to

$$I(a) = \mathrm{i}\int_{|\tilde{l}|<\Lambda} \frac{\mathrm{d}^4\tilde{l}}{(2\pi)^4}\left[\tilde{a}_\mu \frac{\partial}{\partial\tilde{l}_\mu} f_\rho(l) + \frac{\tilde{a}_\mu\tilde{a}_\nu}{2}\frac{\partial^2}{\partial\tilde{l}_\mu\partial\tilde{l}_\nu} f_\rho(l)\ldots\right]. \tag{7.195}$$

Now we can apply the four-dimensional Gauss's integral theorem, leading to

$$I(a) = \frac{\mathrm{i}\tilde{a}_\mu}{(2\pi)^4}\int_{\Omega^{(4)}} \frac{\mathrm{d}^3\Omega^{(4)}}{(2\pi)^3}\left[\tilde{l}_\mu\Lambda^2 f_\rho(\tilde{l}) + \ldots\right] \underset{\Lambda\to\infty}{\cong} \frac{\mathrm{i}\tilde{a}_\mu}{(2\pi)^4}\int_{\Omega^{(4)}} \mathrm{d}^3\Omega^{(4)} A\frac{\tilde{l}_\mu\tilde{l}_\rho}{\tilde{\Lambda}^2}. \tag{7.196}$$

Here we could omit the higher-order terms in the Taylor expansion, because according to the asymptotic behavior (7.194) these terms go to 0 with a power larger than $1/\tilde{\Lambda}^3$ for $\Lambda \to \infty$ and thus do not contribute in this limit. For symmetry reasons we can substitute $\tilde{l}_\mu\tilde{l}_\rho \to \Lambda^2\delta_{\mu\rho}/4$. Since the three-dimensional unit sphere in four-dimensional space has a surface of $\Omega^{(4)} = 2\pi^2$, we finally get

$$I(a) = \frac{\mathrm{i}Aa_\rho}{32\pi^2}, \tag{7.197}$$

where we have Wick rotated $\tilde{a}_\rho$ back to Minkowski space a_ρ.

Since in (7.190) the shift in the integrands of the first to terms is q_2 and for the third and forth two terms q_1, using (7.197) leads indeed to 0.

This is, of course not the desired result, because it would not solve the discrepancy between the otherwise successful PCAC relation between the axial vector current and the pion and the $\pi^0 \to \gamma\gamma$ decay rate. Of course, formally there is nothing wrong with the above calculation, but we have overlooked a subtle hidden problem with writing down the Feynman rules for the triangle diagrams (7.186), because we have arbitrarily chosen the loop momentum in an apparently convenient way. Now the contraction with p^α in (7.189) to check the axial Ward-Takahashi identity for the three-current correlation function on the other hand lead to integrals of type (7.193) with the single integrands linearly divergent. The final result is proportional to the shift a_μ in the arguments of the two linearly divergent integral.

Thus we have to investigate, how the triangle-loop diagram changes if we choose an arbitrary loop momentum. Now, if we choose for the first diagram in (7.186) the loop momentum as

$$l^\mu = k^\mu + q_3^\mu \quad \text{with} \quad q_3 = \lambda_1 q_1 + \lambda_2 q_2 \tag{7.198}$$

with arbitrary real coefficients λ_1 and λ_2, then because we have to keep the Bose symmetry in the exchange of the external photon lines, we must use

$$l^\mu = \bar{k}^\mu + \bar{q}_3^\mu \quad \text{with} \quad \bar{q}_3 = \lambda_1 q_2 + \lambda_2 q_1. \tag{7.199}$$

A reanalysis of the corresponding Feynman integral expressions in the very same way as performed above with the special choice of the loop momentum, finally leads to

$$p_\alpha \mathcal{M}_{AVV}^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma}(\lambda_1 - \lambda_2), \tag{7.200}$$

which is already of the right shape for the expected "anomalous contribution" to the PCAC $\pi^0\gamma\gamma$ amplitude, cf. (7.185), but the coefficient depends on the arbitrary choice of the loop momentum, i.e., the coefficients λ_1 and λ_2.

This means that we have to find a criterion for choosing the loop momentum, and the solution is that the vector current must be conserved at any cost, because otherwise the gauge symmetry of QED would be lost. The corresponding Ward-Takahashi identity,

$$q_{1\mu} \mathcal{M}_{AVV}^{\alpha\mu\nu} \stackrel{!}{=} 0 \tag{7.201}$$

thus must hold in any case. Of course, the convergence properties are the same as for the expression (7.200) and thus it also depends on the choice of the loop momentum, i.e., the parameters λ_1 and λ_2. A completely analogous lengthy calculation as for the contraction (7.200) for the WTI of the axial current leads to

$$q_{1\mu} \mathcal{M}_{AVV}^{\alpha\mu\nu} = -\frac{1}{4\pi^2}\epsilon^{\alpha\nu\rho\sigma} q_{1\rho} q_{2\sigma}(\lambda_1 - \lambda_2 - 1) \stackrel{!}{=} 0 \;\rightarrow\; \lambda_1 - \lambda_2 \stackrel{!}{=} 1. \tag{7.202}$$

Plugging this result into (7.200) we finally get

$$p_\alpha \mathcal{M}_{AVV}^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma}. \tag{7.203}$$

This is indeed in accordance with (7.185), when dressing the photon legs with the corresponding amplitudes, using $\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma} \epsilon_\mu(q_1)\epsilon_\nu(q_2) \sim \epsilon^{\mu\nu\rho\sigma} F_{\rho\mu} F_{\sigma\nu}/4 = -\epsilon^{\rho\mu\sigma\nu} F_{\rho\mu} F_{\sigma\nu}/4$.

In conclusion we note that the renormalization condition in the case of a linearly divergent matrix element of vector and axial-vector currents (current-correlation functions) is undefined without the further constraint that the vector current must be conserved due to electromagnetic gauge symmetry. Without this constraint one can get a renormalization prescription leading to the "normal" Ward-Takahashi identity (WTI) corresponding to the conservation of either current or an arbitrary linear combination of the vector and axial-vector currents. On the other hand the constraint that the WTI for the vector current must be fulfilled, leads to a unique determination of the anomalous Ward-Takashi identity that corresponds to the relation (7.185), which makes the apparent discrepancy between the PCAC conjecture about the pion as a Goldstone boson of the spontaneously broken approximate global $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ symmetry in the light-quark sector of QCD compatible with the observed $\pi^0 \to \gamma\gamma$ decay rate.

7.6.4 The $\mathrm{U}(1)_\mathrm{A}$ anomaly: path-integral analysis

With the hindsight of the somewhat cumbersome perturbative analysis, we investigate the derivation of the Ward-Takahashi identities (WTIs) for the vector and axial-vector current in Quantum

Electrodynamics with massless fermions. The Lagrangian

$$\mathscr{L} = \overline{\psi}(\mathrm{i}\partial\!\!\!/ + q A\!\!\!/)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{7.204}$$

is invariant under the *global* $\mathrm{U}(1)_\mathrm{V}$ transformation

$$\psi \to \psi' = \exp(-\mathrm{i}\alpha)\psi, \quad \overline{\psi} \to \overline{\psi}' = \overline{\psi}\exp(+\mathrm{i}\alpha), \quad A_\mu \to A'_\mu = A_\mu \tag{7.205}$$

as well as under the *global* axial-vector or chiral transformation $\mathrm{U}(1)_\mathrm{A}$

$$\psi \to \psi' = \exp(-\mathrm{i}\beta\gamma_5)\psi, \quad \overline{\psi} \to \overline{\psi}' = \overline{\psi}\exp(-\mathrm{i}\beta\gamma_5), \quad A_\mu \to A'_\mu = A_\mu. \tag{7.206}$$

The corresponding Noether currents of these symmetries are

$$j_\mu = \overline{\psi}\gamma^\mu\psi \quad \text{and} \quad j_{5\mu} = \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{7.207}$$

On the classical level both corresponding charges are conserved due to the equations of motion, i.e., the continuity equations $\partial_\mu j^\mu = \partial_\mu j_5^\mu = 0$. From the perturbative analysis in the previous section we know, however, that the naive WTIs for expectation values of time ordered products of the form $\langle T_c j_{5\mu}(x)\mathcal{O}(x_1,\ldots,x_n)\rangle$, where $\mathcal{O}$ is a gauge-invariant operator (e.g., products of vector or axial-vector currents) is not fulfilled, provided we insist on the validity of the WTI's for the vector currents, which is a necessary condition for the invariance of the theory under the local gauge symmetry $\mathrm{U}(1)_\mathrm{em}$, which must not be broken in any way.

We start the analysis with the Feynman-Kac path-integral formula for the expectation value of time-ordered operators,

$$\langle T_c\mathcal{O}(x_1,\ldots,x_n)\rangle = \frac{1}{Z(0)}\int \mathrm{D}\psi \int \mathrm{D}\overline{\psi} \int \mathrm{D}A \exp(\mathrm{i}S)\mathcal{O}(x_1,\ldots,x_n) \tag{7.208}$$

with the generating functional of disconnected Green's functions with vanishing external sources

$$Z(0) = \int \mathrm{D}\psi \int \mathrm{D}\overline{\psi} \int \mathrm{D}A \exp(\mathrm{i}S). \tag{7.209}$$

We first consider the $\mathrm{U}(1)_\mathrm{V}$ transformations (7.205) but now making $\alpha = \delta\alpha(x)$ "infinitesimal" but dependent on the space-time variables, but not treating the symmetry as the local gauge symmetry but as for the global transformations letting $A'_\mu = A_\mu$. Now we can write (7.208) in terms of the "new" fields

$$\langle T_c\mathcal{O}(x_1,\ldots,x_n)\rangle = \frac{1}{Z(0)}\int \mathrm{D}\psi' \int \mathrm{D}\overline{\psi}' \int \mathrm{D}A \exp(\mathrm{i}S')\mathcal{O}(x_1,\ldots,x_n). \tag{7.210}$$

Now we substitute back the "old" fields. For the action we have

$$S' = S + \mathrm{i}\left\{\overline{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\delta\alpha\right\} \tag{7.211}$$

and for the path-integral measure

$$\mathrm{D}\psi'\mathrm{D}\overline{\psi}' = \mathrm{D}\psi\mathrm{Det}^{-1}\exp(-\mathrm{i}\delta\alpha)\mathrm{D}\overline{\psi}\mathrm{Det}^{-1}\exp(+\mathrm{i}\delta\alpha). \tag{7.212}$$

Of course, the functional determinant (which occur in the denominator, because the Dirac-spinor fields are represented by Grassmann numbers in the path integral) is highly divergent and has to

be regularized, but no matter how we regularize it, the contributions for the transformation of $\mathrm{D}\psi$ and $\mathrm{D}\overline{\psi}$ cancel each other, and thus the path-integral measure for the fermions is invariant undert the $\mathrm{U(1)_V}$ transformations. Thus we have

$$\begin{aligned}\langle T_c \mathcal{O}(x_1,\ldots,x_n)\rangle = \frac{1}{Z[0]}\int \mathrm{D}\psi \int \mathrm{D}\overline{\psi} \int \mathrm{D}A \exp(\mathrm{i}S)\left[1-\mathrm{i}\left\{j^\mu(x)\partial_\mu \delta\alpha(x)\right\}_x\right] \\ \times \mathcal{O}(x_1,\ldots,x_n).\end{aligned} \tag{7.213}$$

With (7.208) we conclude that

$$\begin{aligned}&\frac{1}{Z[0]}\int \mathrm{D}\psi \int \mathrm{D}\overline{\psi} \int \mathrm{D}A \exp(\mathrm{i}S)\left\{j^\mu(x)\partial_\mu(x)\delta\alpha(x)\right\}_x \mathcal{O}(x_1,\ldots,x_n) \\ &\quad = \left\langle T_c \int \mathrm{d}^4x j^\mu(x)\partial_\mu \delta\alpha(x)\mathcal{O}(x_1,\ldots,x_n)\right\rangle = 0.\end{aligned} \tag{7.214}$$

Now we can integrate by parts and take the integral out of the expectation value, which leads to

$$\int \mathrm{d}^4x \delta\alpha(x)\partial_\mu \left\langle T_c j^\mu(x)\mathcal{O}(x_1,\ldots,x_n)\right\rangle = 0. \tag{7.215}$$

Since this holds for any $\delta\alpha(x)$ we have the "normal" Ward-Takahashi identity for the Noether current of the global $\mathrm{U(1)_V}$ symmetry

$$\langle T_c j^\mu(x)\mathcal{O}(x_1,\ldots,x_n)\rangle = 0. \tag{7.216}$$

This is also a necessary condition for the local electromagnetic gauge invariance, which must be preserved in any case, because otherwise the physical interpretability of the formalism in terms of the S matrix breaks down.

Now we consider the chiral $\mathrm{U(1)_A}$ symmetry, which is not really unitary because of the occurance of the γ_5 matrix in (7.206). This is the deeper reason for the occurance of the anomaly, because now the path-integral measure is not invariant as in the case of the $\mathrm{U(1)_V}$ symmetry:

$$\mathrm{D}\psi'\mathrm{D}\overline{\psi}' = \mathrm{D}\psi\mathrm{D}\overline{\psi}\mathrm{Det}^{-2}\exp[-2\mathrm{i}\delta\beta(x)\gamma_5]. \tag{7.217}$$

The functional determinant is quite undefined, because using the general identity

$$\ln\det\hat{\Delta} = \mathrm{Tr}\ln\hat{\Delta} \tag{7.218}$$

in our case leads to

$$\mathrm{Tr}\,\delta\beta(x)\gamma_5 = \mathrm{tr}\,\gamma_5 \int \mathrm{d}^4x \delta\beta(x), \tag{7.219}$$

which is undefined, because for a general function $\delta\beta(x)$ the integral does not converge, while the Dirac trace $\mathrm{tr}\,\gamma_5 = 0$.

This forces us to regularize the functional trace (7.219), but we are forced to do that in a way that preserves the local $\mathrm{U(1)_{em}}$ gauge invariance. A possible choice for a regularized version of (7.219) is

$$T_{\mathrm{reg}}(\Lambda) = \mathrm{Tr}[\delta\beta(x)\gamma_5 \exp(-\not{D}^2/\Lambda^2)]. \tag{7.220}$$

Now we have

$$\begin{aligned}\not{D}^2\psi &= \gamma^\mu\gamma^\nu(\partial_\mu - \mathrm{i}qA_\mu)(\partial_\mu - \mathrm{i}qA_\mu)\\ &= \frac{1}{2}\left[\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right](\partial_\mu - \mathrm{i}qA_\mu)(\partial_\mu - \mathrm{i}qA_\mu)\\ &= (g^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu})(\partial_\mu - \mathrm{i}qA_\mu)(\partial_\mu - \mathrm{i}qA_\mu)\\ &= \mathrm{D}_\mu\mathrm{D}^\mu\psi - \frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}.\end{aligned} \tag{7.221}$$

This implies

$$T_{\mathrm{reg}}(\Lambda) = \int \mathrm{d}^4x\,\mathrm{tr}\left\langle x\left|\beta(x)\gamma_5\exp\left(-\mathrm{D}_\mu\mathrm{D}^\mu - \frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)\right|x\right\rangle. \tag{7.222}$$

Here we have introduced a fictitious one-particle Hilbert space with $|x\rangle$ as generalized time-position eigenvectors ($x\in\mathbb{R}^4$). Inserting a complete set of momentum eigenvectors gives

$$\begin{aligned}T_{\mathrm{reg}}(\Lambda) &= \int \mathrm{d}^4x\beta(x)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\langle x\left|\beta(x)\gamma_5\exp\left(-\frac{\mathrm{D}_\mu\mathrm{D}^\mu}{\Lambda^2} + \frac{q}{2\Lambda^2}\sigma^{\mu\nu}F_{\mu\nu}\right)\right|p\right\rangle\langle p\,|\,x\,\rangle\\ &= \int \mathrm{d}^4x\beta(x)\underbrace{\mathrm{tr}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\langle x\left|\gamma_5\exp\left(-\frac{\mathrm{D}_\mu\mathrm{D}^\mu}{\Lambda^2} + \frac{q}{2\Lambda^2}\sigma^{\mu\nu}F_{\mu\nu}\right)\right|p\right\rangle\exp(-\mathrm{i}p\cdot x)}_{A(x)}.\end{aligned} \tag{7.223}$$

Now using the position representation of the operators in the exponential we get

$$A(x) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\exp(-\mathrm{i}p\cdot x)\,\mathrm{tr}\left[\gamma_5\exp\left(-\frac{\mathrm{D}_\mu\mathrm{D}^\mu}{\Lambda^2} + \frac{q}{2\Lambda^2}\sigma^{\mu\nu}F_{\mu\nu}\right)\exp(\mathrm{i}p\cdot x).\right] \tag{7.224}$$

Now for any function χ we have

$$\mathrm{D}_\mu[\chi(x)\exp(\mathrm{i}p\cdot x)] = \exp(\mathrm{i}p\cdot x)(\mathrm{i}p_\mu + \mathrm{D}_\mu)\chi(x). \tag{7.225}$$

This implies

$$A(x) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_5\exp\left(\frac{(k-\mathrm{i}D)^2}{\Lambda^2} + \frac{q}{2\Lambda^2}\sigma^{\mu\nu}F_{\mu\nu}\right)1\right]. \tag{7.226}$$

Now

$$\begin{aligned}&\mathrm{tr}\,\gamma_5 = \mathrm{tr}\,\gamma_5\gamma^\mu = \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu = \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho = 0,\\ &\mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}.\end{aligned} \tag{7.227}$$

Also all traces with a γ_5 multiplied with a product of an odd number of γ^μ matrices vanish. Thus expanding the operator exponential and applying the Dirac trace gives

$$\begin{aligned}A(x) &= \int\frac{\mathrm{d}^4p}{(2\pi)}\left[\frac{q^2}{8\Lambda^4}\,\mathrm{tr}(\gamma_5\sigma^{\mu\nu}\sigma^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma}) + \mathcal{O}(\Lambda^{-6})\right]\exp\left(\frac{k^2}{\Lambda^2}\right)\\ &= \frac{q^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x).\end{aligned} \tag{7.228}$$

Finally, plugging this result into the path-integral expression analogous to (7.213) we find the anomalous WTI for the $\mathrm{U(1)_A}$-Noether current

$$\partial_\mu\left\langle T_c j_5^\mu(x)\mathcal{O}(x_1,\ldots,x_n)\right\rangle = -\frac{q^2}{16\pi^2}\left\langle T_c\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x)\mathcal{O}(x_1,\ldots,x_n)\right\rangle, \tag{7.229}$$

which is precisely (7.185) again, as it should be.

In the path-integral approach the anomaly is thus due to the non-invariance of the path-integral measure under $\mathrm{U(1)_A}$ transformations [Fuj80, Fuj79].

Appendix A

Variational Calculus and Functional Methods

A.1 The Fundamental Lemma of Variational Calculus

One of the most fundamental mathematical concepts is that of functions. Functions can be seen as unique maps from one set to another, i.e. if A and B are maps then a function $f : A \to B$ means that there is a rule which maps an element of A uniquely to an element in B.

In our context functions are less general defined as maps from a given set of numbers or vectors to the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ or to vectors of such numbers.

Now in modern mathematics one investigates general structures, i.e., sets with operations defined on and mappings between such structures obeying some special features connected to these structures. This point of view makes it possible to look on the various concepts of classical mathematics from a generalised point of view.

On the other hand a physicist needs objects which can be handled for practical purposes but one should have an idea about the structures behind. So in this appendix we like to define some mathematical concepts which are used in the main part. Again we do not care too much about rigour but the ideas should become clear. Here we want to give a sense to functionals and variational calculus from a modern point of view which makes it much easier to handle the physical calculations. The ideas are quite old and brilliantly given in Courant's and Hilbert's books about the mathematical methods in physics. We put the things only in a somewhat more modern language.

A wide range of physics (if not all fundamentals of our modern theories) can be formulated in terms of so called *variational calculus*. The first ingredient is a *functional* which is a map from a certain space of functions to real numbers. In our case the space of functions used can be thought to be the set of all smooth functions with one or more real or complex arguments. The integrals used further on are Lebesgue integrals but it does not matter if one thinks in the more intuitive concept of Riemann integrals because we will not be very strict in our proofs. One should only get an idea how to handle the stuff in practice.

The rest of the chapter will deal with variational calculus used to formulate classical point mechanics. It may be enough to look on one-dimensional problems to keep the story short. The more general case of more dimensions is a straightforward generalisation. In canonical mechanics we define the action functional S which is a map from the smooth functions $x : (t_1, t_2) \to \mathbb{R}$ to

the real numbers given by

$$S[x] = \int_{t_1}^{t_2} \mathrm{d}t L(x, \dot{x}, t). \tag{A.1}$$

Thereby (t_1, t_2) is an open interval of real numbers and $L : \mathbb{R}^3 \to \mathbb{R}$ a smooth function. The Hamilton principle tells us that the particle will move along that special path which minimises the action and fulfils the boundary condition

$$x(t_1) = x_1,\ x(t_2) = x_2. \tag{A.2}$$

So in other words we have to find a minimum of the functional S. This is what is called a *variational problem*, namely to find a (local) extremum of a functional, i.e. a function which minimises or maximises the functional.

To solve this problem let us remember the analogous situation in the case of ordinary functions $f : \mathbb{R} \to \mathbb{R}$. In the context of basic calculus one learns that a necessary condition for $x_0 \in \mathbb{R}$ to be a (local) minimum of f is that the first derivative of f vanishes at the point x_0.

Now we want to apply these ideas to our variational problem. For this purpose we introduce an ordinary function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(\epsilon) = S[\xi + \epsilon\eta] = \int_{t_1}^{t_2} \mathrm{d}t L(\xi + \epsilon\eta, \dot{\xi} + \epsilon\dot{\eta}, t), \tag{A.3}$$

where we have chosen ξ to be a solution of the variational problem and $\eta : (t_1, t_2) \to \mathbb{R}$ an arbitrary smooth function. Now a necessary condition for ξ to be a minimal point of the functional S in the function space of smooth functions is that

$$\forall \eta : (t_1, t_2) \to \mathbb{R} \text{ smooth with } \eta(t_1) = \eta(t_2) = 0 : \left.\frac{dF}{d\epsilon}\right|_{\epsilon=0} = 0. \tag{A.4}$$

If ξ fulfils this necessary condition we call it a *stationary point* of the functional. Now an application of the chain rule of differential calculus and the fact that we can interchange integration with respect to t with differentiation with respect to ϵ leads to the condition

$$\left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\eta = 0 \tag{A.5}$$

to hold for all smooth functions η vanishing at the end points of the interval. In the last step we have partially integrated and used the vanishing of η at the end points of the integral's interval.

Now we claim that this can only be fulfilled for all "allowed" functions η if the expression in the parenthesis vanishes identically, i.e., if

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0. \tag{A.6}$$

This is indeed the well known *Euler-Lagrange equation* for our problem. Now we like to prove our claim. In other words we have to show the following *fundamental lemma of variational calculus*:

If for all smooth functions η vanishing at the end points and a given continuous function f the condition

$$\int_{t_1}^{t_2} \mathrm{d}t f(t)\eta(t) = 0 \tag{A.7}$$

is valid then f is necessarily vanishing identically. It should be emphasised that this is by no means trivial because the restriction of η to be smooth is a very strong restriction because there are not very much smooth functions compared to all continuous functions.

Nevertheless the proof is not too difficult. At first we see that the condition (A.7) is a global one because it contains the integral over the whole interval (t_1, t_2). The only freedom we have is to *vary* the functions η. Since we want to conclude a local property about f, namely to vanish at any given point $\tau \in (t_1, t_2)$, we have to localise the integral by choosing the η cleverly. This will be constructed with help of the function

$$M(x) = \left\{ \begin{array}{c} \exp\left(-\frac{1}{1-x^2}\right) \text{ for } |x| < 1 \\ 0 \text{ for } |x| \geq 1 \end{array} \right\}. \tag{A.8}$$

One can show easily (as an exercise in calculus) that this function is a smooth function defined on $\mathbb{R}$ which is nowhere < 0.

Now let ϵ be an arbitrary positive real number. Then the function

$$M_{\epsilon t}(\tau) = M\left(\frac{\tau - t}{\epsilon}\right) \tag{A.9}$$

is a smooth nowhere negative function which is vanishing outside the interval $(t - \epsilon, t + \epsilon)$. Now assume that f is not 0 everywhere. Suppose without loss of generality that $f(t) > 0$ for a $t \in (t_1, t_2)$. Because f is continuous we know that there exists a neighbourhood of t contained in (t_1, t_2) where f is greater than 0. Now we can chose this neighbourhood to be of the form $(t-\epsilon, t+\epsilon) \subset (t_1, t_2)$ with a sufficiently small positive real number ϵ. Now it is allowed to chose $\eta(\tau) = M_{\epsilon t}(\tau)$ because the function $M_{\epsilon t}$ is a smooth function vanishing outside the neighbourhood, which is lying completely in the interval (t_1, t_2), so that η is vanishing at the endpoints because this has been shown for $M_{\epsilon t}$ before. Now since f is positive in $(t - \epsilon, t + \epsilon)$ and η is nowhere negative and vanishing outside $(t - \epsilon, t + \epsilon)$ we can write

$$\int_{t_1}^{t_2} \mathrm{d}\tau f(\tau) M_{\epsilon t}(\tau) = \int_{t-\epsilon}^{t+\epsilon} \mathrm{d}\tau f(\tau) M_{\epsilon t}(\tau) > 0. \tag{A.10}$$

But this is a contradiction against the condition (A.7) assumed to hold true for all allowed functions η. From this we conclude that f has to vanish everywhere in (t_1, t_2). This finishes our proof that a necessary condition for ξ to be a (local) extremal point of the action functional S is, that it fulfils the Euler Lagrange equations.

A.2 Functional Derivatives

With help of the fundamental lemma of variational calculus we can easily define the functional derivative. Let A be a functional mapping a function space to the set of real or complex numbers. Then the functional derivative at the point x in the function space is the distribution $\delta A / \delta x(t)$ which is uniquely defined by

$$\forall \eta : (t_1, t_2) \to \mathbb{R}(\mathbb{C}) \text{ smooth } : \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon} A[x + \epsilon\eta]\right|_{\epsilon=0} = \int_{t_1}^{t_2} \mathrm{d}t \frac{\delta A}{\delta x(t)} \eta(t). \tag{A.11}$$

The uniqueness of this expression follows directly from the fundamental lemma of variational calculus. With help of this definition we can express the fact that ξ is a stationary point of the action functional by

$$\left.\frac{\delta S}{\delta x}\right|_{x=\xi} = 0. \tag{A.12}$$

From the arguments above one sees immediately that this is nothing else than a short hand notation for the validity of the Euler-Lagrange equation of the variational problem. In physics this determines the equations of motion. The great advantage of the formulation of physical problems in terms of variational principles lies in the fact that the Euler-Lagrange equations do not change their form under general transformations of the variables. This is what is called a *covariant description* of physical laws with respect of a given class of variable transformations. In the case of the description with help of variational principles this class consists of all coordinate transformations which is a rather big one. This makes it possible to prove such far reaching consequences as Noether's theorem and its quantum analogue[1] .

With our definition of functional derivatives it is easy to find that the most rules known from calculus with ordinary functions can be taken over to the case of functionals. For example let us look on the product rule:

$$\frac{\delta}{\delta x(t)}(AB) = \frac{\delta A}{\delta x(t)}B + A\frac{\delta B}{\delta x(t)}. \tag{A.13}$$

This is proven with help of the definition (A.11) by applying the product rule to the function $F(\epsilon) = A[x + \epsilon\eta]B[x + \epsilon\eta]$.

One important example of a functional is

$$A[x] = x(\tau), \tag{A.14}$$

where τ is a fixed point $\tau \in (t_1, t_2)$. Now we calculate the functional derivative of this functional:

$$\frac{d}{d\epsilon}A[x + \epsilon\eta] = \eta(\tau) = \int_{t_1}^{t_2} \mathrm{d}t\eta(t)\delta(t - \tau). \tag{A.15}$$

From the definition of the functional derivative we obtain

$$\frac{\delta A}{\delta x(t)} = \delta(t - \tau). \tag{A.16}$$

In a sloppy form we identify this special kind of functionals with the function x itself. Because of this the functional derivative of a function is defined to be the δ distribution:

$$\frac{\delta x(\tau)}{\delta x(t)} = \delta(t - \tau). \tag{A.17}$$

In the same manner we can define the higher functional derivatives of the functional A. If

$$\begin{aligned} &\forall\eta : (t_1, t_2) \to \mathbb{R}(\mathbb{C}) \text{ smooth:} \\ &\left.\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}A[x + \epsilon\eta]\right|_{\epsilon=0} = \int_{t_1}^{t_2} \mathrm{d}t_1 \ldots \mathrm{d}t_n \frac{\delta^n A}{\delta x(t_1)\ldots\delta x(t_n)}\eta(t_1)\ldots\eta(t_n) \end{aligned} \tag{A.18}$$

[1] Of course one must be sure that the quantisation procedure itself does not destroy the symmetry which holds true for the classical theory. The phenomenon that the quantum theory of a classical system is not symmetric under a symmetry transformation of the classical theory is called *anomaly* and we shall come back in the main text to this important topic.

then we call $\delta^n A/(\delta x(t_1)\dots\delta x(t_n))$ the nth functional derivative of A.

It is also straight forward to define functional derivatives of functionals depending on two or more functions. For instance in the case of a functional depending on to functions we define $\delta S[x,y]/\delta x(t)$ as the functional for which

$$\forall \eta_1,\eta_2 : (t_1,t_2)\to\mathbb{R}(\mathbb{C}) \text{ smooth: } \left.\frac{\partial}{\partial\epsilon_1}A[x+\epsilon_1\eta_1, y+\epsilon_2\eta_2]\right|_{\epsilon_1,\epsilon_2=0} = \int_{t_1}^{t_2}\mathrm{d}t\frac{\delta A}{\delta x(t)}\eta_1(t). \quad \text{(A.19)}$$

As a last example we look on the chain rule for functional derivatives. If $x[y,t]$ is a function which is a functional of y, we can define the functional B with help of another functional A to be

$$B[y] = A[x[y,t]]. \quad \text{(A.20)}$$

Now we like to find the functional derivative of B with respect to y. For this purpose we apply the general definition:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}B[y+\epsilon\eta]\right|_{\epsilon=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}A[x[y+\epsilon\eta,t]]\right|_{\epsilon=0}. \quad \text{(A.21)}$$

Now we expand x in the argument of A with respect to ϵ:

$$x[y+\epsilon\eta,t] = x[y,t] + \epsilon\int_{t_1}^{t_2}\mathrm{d}\tau\frac{\delta x}{\delta y(\tau)}\eta(\tau) + O(\epsilon^2), \quad \text{(A.22)}$$

where we have used the definition for the functional derivative of x with respect to y. Now inserting this into (A.21) we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}B[y+\epsilon\eta]\right|_{\epsilon=0} = \int_{t_1}^{t_2}\mathrm{d}t\int_{t_1}^{t_2}\mathrm{d}\tau\frac{\delta A}{\delta x[y,\tau]}\frac{\delta x[y,\tau]}{\delta y(t)}\eta(t), \quad \text{(A.23)}$$

and since this should hold true for all η we have the desired chain rule

$$\frac{\delta B}{\delta y(t)} = \int_{t_1}^{t_2}\mathrm{d}\tau\frac{\delta A}{\delta x[y,\tau]}\frac{\delta x[y,\tau]}{\delta y(t)}. \quad \text{(A.24)}$$

Appendix B

The Symmetry of Space and Time

In this appendix we shall investigate the symmetries of the special-relativistic space time manifold, the *Minkowski space*, from a mathematical point of view. It should be emphasised that this appendix which bases completely on one of E.P. Wigner's most important papers, written in 1939 about this topic, is on the heart of relativistic quantum field theory since it determines all possible one-particle Hilbert spaces which are consistent with the structure of the Minkowski space. The reason that it is taken to the appendix is, that it is rather involved and not easy to read. The reader not so familiar with group theory and the representation theory of groups can omit this appendix in the first reading. It is also possible to follow the line of reasoning only reading chapter 4. Put it in the other way it might be useful for understanding this appendix to have a physical picture about the mathematics.

B.1 The Lorentz Group

As we have seen in the beginning of chapter 3 the Lorentz group is the invariance group of the Minkowski space which is $\mathbb{R}^4$ with the fundamental bilinear form

$$\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R} : \ (x^\mu, y^\nu) \mapsto xy = g_{\mu\nu} x^\mu y^\nu \tag{B.1}$$

with $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$. This space will be denoted by $\mathbb{R}^{(1,3)}$. The Lorentz group contains all invertible real 4×4-matrices which leave the bilinear form (B.1) invariant. This group is called $\mathrm{O}(1,3)$, that means the orthogonal group of the bilinear form with one positive and three negative eigenvalues.

Let $(L^\mu{}_\nu) \in \mathrm{O}(1,3)$. Then we have

$$\forall x, y \in \mathbb{R} : \ g_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma x^\rho y^\sigma = g_{\rho\sigma} x^\rho x^\sigma. \tag{B.2}$$

Since this has to hold for all $x, y \in \mathbb{R}^{(1,3)}$ it is necessary and sufficient for $\hat{L} \in \mathrm{O}(1,3)$

$$g_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma = g_{\rho\sigma}. \tag{B.3}$$

In matrix notation this means

$$\hat{L}^t \hat{g} \hat{L} = \hat{g} \Rightarrow \hat{L}^{-1} = \hat{g} \hat{L}^t \hat{g}. \tag{B.4}$$

Lemma 4. *The Lorentz group* $\mathrm{O}(1,3)$ *contains a subgroup which is isomorphic to the* $\mathrm{O}(3)$*, i.e., the orthogonal group in three dimensions.*

Proof. A glance at (3.1) shows, that the three-dimensional subspace $x^0 = 0$ of $\mathbb{R}^{(1,3)}$ is isomorphic to the Euclidean $\mathbb{R}^3$ (up to the sign of the scalar product these spaces are identical). The Lorentz transformations of the form

$$\hat{D} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{D}' \end{pmatrix} \text{ with } \hat{D}' \in \mathrm{O}(3) \tag{B.5}$$

build a subgroup acting on this $\mathbb{R}^3$ as $\mathrm{O}(3)$. Q.E.D.

Now we investigate the connected components of the Lorentz group. From (B.4) we see, that because of $\det \hat{g} = -1$

$$\forall \hat{L} \in \mathrm{O}(1,3) : \ (\det \hat{L})^2 = 1 \Rightarrow \det \hat{L} = \pm 1. \tag{B.6}$$

Thus the connected component with the identity of the Lorentz group must be a subgroup of $\mathrm{O}(1,3)$ the *special Lorentz group*, which consists of all $\mathrm{O}(1,3)$-matrices with determinant 1.

On the other hand for the basis vector $e_0 \in \mathbb{R}^4$ we have for $\hat{L} \in \mathrm{O}(1,3)$:

$$1 = e_0 e_0 = (\hat{L}e_0)(\hat{L}e_0) \Rightarrow \left(L^0{}_0\right)^2 - \sum_{a=1}^{3} \left(L^a{}_0\right)^2 \Rightarrow \left|L^0{}_0\right| \geq 1. \tag{B.7}$$

Thus the Lorentz transformations can be classified in such with $L^0{}_0 \geq 1$, the *orthochronous* Lorentz transformations, and such with $L^0{}_0 \leq -1$, the *antichronous* Lorentz transformations.

It is clear, that the connected component of the Lorentz group containing the identity has to lay in the orthochronous class of the group.

Lemma 5. *The orthochronous Lorentz transformations build an invariant subgroup of the Lorentz group, which is denoted by* $\mathrm{O}(1,3)^{\uparrow}$. *The orthochronous Lorentz transformations with determinant* 1 *build a subgroup* $\mathrm{SO}(1,3)^{\uparrow}$ *of the* $\mathrm{O}(1,3)^{\uparrow}$.

Proof. The Minkowski space $\mathbb{R}^{(1,3)}$ is divided in three classes of vectors, namely

- $x \in \mathbb{R}^{(1,3)}$ with $x^2 < 0$: space-like vectors
- $x \in \mathbb{R}^{(1,3)}$ with $x^2 = 0$: light-like vectors
- $x \in \mathbb{R}^{(1,3)}$ with $x^2 > 0$: time-like vectors.

The light-like vectors define the *light-cone*, shown in a $1+2$-dimensional space-time in figure 3.1, this is the hypersurface in $\mathbb{R}^{(1,3)}$ with

$$\left(x^0\right)^2 - \sum_{a=1}^{3} (x^a)^2 = 0. \tag{B.8}$$

A time-like vector $x \in \mathbb{R}^{(1,3)}$ is said to be in the forward or backward light-cone if $x^0 > 0$ or $x^0 < 0$ respectively.

To an arbitrarily given normalised time-like vector we find by applying of Schmidt's orthonormalisation theorem three space-like vectors which are orthonormalised and orthogonal to the time-like vector. But such four vectors build a basis which can be obtained by operating with an appropriate $\mathrm{O}(1,3)^{\uparrow}$ matrix on the canonical basis of $\mathbb{R}^{(1,3)}$. This matrix has columns given by the column vectors of the new basis. Thus if the time-like vector is in the forward (backward) light-cone the transformation is orthochronous (antichronous).

The same is true for any given Lorentz transformation: It is orthochronous if and only if it leaves the canonical time-like basis vector of $\mathbb{R}^{(1,3)}$ in the forward light-cone. From this it is clear that the composition of two orthochronous Lorentz transformations is orthochronous and the inverse of a orthochronous Lorentz transformation is orthochronous too. Thus the orthochronous Lorentz transformations build a subgroup of $\mathrm{O}(1,3)$, further on denoted by $\mathrm{O}(1,3)^{\uparrow}$.

Since the proper Lorentz transformations build the subgroup $\mathrm{SO}(1,3)$ the $\mathrm{SO}(1,3)^{\uparrow}$ is also a subgroup, because it is given as the intersection $\mathrm{O}(1,3)^{\uparrow} \cap \mathrm{SO}(1,3)$ of two subgroups.

Q.E.D.

Now we define the boost in 1-direction as

$$\hat{B}_1(\lambda) = \begin{pmatrix} \hat{M} & 0 \\ 0 & 1 \end{pmatrix} \text{ with } \hat{M} = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{B.9}$$

The physical meaning of this boosts is described in chapter 3. By a simple calculation with using the addition formula for the hyperbolic functions we find that the boost in the 1-direction builds a one parameter subgroup of the Lorentz group: $\hat{B}_1(\lambda_1)\hat{B}_1(\lambda_2) = \hat{B}_1(\lambda_1 + \lambda_2)$.

Since the mapping $\hat{B}_1 : \mathbb{R} \to \mathrm{SO}(1,3)^{\uparrow}$ is invertible, continuous and open this subgroup is isomorphic to the additive group of $\mathbb{R}$ as topological groups.

Theorem 7. *Each proper orthochronous Lorentz transformation $\hat{L} \in \mathrm{SO}(1,3)^{\uparrow}$ can be written in the form*

$$\hat{L} = \hat{D}_1\hat{B}_1\hat{D}_2 \text{ with } \hat{D}_1, \hat{D}_2 \in \mathrm{SO}(3),\ \hat{B}_1 \text{ boost in 1-direction.} \tag{B.10}$$

Proof. The proof is done the same way one proves the analogous theorem about the parameterisation of the rotation group with Euler angles.

Now we write $\hat{L} \in \mathrm{SO}(1,3)^{\uparrow}$

$$\hat{L} = \begin{pmatrix} a & \vec{x}^t \\ \vec{y} & \hat{A} \end{pmatrix} \Rightarrow \hat{L}^t\hat{g}\hat{L} = \begin{pmatrix} a^2 - \vec{y}^2 & a\vec{x}^t - \vec{y}^t\hat{A} \\ a\vec{x} - \hat{A}^t\vec{y} & \vec{x}\otimes\vec{x} - \hat{A}^t\hat{A} \end{pmatrix} = \hat{g}, \tag{B.11}$$

where the equation says that $\hat{L} \in \mathrm{O}(1,3)$. From this we find

$$a^2 = 1 + \vec{y}^2,\ a\vec{x} = \hat{A}^t\vec{y},\ \hat{A}^t\hat{A} = \vec{x}\vec{x}^t + 1. \tag{B.12}$$

Now we make the ansatz

$$\hat{D}_1^t\hat{L} = \hat{B}_1\hat{D}_2 \tag{B.13}$$

and prove, that this can be fulfilled with $\hat{B}_1$ a boost in 1-direction and $\hat{D}_1$ and $\hat{D}_2$ rotation matrices.

Now we operate with this ansatz on the time-like canonical basis vector:

$$\hat{D}_1^t\hat{L}e_0 = \begin{pmatrix} a \\ \hat{D}_1'^t\vec{y} \end{pmatrix}. \tag{B.14}$$

Since $\hat{L}$ is assumed to be orthochronous we have together with (B.12) $a \geq 1$. Thus we can set $a = \cosh\lambda$ with λ defined up to its sign. Using (B.12) again we find $\vec{y}^2 = \cosh^2\lambda - 1 = \sinh^2\lambda$. Thus we find $\hat{D}_1' \in \mathrm{O}(3)$ with

$$\hat{D}_1'\vec{y} = \begin{pmatrix} \sinh\lambda \\ 0 \\ 0 \end{pmatrix}. \tag{B.15}$$

Using now $\hat{D}_2 = \hat{B}_1(\lambda)^{-1}\hat{D}_1^t\hat{L}$ we have by construction $\hat{D}_2 e_0 = e_0$ leading to $\hat{D}_2 \in \mathrm{O}(3)$. Because the three determinants defining $\hat{D}_2$ have determinant 1 that is also the case for $\hat{D}_2$. So we have $\hat{D}_2 \in \mathrm{SO}(3)$. Q.E.D.

Corollary 1. *The proper orthochronous Lorentz group* $\mathrm{SO}(1,3)^\uparrow$ *is the connected subgroup of the Lorentz group containing the identity. It is a six-dimensional linear Lie group, which can be parameterised with help of generalised Euler angles:*

$$\hat{L} = \hat{O}_3(\alpha)\hat{O}_1(\beta)\hat{O}_3(\gamma)\hat{B}_1(\lambda)\hat{O}_1(\delta)\hat{O}_3(\epsilon). \tag{B.16}$$

Herein $O_n(\alpha) \in \mathrm{SO}(3)$ *denotes the rotation around the n-axis with an angle* α*.*

Proof. We have just to parameterise the rotations in (B.10) with their Euler angles and use the fact that a rotation around the 3-axis commutes with a boost in 1-direction. Since the matrices used build one parameter subgroups of the proper orthochronous Lorentz group $\mathrm{SO}(1,3)^\uparrow$ they are all differentiable. So it is a Lie group. Since we use all the time matrix representations this group is by definition a linear Lie group.

Q.E.D.

Theorem 8. *The proper orthochronous Lorentz group* $\mathrm{SO}(1,3)^\uparrow$ *is isomorphic to the factor group* $\mathrm{SL}(2,\mathbb{C})/\{1,-1\}$*:*

$$\mathrm{SO}(1,3)^\uparrow \cong \mathrm{SL}(2,\mathbb{C})/\{1,-1\}. \tag{B.17}$$

Proof. First we look at the rotation group as a subgroup of the $\mathrm{SO}(1,3)^\uparrow$. Its universal covering group is the SU(2) describing the spin 1/2 in quantum mechanics. The fundamental representation of this group is given by the operation of unitary 2×2 matrices with determinant 1 on $\mathbb{C}^2$. In our context the $\mathbb{C}^2$-vectors are called *spinors*. We denote these spinors as column vectors with components with upper indices: $\psi = (\psi^1, \psi^2)^t$ and define its hermitian conjugate as the row vector $\psi^\dagger = (\psi^{1*}, \psi^{2*})$. For an arbitrary SU(2)-matrix U we have

$$\forall \psi_1, \psi_2 \in \mathbb{C}^2 : \ (U\psi_1)^\dagger (U\psi_2) = \psi_1^\dagger \psi_2. \tag{B.18}$$

The transformation law of the conjugate complex spinor is given by

$$(U\psi)^* = U^*\psi^* = (U^\dagger)^t\psi^* = (U^{-1})^t\psi^*. \tag{B.19}$$

One says that the conjugate complex spinor is transformed *contragredient* to the spinor itself. In the case of SU(2) the dual spinor $\psi^d = \epsilon\psi$ with the matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \epsilon^t = -\epsilon, \quad \epsilon^2 = -1, \quad \epsilon^t = \epsilon^\dagger = \epsilon^{-1} \tag{B.20}$$

transforms under SU(2) transformations also according to the contragredient transformation, which tells us, that for the SU(2) the fundamental representation and its conjugate complex representation are unitary isomorphic. This is proved immediately by the fact that

$$\forall U \in \mathrm{SU}(2) : \ U^* = \epsilon U \epsilon^{-1}. \tag{B.21}$$

As one knows from the representation theory of angular momenta in quantum mechanics, each irreducible representation of the SU(2) can be given as the irreducible part of tensor representations of the fundamental representation, this is true for all complex representations of the SU(2).

We define the spinor components with lower indices by

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta. \tag{B.22}$$

Because of $(\epsilon^{\alpha\beta}) = (\epsilon_{\alpha\beta})$ we have

$$\psi^\beta = \epsilon^{\alpha\beta}\psi_\alpha = -\epsilon^{\beta\alpha}\psi_\alpha, \tag{B.23}$$

where we have used the properties of ϵ given in (B.20).

All this shows that the skew-symmetric product

$$\chi^\alpha\psi_\alpha \tag{B.24}$$

is invariant under SU(2) transformations.

Now we seek for all transformations which leave this product invariant. This needs not to be a unitary transformation. So let $T^\alpha{}_\beta$ be a complex 2×2 matrix. Then the condition for this matrix to leave the product (B.24) invariant is

$$T^\alpha{}_\epsilon T^\beta{}_\delta \epsilon_{\alpha\beta} = \epsilon_{\epsilon\delta} = \det T \epsilon_{\epsilon\delta} \Rightarrow \det(T) = 1. \tag{B.25}$$

In the last step of the equation above we have used the fact that ϵ is the skew-symmetric matrix in $\mathbb{C}^2$ and the representation of the determinant. We conclude that the only restriction on T to leave the product (B.24) invariant is that its determinant is 1. This means that this product is left invariant under all $\mathrm{SL}(2,\mathbb{C})$ transformations.

Since these matrices have no further restrictions the conjugate complex representation is no longer equivalent. So we define $\mathrm{SL}(2,\mathbb{C})$ spinors which transform under the conjugate complex representation by setting a dot over the indices of the components. This means

$$\psi'^{\dot\beta} = (T^{\dot\beta}{}_{\dot\alpha})^* \psi^{\dot\alpha}. \tag{B.26}$$

Now we can map symmetric second-rank SU(2)-tensors to $\mathbb{R}^3$ vectors. For $\vec{x} = (x, y, z)^t \in \mathbb{R}^3$ this mapping is given by

$$\psi^{11} = -x + \mathrm{i}y,\ \psi^{22} = x + \mathrm{i}y,\ \psi^{12} = z,\ \det(\psi^{\alpha\beta}) = -\vec{x}^2, \tag{B.27}$$

and the inverse transformation is

$$\begin{aligned} x &= \frac{1}{2}(\psi^{22} - \psi^{11}) = \frac{1}{2}(\psi_1^2 + \psi_2^1), \\ y &= \frac{1}{2\mathrm{i}}(\psi^{11} + \psi^{22}) = \frac{1}{2\mathrm{i}}(\psi_1^2 - \psi_2^1), \\ z &= \frac{1}{2}(\psi^{12} + \psi^{21}) = \frac{1}{2}(\psi_1^1 - \psi_2^2). \end{aligned} \tag{B.28}$$

The operation with SU(2) matrices on the second-rank spinors give the well known homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Now we look at $\mathbb{R}^3$ as a part of the Minkowski space $\mathbb{R}^{(1,3)}$. To find an analogous homomorphism for the proper orthochronous Lorentz group we have to include the component x^0 in the mapping. It should be invariant under the SU(2) representations since rotations operate only on the space-like component in the given frame. Such an invariant of the second-rank spinor is given by its trace. But this is fixed by the mapping of $\vec{x}$.

On the other hand by using $\mathrm{SL}(2,\mathbb{C})$-spinors we have the dotted indices, and under the subgroup $\mathrm{SU}(2)$ an upper dotted index transforms like a lower normal index. Thus the mapping

$$\begin{aligned} x^0 &= \frac{1}{2}(\psi^{1\dot{1}} + \psi^{2\dot{2}}), \quad x^1 = \frac{1}{2}(\psi^{2\dot{1}} + \psi^{1\dot{2}}), \\ x^2 &= \frac{\mathrm{i}}{2}(\psi^{1\dot{2}} - \psi^{2\dot{1}}), \quad x^3 = \frac{1}{2}(\psi^{1\dot{1}} - \psi^{2\dot{2}}) \end{aligned} \tag{B.29}$$

is the searched generalisation of the SU(2) case. The inverse is given by

$$\begin{aligned} \psi^{1\dot{1}} &= x^0 + x^3, \quad \psi^{2\dot{2}} = x^0 - x^3, \\ \psi^{1\dot{2}} &= x^1 - \mathrm{i}x^2, \quad \psi^{2\dot{1}} = x^1 + \mathrm{i}x^2 \end{aligned} \tag{B.30}$$

since $\det(\psi^{\alpha\dot{\beta}}) = x^2 = (x^0)^2 - \vec{x}^2$ is left invariant under $\mathrm{SL}(2,\mathbb{C})$. Thus we have a homomorphism $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3)$. We shall show below that it is a homomorphism

$$\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)^{\uparrow}. \tag{B.31}$$

The fundamental representation of the $\mathrm{SL}(2,\mathbb{C})$ is not isomorphic to its complex conjugate as can be seen by the example $S = \mathrm{diag}(2\mathrm{i}, 1/(2\mathrm{i}))$. Since eigenvalues are left unchanged under equivalence transformations and the conjugate complex of S has evidently not the same eigenvalues as S.

In the case of SU(2)-matrices this argument does not hold since all its eigenvalues are of modulus one and thus by the complex conjugation they are only interchanged, which can be described by equivalence transformations with permutation matrices.

To finish the proof of our theorem we use the short hand notation $\hat{\psi}$ for the matrix $(\psi^{\alpha\dot{\beta}})$. Then the $\mathrm{SL}(2,\mathbb{C})$ operation on our spinor of rank two can be written as

$$\hat{\psi}' = A\hat{\psi}A^{\dagger}. \tag{B.32}$$

Since $\det\hat{\psi} = \det\hat{\psi}' = x^2$ we have $\phi(\mathrm{SL}(2,\mathbb{C})) \subseteq \mathrm{O}(1,3)$. Setting

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det A = ad - bc = 1 \tag{B.33}$$

and taking $x = e_0$ in (B.30) we find with $L = \phi(A)$

$$L^0{}_0 = \frac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2) > 0 \tag{B.34}$$

which shows that $\phi(\mathrm{SL}(2,\mathbb{C})) \subseteq \mathrm{O}(1,3)^{\uparrow}$. Now we have $\phi(\hat{1}) = 1$. Since ϕ is continuous and $\det L = \pm 1$ we find that $\phi(\mathrm{SL}(2,\mathbb{C})) \subseteq \mathrm{SO}(1,3)^{\uparrow}$. That it is a homomorphism we can see immediately from (B.32).

Now we have to show that $\phi(\mathrm{SL}(2,\mathbb{C})) = \mathrm{SO}(1,3)^{\uparrow}$, i.e., that the homomorphism is an epimorphism. Since the restriction of ϕ to SU(2) is an epimorphism onto SO(3) due to theorem 8 we have only to show, that all boosts in 3-direction are in its image.

Calculating $L = \phi(A)$ explicitly with A given by (B.33), we find that we have only to set $b = c = 0$ and $a = 1/d = \exp(-\lambda)$ to obtain $L = B_3(\lambda)$ for any given $\lambda \in \mathbb{R}$. This shows that indeed ϕ is an epimorphism.

To finish the proof we have to investigate, which $\mathrm{SL}(2,\mathbb{C})$ matrices are mapped to 1. An explicit calculation shows that $\phi(A)$ is fulfilled only for $b = c = 0$ and $|a| = 1$. Thus we have $\phi^{-1}(\{1\}) = \ker\phi = \{1, -1\}$. From the epimorphism theorem of elementary group theory this finishes the proof.

Q.E.D.

Theorem 9. *The proper orthochronous Lorentz group* $\mathrm{SO}(1,3)^{\uparrow}$ *is simple. Due to the last theorem this means that the only nontrivial invariant subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *is* $\{-1, 1\}$.

Proof. Suppose H is an invariant subgroup of $\mathrm{SL}(2,\mathbb{C})$ which contains a matrix $A \neq \pm 1$. To proof the theorem we have to show that from this follows $H = \mathrm{SL}(2,\mathbb{C})$.

We start with the following classifications of $\mathrm{SL}(2,\mathbb{C})$ matrices.

1. $B \in \mathrm{SL}(2,\mathbb{C})$ has two distinct eigenvalues t and $1/t$. Then the matrix is equivalent to the diagonal matrix $\mathrm{diag}(t, t/t)$.
2. $B \in \mathrm{SL}(2,\mathbb{C})$ has two equal eigenvalues ± 1 and is equivalent to $\mathrm{diag}(\pm 1, \pm 1)$.
3. $B \in \mathrm{SL}(2,\mathbb{C})$ has a single eigenvalue ± 1 and is thus equivalent to one of the Jordan forms

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \tag{B.35}$$

   respectively.

Let

$$X = \begin{pmatrix} x & y \\ z & x \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}). \tag{B.36}$$

Now we look at the matrix $Y = A(XA^{-1}X^{-1})$. Since H is an invariant subgroup the matrix in parentheses is also contained in H and thus $Y \in H$. A little bit of algebra gives

$$s = \mathrm{Tr}\, Y = 2 + b^2z^2 + c^2y^2 - [(a-d)^2 + 2bd]yz, \tag{B.37}$$

where we used (B.33) for the matrix A. Because $A \neq \pm 1$ it cannot be $a = d$ simultaneously with $b = c = 0$. Thus s is not constant with respect to y and z. Now y and z can be chosen arbitrarily since it is always possible to adjust x such that $\det X = 1$. So we can adjust s to any value in $\mathbb{C}$ we like.

The eigenvalues of Y are given as the zeros of the characteristic polynom and after a little calculation we find

$$\lambda_Y = \frac{s}{2} \pm \frac{\sqrt{s^2 - 4}}{2}. \tag{B.38}$$

Thus H contains any matrix with different eigenvalues, i.e., all $\mathrm{SL}(2,\mathbb{C})$ matrices of class one. Since H is an invariant subgroup of $\mathrm{SL}(2,\mathbb{C})$ it contains the unity matrix as well as

$$-1 = \mathrm{diag}(t, 1/t)\mathrm{diag}(-1/t, -t) \tag{B.39}$$

for any number $t \neq \pm 1$. The third class of $\mathrm{SL}(2,\mathbb{C})$-matrices can be written as

$$\begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \pm t & 0 \\ 1/t & \pm 1/t \end{pmatrix}. \tag{B.40}$$

From this we conclude that the invariant subgroup H contains all $\mathrm{SL}(2,\mathbb{C})$-matrices, which shows that the only nontrivial invariant subgroup is $\{1,-1\}$ and from theorem 2 we see that the proper orthochronous Lorentz group $\mathrm{SO}(1,3)^{\uparrow}$ contains no nontrivial invariant subgroup, i.e., it is a simple linear Lie group, which was claimed by the theorem.

Q.E.D.

B.2 Representations of the Lorentz Group

In this section we investigate the finite-dimensional irreducible representations of the proper orthochronous Lorentz group $\mathrm{SO}(1,3)^{\uparrow}$ and its covering group, the $\mathrm{SL}(2,\mathbb{C})$.

Theorem 10. *The Lie algebra of the* $\mathrm{SL}(2,\mathbb{C})$*, the* $\mathrm{sl}(2,\mathbb{C})$*, is algebraically isomorphic to* $\mathrm{su}(2)\times\mathrm{su}(2)$*. Each irreducible representation of* $\mathrm{sl}(2,\mathbb{C})$ *can be characterised by two numbers* k *and* k' *with possible values* $0, 1/2, 1, 3/2, \ldots$.

For $k+k'$ *"half-integer" the representations are spinor representations of* $\mathrm{sl}(2,\mathbb{C})$*. For* $k+k'$ *integer it is equivalent to an irreducible tensor representation of* $\mathrm{so}(1,3)$.

The exponential $\exp:\mathrm{sl}(2,\mathbb{C})\to\mathrm{SL}(2,\mathbb{C})$ *is not surjective, but its image generates the* $\mathrm{SL}(2,\mathbb{C})$.

Proof. As the first step we derive the Lie algebra, which is given by six linearly independent matrices in the tangent space of the group at the identity. These are given simply as derivatives of the three independent rotations and the three independent boosts, where the boosts are given like in (3.5) together with (3.10) and its analogies for the boost in the other two directions. Together with the familiar angular momentum matrices for the rotation group we find by direct calculation the Lie algebra $\mathcal{L}\mathrm{SO}(1,3)^{\uparrow}=\mathcal{L}\mathrm{SL}(2,\mathbb{C})=\mathrm{sl}(2,\mathbb{C})$ to be

$$[J_j,J_k]=\mathrm{i}\epsilon_{jkl}J_l,\ [K_j,K_k]=-\mathrm{i}\epsilon_{jkl}J_l,\ [J_j,K_k]=\mathrm{i}\epsilon_{jkl}K_l. \tag{B.41}$$

We find immediately that the rotation algebra is a sub algebra of $\mathrm{sl}(2,\mathbb{C})$ as we know from lemma 1 about the $\mathrm{O}(1,3)^{\uparrow}$. Now we choose another basis for the Lie algebra defined with help of the above given by

$$A_j=\frac{1}{2}(J_j+\mathrm{i}K_j),\ B_j=\frac{1}{2}(J_j-\mathrm{i}K_j)\Rightarrow \tag{B.42}$$

$$\Rightarrow [A_j,A_k]=\mathrm{i}\epsilon_{jkl}A_l,\ [B_j,B_k]=\mathrm{i}\epsilon_{jkl}B_l,\ [A_j,B_k]=0. \tag{B.43}$$

Thus the $\mathrm{sl}(2,\mathbb{C})$ is the direct sum of two $\mathrm{su}(2)$ Lie algebras: $\mathrm{sl}(2,\mathbb{C})=\mathrm{su}(2)\oplus\mathrm{su}(2)$.

We can immediately apply the representation theory of the $\mathrm{su}(2)$ to our case. There are two commuting Casimir operators, namely $\vec{A}^2=A_iA_i$ and $\vec{B}^2=B_iB_i$. Each irreducible representation can be uniquely determined by the eigenvalues of the two Casimir operators $k(k+1)$ and $k'(k'+1)$ with k and k' running over the range claimed in the theorem.

The vector space, the group operates on by the representation (k,k'), is spanned by the simultaneous eigenvectors of A_3 and B_3. The eigenvalues of A_3 are $-k,-(k-1),\ldots,k$ and those of B_3 are $-k',-(k'-1),\ldots,k'$. The dimension of the representation (k,k') is $(2k+1)(2k'+1)$.

The fundamental representations of $\mathrm{sl}(2,\mathbb{C})$ which can be obtained as the tangent space of the fundamental representations of $\mathrm{SL}(2,\mathbb{C})$ described in the proof of theorem 3, are in this context $(1/2,0)$ and $(0,1/2)$. This can be seen from the fact that these are the only two-dimensional

representations of the algebra. By definition the first one is given by the spinors with normal indices, the second one by the spinors with dotted indices.

The proof of theorem 3 shows also that the fundamental representation of the $\mathrm{SO}(1,3)^{\uparrow}$, i.e., the tangent space of $\mathrm{SO}(1,3)^{\uparrow}$ is $(1/2,1/2)$.

From the Clebsch-Gordan theorem for the rotation algebra we know that each irreducible representation of su(2) can be given as the irreducible part of a spinor representation with an appropriate rank of spinors. The same is thus true for the $\mathrm{sl}(2,\mathbb{C})$. Further we know that the integer-valued irreducible representations are equivalent to irreducible tensor representations.

Since the one-parameter subgroups of $\mathrm{SL}(2,\mathbb{C})$, given by the boosts and rotations, are of course in the image of the exponential function the theorem is proven by application of theorem 2 and corollary 1.

An example for a $\mathrm{SL}(2,\mathbb{C})$ matrix which is not contained in the image of the exponential function $\exp : \mathrm{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ is given by the "null rotations" described below in the context of massless states.

Q.E.D.

B.3 Representations of the Full Lorentz Group

So far we have only given representations of $\mathrm{SO}(1,3)^{\uparrow}$, which is the invariant subgroup of $\mathrm{O}(1,3)$ connected with the identity, and its universal covering group $\mathrm{SL}(2,\mathbb{C})$.

Now we want to find the representations of other subgroups of $\mathrm{O}(1,3)$ and the full Lorentz group itself. For this purpose we prove

Theorem 11. *Let G be a group, G_1 an invariant subgroup such that the factor group G/G_1 is isomorphic to the group of two elements $\mathbb{Z}_2$. Then there are two possibilities for a given irreducible finite-dimensional representation $\Phi : G \to \mathrm{GL}(V)$:*

(1) If the representation of G_1, which is induced by Φ is irreducible, then there exists one and only one other irreducible representation Φ' of G which is inequivalent to the representation Φ and induces the same irreducible representation of G_1 as Φ. If $G_2 \neq G_1$ is the coset of G_1 in G then we have $\forall g_2 \in G_2 :\ \Phi(g_2) = -\Phi'(g_2)$.

(2) If Φ induces a reducible representation of G_1 then this representation is the direct sum of two inequivalent irreducible representations of G_1 which both have the same dimension. These two irreducible representations of G_1 determine Φ up to equivalence uniquely.

Proof. (1) Let Φ induce an irreducible representation of G in V. Let Φ' be also an irreducible representation of G in V with $\Phi(g_1) = \Phi'(g_1)$ for all $g_1 \in G_1$.

For $g_2 \in G_2$ we have for $g_1 \in G_1$ (using the fact that G_1 is invariant subgroup of G) $\bar{g}_1 = g_2^{-1} g_1 g_2 \in G_1$ and thus $\Phi(\bar{g}_1) = \Phi'(\bar{g}_1)$. This implies that $\Phi(g_1)\Phi'^{-1}(g_1)$ is commuting with all $\Phi(g_1)$ with $g_1 \in G_1$. Since the representation of G_1 which is induced by Φ is irreducible we have by Schur's lemma $\Phi(g_2) = \lambda \Phi'(g_2)$ with a $\lambda \in \mathbb{C}$.

Because we have by assumption only one coset of G_1 in G, namely G_2, for all $g_2' \in G_2$ there is a $g_1 \in G_1$ such that $g_2' = g_1 g_2$ with a fixed $g_2 \in G_2$. Thus λ is independent of the choice of $g_2 \in G$.

Since with $g_2 \in G_2$ also $g_2^{-1} \in G_2$, we have necessarily $\lambda = 1/\lambda$ and thus $\lambda = \pm 1$.

Now we assume that we have given the irreducible representation of G_1 which is induced by

Φ. Then we can construct an representation of G by using $\lambda = 1$ which leads back to the representation Φ or by using $\lambda = -1$.

These both representations are inequivalent, since because $\Phi(g_1) = \Phi'(g_1)$ for all $g_1 \in G$ defines an irreducible representation of G_1 on V from $\Phi(g) = T\Phi'(g)T^{-1}$ we have $T \propto \mathrm{id}_V$ from Schur's Lemma. Thus these representations cannot be equivalent. This proves the first claim of the theorem.

(2) Now we suppose that Φ induces a reducible representation of G_1 in V. Then there exists a proper subspace V_1 of V which is invariant under the group operations of G_1 induced with help of Φ.

Now we define for an arbitrary $g_2 \in G_2$ the subspace $V_2 = \Phi(g_2)V_1$. This subspace is independent of the choice of $g_2 \in G_2$. To this end let $g_2' \in G_2$. Since by assumption there exists an element $g_1 \in G_1$ with $g_2' = g_2 g_1$ we have

$$\Phi(g_2')V_1 = \Phi(g_2)\Phi(g_1)V_1 = V_2,$$

where we have used that $\Phi(g_1)$ leaves the subspace V_1 invariant and is an isomorphism in V_1.

Now we show that V_2 is also an invariant irreducible space with respect to the group operations of G_1 induced by Φ. To this end take an arbitrary element $g_1 \in G_1$ and an arbitrary element $g_2 \in G_2$ and calculate

$$\Phi(g_1)V_2 = \Phi(g_1)\Phi(g_2)V_1 = \Phi(g_2)\Phi(\underbrace{g_2^{-1}g_1g_2}_{\in G_1})V_1 = \Phi(g_2)V_1 = V_2,$$

where we have used that G_1 is an invariant subgroup of G.

Now G operates via Φ invariantly on $V' = \mathrm{span}(V_1 \cup V_2)$. Since Φ is an irreducible representation of G on V we have necessarily $V' = V$.

The next step is to prove $V'' = V_1 \cap V_2 = \{0\}$. But this is clear since this is a proper subspace of V which is left invariant under the G-operations defined by the irreducible representation in V. But by definition of an irreducible representation this implies $V'' = \{0\}$.

Now the representation of G_1 induced by Φ in V_2 is equivalent to the representation of G_1 induced by the conjugate representation $\Phi'(g_1) = \Phi(g_2^{-1}g_1g_2)$ with $g_2 \in G_2$ on V_1. We conclude from this that the operation of G_1 on V_2 is also irreducible and that $\Phi(g_2)V_1 = V_2$.

Now suppose the both constructed irreducible representations of G_1 were equivalent. This would imply that the representation Φ' restricted to G_1 would be equivalent to the representation of G_1 determined by Φ restricted to G_1. The equivalence transformation would be given by $\Phi(g_2)$ with an arbitrary $g_2 \in G_2$. But then using again Schur's lemma this would mean that $\Phi(g_2)$ is $\propto 1$ which is impossible since it interchanges the two direct summands V_1 and V_2 of V.

Q.E.D.

In our case of the Lorentz group we use this theorem in the following way

1. $G = \mathrm{O}(1,3)^{\uparrow}$ and $G_1 = \mathrm{SO}(1,3)^{\uparrow}$.

   In this case we have $G/H = \{1, P\} \cong \mathbb{Z}_2$, where P is $\mathrm{diag}(1,-1,-1,-1)$, i.e., the parity operator which describes spatial reflections.

   First we look at the fundamental representations of $\mathrm{SL}(2,\mathbb{C})$, which are $(1/2,0)$ and $(0,1/2)$. From our theorem part (2) we learn that the corresponding irreducible representation of

$\mathrm{O}(1,3)^\uparrow$ is operating on the direct sum $(1/2,0)\oplus(0,1/2)$. From the theorem it follows that this is the only possibility since the two representations of the invariant subgroup have to be inequivalent and be of the same dimension. The only two inequivalent two-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ are $(1/2,0)$ and $(0,1/2)$.

Since we have $\mathrm{O}(1,3)^\uparrow \cong \mathrm{SL}(2,\mathbb{C})/\{1,-1\}$ and in space-time, i.e., as operator in $\mathrm{O}(1,3)^\uparrow$ it is $P^2=1$. Looking on the extension of $\mathrm{SL}(2,\mathbb{C})$, which is the covering group of $\mathrm{O}(1,3)^\uparrow$ to a covering group of $\mathrm{O}(1,3)^\uparrow$ there are two possibilities, namely $P^2=\pm 1$. This shows that the covering of the $\mathrm{O}(1,3)^\uparrow$ is not unique (the deeper reason is that this group is not connected).

In the first case, i.e., $P^2=+1$, this gives the only possibility to extend the $\mathrm{SL}(2,\mathbb{C})$ to a covering of $\mathrm{O}(1,3)^\uparrow$ by introducing a $(1/2,0)$ spinor ψ (a $\mathbb{C}^2$ spinor as described in the last section with components with normal indices) and a $(0,1/2)$ spinor (i.e. a $\mathbb{C}^2$ spinor with dotted indices) χ. Then the parity operator acts in the form

$$P\psi^\alpha = \pm\chi_{\dot\alpha},\ P\chi_{\dot\beta} = \pm\psi^\beta, \tag{B.44}$$

which fulfils for both signs $P^2=1$.

For $P^2=-1$ we have

$$P\psi^\alpha = \pm\mathrm{i}\chi_{\dot\alpha},\ P\chi_{\dot\alpha} = \pm\mathrm{i}\chi^\alpha, \tag{B.45}$$

where again for both sign conventions $P^2=-1$ is fulfilled.

Now we look at the fundamental representation $(1/2,1/2)$. Here applies part (1) of the theorem. Since we can represent the parity operator P by the unity operator (trivial representation of space reflections) there must be another inequivalent representation of $\mathrm{O}(1,3)^\uparrow$ whose restriction to $\mathrm{O}(1,3)^\uparrow$ is the $(1/2,1/2)$ representation. The theorem tells us that this is the fundamental representation of space reflections with $P=\mathrm{diag}(1,-1,-1,-1)$.

2. $G=\mathrm{O}(1,3)$ and $G_1=\mathrm{O}(1,3)^\uparrow$.

   In this case the theorem applies again since here we have $G/G_1=\{-1,1\}$.

   We start again with the fundamental representation of the covering group $\mathrm{SL}(2,\mathbb{C})$ and try to extend it to covering groups of $\mathrm{O}(1,3)$. Since $-1\in\mathrm{O}(1,3)$ commutes with all $\mathrm{O}(1,3)$ matrices the same is the case for the covering group. Since we have $(-1)^2=1$ in $\mathrm{O}(1,3)$ the total space-time reflection P_{tot} in the covering has to fulfil $P_{\mathrm{tot}}^2=\pm 1$. Both possibilities can be realized extending the representations $(1/2,0)$ and $(0,1/2)$.

   In the first covering we have the pair $P_{\mathrm{tot}}\psi=\pm\psi$ of inequivalent representations and in the second covering the pair $P_{\mathrm{tot}}\psi=\pm\mathrm{i}\psi$.

   It should be mentioned that this possibilities are restricted to the classical theory since in quantum theory time reflections are necessarily represented with anti-unitary operators.

Taking both cases together we find that there are four inequivalent covering groups of the whole Lorentz group $\mathrm{O}(1,3)$ and each of them can be realized with two spinor representations. To distinguish the spinors of the $(1/2,0)$ and $(0,1/2)$ which are irreducible representations of the proper orthochronous Lorentz group $\mathrm{SO}(1,3)^\uparrow$ with these of the representation $(1/2,0)\oplus(0,1/2)$ which leads to irreducible representations of the possible covering groups of the proper Lorentz group $\mathrm{O}(1,3)$, we call the former spinors *Weyl spinors* and the latter *Dirac spinors*. In chapter

4 we shall use these results to construct physically different types of spinors, i.e., spinor fields which describe different sorts of particles.

We have found all finite-dimensional representations of the Lorentz group and it should be emphasised that there is no unitary finite representation (except the trivial one). This is due to the fact that the Lorentz group is not compact.

B.4 Unitary Representations of the Poincaré Group

Now we are ready to give the physically relevant unitary representations of the *Poincaré group*, which is the semi-direct product of the different types of the Lorentz group with space-time translations, which is given by the operation on the space-time vector $x \in \mathbb{R}^{(1,3)}$:

$$T(a, L)x = Lx + a \text{ with } L \in G \subseteq \mathrm{O}(1,3),\ a \in \mathbb{R}^{(1,3)}. \tag{B.46}$$

The group operation can be found by applying two such transformations to x and the result is

$$T(a_2, L_2)T(a_1, L_1) = T(L_2 a + b, L_2 L_1). \tag{B.47}$$

It is trivial to show that this is indeed a group multiplication.

Now we have seen that we can restrict our search for unitary representations of this group to $G = \mathrm{SO}(1,3)^{\uparrow}$ since the possible extensions to greater parts of the Lorentz group is constructed in the last section. Then the group defined by (B.47) is called *proper orthochronous Poincaré group* and is denoted by $\mathrm{P}_{+}^{\uparrow}$.

Since this group is by no means compact and we need unitary representations of $\mathrm{P}_{+}^{\uparrow}$ for defining quantum mechanical observables by hermitian operators which are the generators of $\mathrm{P}_{+}^{\uparrow}$, we have to use an infinite-dimensional Hilbert space $\mathscr{H}$ the representations operate on.

Since $\mathrm{SO}(1,3)^{\uparrow}$ is six-dimensional and the space-time translations four-dimensional, $\mathrm{P}_{+}^{\uparrow}$ is ten-dimensional, and thus we define the operators representing the ten following observables of a fundamental quantum object (here called "particles" for abbreviation) by looking the (physically meaningful) unitary representations of $\mathrm{P}_{+}^{\uparrow}$ on $\mathscr{H}$: Energy, momentum, angular momentum, centre of mass coordinates.

Now we look on the unitary representations of $\mathrm{P}_{+}^{\uparrow}$. Since we can do this by calculating the operation of unitary operators of the factors in the semi-direct product of this group by searching a generalised basis of the Hilbert space consisting of the simultaneous generalised eigenvectors of a complete set of hermitian generators of the group. For this purpose we have to find such a complete set, which is linearly independent and pairwise commuting.

We start with the generators of translations $\mathbf{p}$ which are commuting since the translation group (here seen as subgroup of $\mathrm{P}_{+}^{\uparrow}$). From this we find immediately that the unitary operator for translations is given by

$$\mathbf{U}(a) = \exp(\mathrm{i}a\mathbf{p}) \text{ with } a \in \mathbb{R}^{(1,3)}. \tag{B.48}$$

The $\mathbf{p}$ are hermitian commuting operators. Now we define $|p, \alpha\rangle$ as the simultaneous eigenket of the momenta $\mathbf{p}$, where α labels the degeneracy of the eigenket.

This defines the operation of the translation group on the generalised eigen-kets of the momentum operators:

$$\mathbf{U}(a)\,|p, \alpha\rangle = \exp(\mathrm{i}pa)\,|p, \alpha\rangle\,. \tag{B.49}$$

The spectrum of $\mathbf{p}$ is $\mathbb{R}^4$ as one can show in the same way as in nonrelativistic quantum mechanics.

Now we investigate the unitary operators $\mathbf{U}(\hat{L})$ representing $\mathrm{SL}(2,\mathbb{C})$ transformations of a given irreducible representation shown in the last section.

Since $\mathbf{p}$ has to be a vector operator as can be shown by the usual method with help of the commutator relations following from (B.47) for "infinitesimal" transformations, we have

$$\mathbf{p}_\mu \mathbf{U}(\hat{L})\,|p,\alpha\rangle = \mathbf{U}(\hat{L}) L_\mu{}^\nu \mathbf{p}_\nu\,|p,\alpha\rangle = L_\mu{}^\nu p_\nu \mathbf{U}(\hat{L})\,|p,\alpha\rangle\,. \tag{B.50}$$

Comparing the very lefthand side with the very righthand side of this equation we conclude, that $\mathbf{U}(\hat{L})\,|p,\alpha\rangle$ is a simultaneous eigenket of $\mathbf{p}$ with the eigenvalues $\hat{L}p$.

Since $|p,\alpha\rangle$ spans by definition the irreducible eigenspace of $\mathbf{p}$, further on denoted by $\mathrm{Eig}(\mathbf{p},p)$, it must hold

$$\mathbf{U}(\hat{L})\,|p,\alpha\rangle = \sum_\beta Q_{\beta\alpha}(p,\hat{L})\left|\hat{L}p,\beta\right\rangle. \tag{B.51}$$

Since we are only interested in unitary representations of the Poincaré group $Q_{\alpha\beta}$ has to be a unitary matrix, which means $(Q^{-1})_{\alpha\beta} = Q^*_{\beta\alpha}$.

Now the representation can only be irreducible, if it is possible to transform each $|p,\alpha\rangle$ to any $|p',\alpha\rangle$ contained in the generalised basis of the Hilbert space. Since the translation operators do not change the momentum eigenvalue of the ket, this change must be possible by applying $\mathrm{SO}(1,3)^\uparrow$-transformation (B.51). Thus the spectrum of the momenta building the eigen-kets which span the irreducible Hilbert space have to be in a manifold, on which the *proper orthochronous Lorentz group operates transitively.* These manifolds are given by the condition

$$(p')^2 = p^2,\ \mathrm{sign}(p')^0 = \mathrm{sign}\,p^0 \tag{B.52}$$

with an arbitrary vector of reference p contained in this manifold. We conclude that these manifolds are given in the following form

$$p^2 = m^2 > 0,\ p^0 > 0, \tag{B.53}$$
$$p^2 = m^2 > 0,\ p^0 < 0, \tag{B.54}$$
$$p^2 = 0,\ p^0 > 0, \tag{B.55}$$
$$p^2 = 0,\ p^0 < 0, \tag{B.56}$$
$$p = 0, \tag{B.57}$$
$$p^2 = m^2 < 0. \tag{B.58}$$

Thus to specify an irreducible representation we need at least the class of momenta given by one of these manifolds. The discussion about the causality of waves describing free particles, which we aim to classify by finding all unitary irreducible representation of the Poincaré groups, shows that only the classes (B.53-B.56) lead to causal fields (at least in the quantised form which gives the possibility give to solve the problem with the negative energies in the cases (B.54) and (B.56) in terms of the Feynman-Stueckelberg formalism).

Now we go further in the classification of the irreducible unitary representations. To this end we have to investigate the possible realisations of the matrices $Q_{\beta\alpha}$ in (B.51). Using this equation for the composition of two Lorentz transformations leads to

$$Q_{\gamma\alpha}(\hat{L}_2\hat{L}_1,p) = \sum_\beta Q_{\gamma\beta}(\hat{L}_2,\hat{L}_1 p) Q_{\beta\alpha}(\hat{L}_1,p). \tag{B.59}$$

This equation is the property for a group homomorphy $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathrm{Eig}(\mathbf{p}, p))$ if $\hat{L}$ is restricted to the subgroup of $\mathrm{SO}(1,3)^\uparrow$ which leaves the vector p invariant, which is called *little group with respect to p*. We denote this little group for abbreviation with $K(p)$ (this is no standard notation in literature but convenient for our purposes). Now we chose a standard vector p_0 in the appropriate manifold (B.53-B.56).

Here we give the formal definition for the little group with respect to the standard momentum vector p_0.

$$K(p_0) = \{\hat{L} \in \mathrm{SO}(1,3)^\uparrow | \hat{L}p_0 = p_0\}. \tag{B.60}$$

It is trivial to show that this defines a sub group of the $\mathrm{SO}(1,3)^\uparrow$.

No we show how to obtain the irreducible representation of the whole group supposed the $Q_{\beta\alpha}(\hat{K}, p_0)$ build an irreducible representation of the little group $K(p_0)$. Since the representation is irreducible each vector p which may occur in the set of momentum eigen-kets of the representation (which is necessarily given by one of the manifolds implicitly defined by (B.53-B.56)) can be obtained by operating with a certain given $\mathrm{SO}(1,3)^\uparrow$ matrix on p_0, because these manifolds are those on which the $\mathrm{SO}(1,3)^\uparrow$ operates transitively. In a more compact notation this can be described by

$$\forall p \in M \exists \Lambda(p) \in \mathrm{SO}(1,3)^\uparrow : \; p_0 = \Lambda(p)p, \tag{B.61}$$

where M is one of the manifolds (B.53-B.56) which describe possible causal fields, if the Hilbert space $\mathscr{H}$ is realized as the function space L^2.

The only restriction we want to make about $\Lambda : M \to \mathrm{O}(1,3)^\uparrow$ is that it is a continuously differentiable mapping with $\Lambda(p_0) = 1$.

We start with the case (B.53). As the standard vector we chose that of the rest frame momentum of a particle with mass m, namely $p_0 = (m, 0, 0, 0)^t$. The manifold M can now be parameterised with help of the spatial part $\vec{p} \in \mathbb{R}^3$ of the momentum:

$$p = \begin{pmatrix} \sqrt{m^2 + \vec{p}^2} \\ \vec{p} \end{pmatrix}. \tag{B.62}$$

From our physical intuition it is clear that the change from the rest frame momentum p_0 to the general momentum (B.62) should be given by the boost in direction $\vec{n} = \vec{p}/|\vec{p}|$ with velocity (measured in units of the light velocity) $\beta = \vec{n}\vec{p}/\sqrt{m^2 + \vec{p}^2}$. The appropriate matrix is thus given by

$$\Lambda^{-1}(p) = \begin{pmatrix} \gamma & \gamma\beta\vec{n}^t \\ \gamma\beta\vec{n} & (\gamma - 1)\vec{n} \otimes \vec{n} + 1 \end{pmatrix} \text{ with } \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{B.63}$$

It is easy to verify that (B.63) indeed fulfils (B.61) $\Lambda^{-1}(p)p_0 = p$. It is also continuously differentiable since this is the case for each single matrix element of $\Lambda(p)$.

Now for all $\hat{L} \in \mathrm{SO}(1,3)^\uparrow$ and for all $p \in M$ the matrix $\hat{K}(L, p) = \Lambda(\hat{L}p)\hat{L}\Lambda^{-1}(p) \in K(p_0)$. This can be proven simply by applying the definition (B.61) of $\Lambda(p)$ twice:

$$\hat{K}(\hat{L}, p) = \Lambda(\hat{L}p)\hat{L}\Lambda^{-1}(p)p_0 = \Lambda(\hat{L}p)\hat{L}p = p_0. \tag{B.64}$$

Together with the given choice of $\Lambda(p)$ we have thus a unique decomposition of any $\mathrm{O}(1,3)^\uparrow$-matrix

$$\hat{L} = \Lambda^{-1}(\hat{L}p)\hat{K}(\hat{L}, p)\Lambda(p). \tag{B.65}$$

where $\hat{K}(\hat{L},p) \in K(p_0)$.

Now we chose the base kets $|p,\alpha\rangle$ in the following way

$$|p,\alpha\rangle = \tilde{U}^{-1}(\Lambda(p))\,|p_0,\alpha\rangle\,. \tag{B.66}$$

With respect to this so called *Wigner basis* together with (B.51) we obtain

$$\mathbf{U}[\hat{K}(\hat{L},p)]\,|p_0,\alpha\rangle = \sum_\beta Q_{\beta\alpha}[\hat{K}(\hat{L},p),p_0]\,|p_0,\beta\rangle\,. \tag{B.67}$$

Since $\hat{K}(\hat{L},p)$ is in the little group with respect to p_0 together with (B.59) this transformation law is completely determined by choosing an arbitrary representation of this little group $K(p_0)$ operating on the simultaneous eigenspaces of the $\mathbf{p}$ with eigenvalues p_0 whose base kets we have denoted with $|p_0,\alpha\rangle$.

We have to show now how to determine the $Q_{\beta\alpha}(\hat{L},p)$ for all $\mathrm{O}(1,3)^\uparrow$-matrices, which are not contained in $K(p_0)$. To this end we apply (B.65) to (B.51)

$$\sum_\beta Q_{\beta\alpha}(\hat{L},p)\left|\hat{L}p,\beta\right\rangle = \mathbf{U}[\Lambda^{-1}(\hat{L}p)\hat{K}(\hat{L},p)\Lambda(p)]\,|p,\alpha\rangle\,. \tag{B.68}$$

Multiplying this with $\mathbf{U}[\Lambda(\hat{L}p)]$ from the left and using the fact that U is a representation of $\mathrm{P}_+^\uparrow$ we find with help of the definition (B.61) for $\Lambda(p)$:

$$\mathbf{U}[\hat{K}(\hat{L},p)]\,|p_0,\alpha\rangle = \sum_\beta Q_{\beta\alpha}(K(\hat{L},p),p_0)\,|p_0,\beta\rangle \tag{B.69}$$

Comparing this with (B.67) we have

$$Q_{\beta\alpha}(\hat{L},p) = Q_{\beta\alpha}[\hat{K}(\hat{L},p),p_0] \tag{B.70}$$

which shows that the $Q_{\beta\alpha}$ are completely determined by a unitary representation of the little group $\hat{K}(p_0)$ since, given this representation, all other $Q_{\beta\alpha}$ are determined by (B.70) and then it is given, because $\hat{K}(\hat{L},p) \in K(p_0)$.

The same time it is clear that the unitary representation $\mathbf{U}$ given by of (B.49), (B.51) and (B.70) with help of a unitary representation $Q_{\beta\alpha}$ of the little group $K(p_0)$ is irreducible if and only if the representation of the little group is irreducible. It is also clear that constructing an unitary function $\mathbf{U}$ in this way we obtain indeed an irreducible representation of $\mathrm{P}_+^\uparrow$ and thus all unitary representations of $\mathrm{P}_+^\uparrow$ can be constructed in this way.

Now we can find the physically relevant irreducible unitary representations, defining the one-particle Hilbert spaces of elementary particles, $\mathrm{P}_+^\uparrow$ by giving the irreducible representations of the various classes of such representations defined by the manifolds (B.53-B.56).

To this end we investigate the Lie algebra of the little group defined by the standard vector p_0, which build a subalgebra of the $\mathrm{O}(1,3)^\uparrow = \mathrm{sl}(2,\mathbb{C})$. Since the Lie algebra is the same for all representations we can find the general structure of Lie algebra $\mathcal{K}(p_0) := k(p_0)$ by investigating the fundamental representation of $\mathrm{O}(1,3)^\uparrow$ operating on $\mathbb{R}^{(1,3)}$. This gives also a nice picture about the geometric content of the little group.

Now it is convenient to parameterise the Lie algebra operation on the $\mathbb{R}^{(1,3)}$-vectors with help of antisymmetric matrices $\delta\omega$:

$$\delta x^\mu = \delta\omega^\mu{}_\nu x^\nu \text{ with } \delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}. \tag{B.71}$$

The six corresponding matrices building a basis of the Lie algebra are given by $\hat{M}^{\mu\nu} = -\hat{M}^{\nu\mu}$ and are, for $\mu = 0$, the three independent generators for boosts, while the purely spatial components define the three generators for rotations.

Now a Lie algebra element is in $k(p_0)$ if it leaves the given vector p_0 unchanged:

$$\delta p_0^\mu = 0 = \delta\omega^\mu{}_\nu p_0^\nu \Leftrightarrow (\delta\omega^\mu{}_\nu) \in k(p_0). \tag{B.72}$$

The general solution of this condition can be parameterised with a vector $\delta\Phi$ by

$$\delta\omega_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\delta\Phi^\rho p_0^\sigma. \tag{B.73}$$

Thus in a general representation the generators $\mathbf{W}_\rho$ of $k(p_0)$ are given with help of the hermitian operators $\mathbf{M}^{\mu\nu} = -\mathbf{M}^{\nu\mu}$ representing the Lie algebra of the $\mathrm{O}(1,3)^\uparrow$ in the Hilbert space

$$\mathbf{W}_\rho = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{M}^{\mu\nu}\mathbf{p}^\sigma, \tag{B.74}$$

if we restrict the operation to $\mathrm{Eig}(\mathbf{p}, p_0)$. $\mathbf{W}_\rho$ is known as the *Pauli-Lubanski vector*. Since we have the restriction $\mathbf{W}_\mu\mathbf{p}^\mu = 0$ which follows directly from (3.32) together with the commutativity of the four $\mathbf{p}$. Thus the little group is in general three-dimensional. As we shall see this is the case for the "causal" representations given by the manifolds (B.53-B.56).

B.4.1 The Massive States

Let us start with the case (B.53), i.e., the standard vector of the little group p_0 should be in the forward light-cone defined by $p_0^2 = m^2 > 0$ with positive time component. To keep the story simple we chose $p_0 = (m, 0, 0, 0)^t$ with $m > 0$. It is clear that in this case $K(p_0) = \mathrm{O}(3)$, operating on the three space components of the frame defined by p_0. The irreducible representations of the rotation group O(3) or its covering group SU(2) is well known from the angular momentum algebra in quantum mechanics. Since this SU(2) operates in the rest frame basis of the Hilbert space, the little group for massive particles is the intrinsic angular momentum of the particles, i.e., the *spin*.

We conclude: For massive particles any irreducible representation of $\mathrm{P}_+^\uparrow$ is defined by the eigenvalues of the Casimir operators $\mathbf{p}^2$ and the spin square $\mathbf{s}^2$ with eigenvalues m^2 with $m \in \mathbb{R}^+$ and $s(s+1)$ with $s = k/2,\ k \in \mathbb{N}$. But these Casimir operator eigenvalues are not uniquely determining the representation because to each $m^2 > 0$ and s there are two inequivalent irreducible representations of $\mathrm{P}_+^\uparrow$, namely those with $p_0 = (+m, 0, 0, 0)^t$ with the manifold M given by (B.53) and those with $p_0 = (-m, 0, 0, 0)^t$ with the manifold M given by (B.54).

We want to prove this with help of the formalism developed above and the same time to give the relation between the Pauli-Lubanski vector and the spin. Because of (B.74) our choice of $p_0 = (\pm m, 0, 0, 0)^t$ leads to

$$\mathbf{W}_0 \left|p_0, \alpha\right\rangle = 0. \tag{B.75}$$

With help of the Lie algebra of $\mathrm{P}_+^\uparrow$ one calculates the commutator relations of $\mathbf{M}^{\mu\nu}$ and $\mathbf{P}^\sigma$ and with these

$$[\mathbf{W}_\alpha, \mathbf{W}_\beta] = -\mathrm{i}\epsilon_{\alpha\beta\nu\rho}\mathbf{W}^\nu\mathbf{P}^\rho, \tag{B.76}$$

and with $a, b, c \in \{1, 2, 3\}$ we obtain finally

$$[\mathbf{W}_a, \mathbf{W}_b] \left|p_0, \alpha\right\rangle = \mathrm{i}\epsilon_{abc} m W^c \left|p_0, \alpha\right\rangle. \tag{B.77}$$

Thus the operators

$$\mathbf{S}^a = \frac{\mathbf{W}^a}{m} \tag{B.78}$$

fulfil the algebra of angular momentum in the subspace spanned by $|p_0, \alpha\rangle$, and since we have shown above that the three independent components (which are in our case the operators $\mathbf{W}^a$ with $a \in \{1, 2, 3\}$) span the representation of the Lie algebra of the little group with respect to p_0 on the subspace $\mathrm{Eig}(\mathbf{p}, p_0)$, this shows formally what we stated above, namely that the little group in our case is the SU(2) representing the spin of the particle in its rest frame.

Since $\mathbf{W}^2$ is a Casimir operator of the Poincaré group it follows from the irreducibility, which is given if the $|p_0, \alpha\rangle$ span an irreducible representation space of the spin group, it must be $\propto \mathbf{1}$. From (B.78) we find from the known spectrum of the spin operators (B.78)

$$\mathbf{W}^2 = -m^2 s(s+1)\mathbf{1}. \tag{B.79}$$

Thus the representation for the cases (B.53) and (B.54) are uniquely determined by the operating of the Lie algebra operators of the Poincaré group on the Wigner basis:

$$\mathbf{p}^2 \,|m, s; p, \sigma\rangle = m^2 \,|m, s; p, \sigma\rangle \tag{B.80}$$

$$\mathbf{W}^2 \,|m, s; p, \sigma\rangle = -s(s+1)m^2 \,|m, s; p, \sigma\rangle \tag{B.81}$$

$$\mathbf{p}_\mu \,|m, s; p, \sigma\rangle = p_\mu \,|m, s; p, \sigma\rangle \;\text{ with } p^2 = m^2,\; p^0 > 0 \text{ or } p^0 < 0 \tag{B.82}$$

$$\mathbf{U}(a) \,|m, s; p, \sigma\rangle = \exp(\mathrm{i}\mathbf{p}a) \,|m, s; p, \sigma\rangle \tag{B.83}$$

$$\mathbf{U}(\hat{L}) \,|m, s; p, \sigma\rangle = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma'\sigma}[K(\hat{L}, p)] \left|m, s; \hat{L}p, \sigma'\right\rangle \tag{B.84}$$

$$\text{where } K(\hat{L}, p) = \Lambda(\hat{L}p)\hat{L}\Lambda^{-1}(p) \tag{B.85}$$

with $\Lambda(p)$ given by (B.63). Herein we have made use of the known rotation matrices in the representation $D^{(s)}$ and the properties calculated above about the action of the Lorentz group described with help of the irreducible representation of the little group.

B.4.2 Massless Particles

Now we look on the cases (B.55) and (B.56). As the standard vector of the little group we use in the former case (the latter can be treated analogously):

$$p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{B.86}$$

Here the little group is not so simple to determine as in the massive case. For sure there are the rotations around the 3-axis as a subgroup.

To find the little group to the light-like standard vector we use the $\mathrm{SL}(2, \mathbb{C})$ representation. The standard vector is mapped to a mixed spinor of rank two with help of the rule (B.30). Since

$$\det(p_0^{\alpha\dot{\beta}}) = p_0^2 = 0, \tag{B.87}$$

we can express this with help of a spinor κ

$$p_0^{\alpha\dot{\beta}} = \kappa^\alpha \kappa^{*\dot{\beta}} \tag{B.88}$$

which is determined, up to an arbitrary phase, to be

$$(\kappa^\alpha) = \sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}. \tag{B.89}$$

The little group is thus represented by those $\mathrm{SL}(2,\mathbb{C})$ matrices for which the spinor (B.89) is eigenvector with a phase $\exp(\mathrm{i}\alpha/2)$ with $\alpha \in \mathbb{R}$ as eigenvalue. As one can show easily by direct calculation the most general matrix fulfilling these requirements is given by

$$A(b,\alpha) = \begin{bmatrix}\exp\left(\mathrm{i}\frac{\alpha}{2}\right) & b\exp\left(-\mathrm{i}\frac{\alpha}{2}\right)\\ 0 & \exp\left(-\mathrm{i}\frac{\alpha}{2}\right)\end{bmatrix} \text{ with } b \in \mathbb{C},\ \alpha \in \mathbb{R}. \tag{B.90}$$

Thus, also in this case the little group $k(p_0)$ is three-dimensional. The independent real parameters in (B.90) are $\mathrm{Re}\, b$, $\mathrm{Im}\, b$, and α. Applying the $\mathrm{SL}(2,\mathbb{C})$ transformations to the mixed second-rank spinor one sees that for $b = 0$ we obtain the rotations around the 3-axis with angle α. The two other parameters belong to so called "null rotations".

To identify the little group we multiply two group elements given by (B.90) leading to the multiplication law

$$A(b',\alpha')A(b,\alpha) = A[b' + \exp(\mathrm{i}\alpha')b, \alpha' + \alpha], \tag{B.91}$$

showing that it is the $\mathrm{ISO}(\mathbb{R}^2)$ here represented as $\mathrm{ISO}(\mathbb{C})$, i.e., the symmetry group of $\mathbb{R}^2$ or $\mathbb{C}$ as affine point spaces. This group is the semi-direct product of rotations around the origin of $\mathbb{C}$, i.e., the U(1) and translations in $\mathbb{C}$, which is the additive group of $\mathbb{C}$. This is just the same construction as used by building $\mathrm{P}_+^\uparrow$ as the semi-direct product of $\mathrm{SO}(1,3)^\uparrow$ and the translations.

Thus the unitary irreducible representations of this group can be obtained in the same way as we found those of $\mathrm{P}_+^\uparrow$. But the classification of the representations due to the manifolds, the rotations operate transitively on, are much simpler determined since the Euclidean metric is positive definite: namely those with $c = 0$ and $|c| = r \in \mathbb{R}$.

Thus there are only two classes given by the standard vectors of the little group $k'(c_0)$, namely $c_0 = 0 \in \mathbb{C}$ and $c_0 = r$.

In the latter case the little group $k'(c_0)$ is trivial group, i.e., the identity. Nevertheless we have a continuous set c, namely the circle of radius r in the complex plane. With regard to the massless representations of the Poincaré group this corresponds a continuous inner spin-like degree of freedom, which has been never observed so far. We thus exclude these from the possible representations describing particles in nature.

In the former case, $c_0 = 0$, the little group $k'(0)$ is U(1) parameterised by α which corresponds to the rotations around the 3 axis in $k(p_0)$.

Since the U(1) is abelian all irreducible representations are one-dimensional

$$d(\alpha) = \exp(\mathrm{i}\lambda\alpha) \text{ with } \lambda \in \mathbb{R}, \tag{B.92}$$

classifying all covering groups of U(1) by a real number λ. For $\lambda \notin \mathbb{Q}$ it is isomorphic to $\mathbb{R}$ as the covering group.

Now the subgroup SO(3) of rotations of $\mathrm{SO}(3,1)^\uparrow$ should be represented by the representations of SU(2). The rotations around the three-axis, represented by the subgroup of the SU(2), are

coverings of U(1) corresponding to values $\lambda \in \{0, 1/2, 1, 3/2, \ldots\}$. In this way the values of λ are restricted to these half-integer values[1].

We can calculate now the Wigner transformations (B.61) for this case. The Lie algebra of the little group spanned by the Pauli-Lubanski vector operator (B.74) leads to infinitesimal transformations which can be mapped to our parameterisation given by the $\mathrm{SL}(2,\mathbb{C})$ matrices (B.90). A little bit of algebra leads to

$$\delta b = \delta\Phi^2 + i\delta\Phi^1,\ \delta\alpha = \delta\Phi^3 - \delta\Phi^0, \tag{B.93}$$

showing that $\mathbf{W}_3$ generates the rotations around the 3-axis and $\mathbf{W}_1$ and $\mathbf{W}_2$ the null-rotations. Since the little group does not contain null-rotations (corresponding to translations of $I\mathrm{SO}(\mathbb{C})$) these are represented trivially on the subspace $\mathrm{Eig}(\mathbf{p}, p_0)$ which transforms under the operations of the little group $k(p_0)$:

$$\mathbf{W}_1 |p_0, \lambda\rangle = \mathbf{W}_2 |p_0, \lambda\rangle = 0. \tag{B.94}$$

Since we have

$$\mathbf{W}\mathbf{p} |p_0, \lambda\rangle = (\mathbf{W}_0 - \mathbf{W}_3) |p_0, \lambda\rangle = 0 \tag{B.95}$$

and from (B.76) for $\alpha = 0$ and $\beta = 3$ we find together with (B.94) that $[\mathbf{W}_0, \mathbf{W}_3] = 0$ on the subspace $\mathrm{Eig}(\mathbf{p}, p_0)$ the little group operates on. Thus we have

$$\mathbf{W}^2 |p_0, \lambda\rangle = (\mathbf{W}_0^2 - \mathbf{W}_3^2) |p_0, \lambda\rangle = 0. \tag{B.96}$$

Since $\mathbf{W}^2$ is a Casimir operator of the Poincaré group and the representation is irreducible due to Schur's lemma we have $\mathbf{W}^2 = 0$ on the whole Hilbert space.

Since we have $\mathbf{W}^2 = 0$ and $\mathbf{W}\mathbf{p} = 0$ and two orthogonal light-like vectors are proportional, we have $\mathbf{W} = \mu\mathbf{p}$. Using (B.94) together with (B.93) and (B.92) with $\delta b = 0$ we obtain

$$(1 - \mathrm{i}\delta\Phi\mathbf{W}) |p_0, \lambda\rangle = (1 - \mathrm{i}\delta\alpha\mathbf{W}_3) |p_0, \lambda\rangle = (1 + \mathrm{i}\delta\alpha\lambda) |p_0, \lambda\rangle . \tag{B.97}$$

Together with our choice $p_0 = (1, 0, 0, 1)^t$ we obtain $\mu = \lambda$. With the definition of the Pauli Lubanski vector we have

$$\mathbf{W}_\mu = \frac{1}{2}\epsilon_{\rho\sigma\nu\mu}\mathbf{M}^{\rho\sigma}\mathbf{P}^\nu = \lambda\mathbf{p}_\mu. \tag{B.98}$$

Especially for the 0 component we get

$$\mathbf{W}_0 = \sum_{k=1}^{3} \mathbf{S}^k\mathbf{p}^k \text{ with } \mathbf{S}^k = \frac{1}{2}\epsilon_{ijk0}\mathbf{M}^{ij}, \tag{B.99}$$

where $\tilde{\mathbf{S}}$ is the spin operator of the system.

From this we find

$$\boldsymbol{\lambda} = \frac{\tilde{\mathbf{S}}\tilde{\mathbf{p}}}{\|\tilde{\mathbf{p}}\|} \tag{B.100}$$

as the operator with eigenvalue λ, which is known as *helicity*, which is, as we have seen here, a Poincaré-invariant quantum number for massless particles only.

[1] Indeed, up to now the only massless elementary particles are the gauge bosons of the standard model, namely the photon (describing electromagnetic interactions) and the gluons (describing strong interactions). Those all have $\lambda = 1$. To a good approximation also the neutrinos can be described as massless particles, but they are doubtlessly corresponding to $\lambda = 1/2$. On the other hand nowadays, from the observation of neutrino oscillations there is no doubt that at least two of the three standard-model neutrinos must have a mass different from 0.

Since from (B.100) we see that the helicity is the projection of the spin in direction of the momentum of the particle. Since it is a good quantum number for massless particles this defines a definite *chirality* on them.

The physical applications of this appendix, especially in the quantised theory are given in chapter 4.

B.5 The Invariant Scalar Product

In this last short section we define the invariant scalar product in the momentum representation of the one-particle irreducible Hilbert spaces. We denote the irreducible Hilbert spaces by $\mathscr{H}(m, s, \pm)$, where $m \geq 0$ is the quantum number of the Casimir operator $\mathbf{p}^2$ which is $m^2 > 0$ for the physical representations, s is the Casimir operator of the little group, which is $\vec{\boldsymbol{\sigma}}^2$ (spin squared) for the massive and $|\lambda|$ for the massless states, and $\pm$ denotes if we are in the space with positive or negative energy respectively.

The Wigner base kets are denoted by $|m, s, \pm; p, \sigma\rangle$ (or for short hand notation, if we fix the representation and there is no danger of confusion $|p, \sigma\rangle$). The momenta fulfil the energy momentum relation which is given by the *on-shell condition* $p^2 = m^2$ leading to $p^0 = \pm\sqrt{m^2 + \vec{p}^2}$ for the positive or negative energy representations respectively.

Because the representation is constructed to be unitary, the Wigner basis must be orthogonal. This is written as

$$\left\langle p', \sigma' \,\middle|\, p, \sigma \right\rangle = A(\vec{p})\delta^{(3)}(\vec{p} - \vec{p}')\delta_{\sigma\sigma'}. \tag{B.101}$$

From the unitarity of the representation we have on one hand

$$\left\langle p', \sigma' \,\middle|\, \mathbf{U}^\dagger(\hat{L})\mathbf{U}(\hat{L}) \,\middle|\, p, \sigma \right\rangle = \left\langle p', \sigma' \,\middle|\, p, \sigma \right\rangle. \tag{B.102}$$

On the other hand with (B.84) the unitarity of the rotation matrices $D^{(s)}_{\sigma\sigma'}$ we have

$$\left\langle \hat{L}p', \sigma' \,\middle|\, \hat{L}p, \sigma \right\rangle = \left\langle p', \sigma' \,\middle|\, p, \sigma \right\rangle. \tag{B.103}$$

Thus the function $A(\vec{p})\delta^{(3)}(\vec{p} - \vec{p}')$ has to be a $\mathrm{SO}(1,3)^\uparrow$ scalar distribution. To find this distribution we use the fact that $\delta^{(4)}(p - p')$ is a $O(1,3)^\uparrow$ scalar distribution and write it in the following form

$$\begin{aligned} \delta^{(4)}(p - p') &= \pm\Theta(\pm p^0)2p^0\delta[(p^0)^2 - (p'^0)^2]\delta^{(3)}(\vec{p} - \vec{p}') = \\ &= \pm\Theta(\pm p^0)\delta(m^2 - m'^2)\delta^{(3)}(\vec{p} - \vec{p}') \text{ with } p^2 = m^2,\ p'^2 = m'^2. \end{aligned} \tag{B.104}$$

Since in the irreducible subspace we have $m^2 = m'^2$ and $\operatorname{sign} p^0 = \operatorname{sign} p'^0$ fixed, we define using the fact that $\Theta(\pm p^0)\delta(m^2 - m'^2)$ is a $\mathrm{SO}(1,3)^\uparrow$ scalar distribution

$$A(\vec{p}) = (2\pi)^3 2\omega(\vec{p}), \tag{B.105}$$

where the factor $(2\pi)^3$ is an arbitrary factor, introduced by convention.

Thus the relativistic invariant scalar product in momentum representation is given with help of the completeness relation

$$\sum_{\sigma=-s}^{s} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})} \left|p, \sigma\right\rangle \left\langle p, \sigma\right| = 1. \tag{B.106}$$

With help of this the invariant scalar product looks in momentum representation like

$$\langle \phi \,|\, \psi \rangle = \sum_{\sigma=-s}^{s} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega(\vec{p})} \phi^*_\sigma(p)\psi_\sigma(p), \tag{B.107}$$

where the wave functions are defined as

$$\psi_\sigma(p) = \langle p, \sigma \,|\, \psi \rangle \,. \tag{B.108}$$

Appendix C

Formulae

C.1 Amplitudes for various free fields

For using in calculations of S-matrix elements we have to normalise the plane wave solutions for free fields to one particle leading to the correct amplitudes which have to be used in the momentum Feynman rules. The outcome of physical quantities is further independent of the phase we chose for these amplitudes.

For *scalar fields* we have

$$(\Box + m^2)\Phi_{\vec{p}} = 0. \tag{C.1}$$

The positive energy solution with positive energy (i.e. in-fields) is given by

$$\Phi_{\vec{p}}(x) = N(\vec{p}) \exp(-ipx) \text{ with } p^2 = m^2,\ p_0 = +\omega(\vec{p}) := \sqrt{\vec{p}^2 + m^2}. \tag{C.2}$$

The correct normalisation is given by the normalisation condition for the energy of one particle.

$$E(\vec{p}, \vec{p}') = \int d^3\vec{x}[\dot{\Phi}'^*\dot{\Phi} + (\nabla\Phi')^*(\nabla\Phi) + m^2\Phi'^*\Phi] = |N(\vec{p})|^2 2\omega^2(\vec{p})(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}'). \tag{C.3}$$

Here we have used the time-time component of the canonical energy momentum tensor for the field defined in chapter 3 with help of Noether's theorem. Now a particle with three-momentum $\vec{p}$ should carry an energy $\omega(\vec{p})$. Thus we have to set

$$N(\vec{p}) = \frac{1}{\sqrt{2\omega(\vec{p})(2\pi)^3}}, \tag{C.4}$$

which is the amplitude to be used in momentum space Feynman diagrams for the external legs of scalar bosons.

For spin-1/2-fields (Dirac-spinors) we have defined the amplitudes $u_\pm(p, \sigma)$ in chapter 4 with help of the amplitudes for the particles at rest and the normalisation

$$\bar{u}_\pm(\pm p, \sigma)u_\pm(\pm p, \sigma) = \pm 2m. \tag{C.5}$$

Then we write for the plane wave of an incoming particle

$$\psi_{\vec{p},\sigma}(x) = N(\vec{p})u_+(p, \sigma) \exp(-ipx) \tag{C.6}$$

The same argument as given above for the scalar particle leads to

$$N(\vec{p}) = \frac{1}{\sqrt{2\omega(2\pi)^3}}. \tag{C.7}$$

C.2 Dimensional regularised Feynman-integrals

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^{\alpha}} = \frac{\mathrm{i}}{(4\pi)^{\omega}} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{1}{(q^2 + m^2)^{\alpha - \omega}}. \tag{C.8}$$

$$\int \frac{d^{2\omega}}{(2\pi)^{2\omega}} \frac{p_\mu}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^{\alpha}} = -\frac{\mathrm{i}}{(4\pi)^{\omega}} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{q_\mu}{(q^2 + m^2)^{\alpha - \omega}}. \tag{C.9}$$

$$\begin{aligned} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\mu p_\nu}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^{\alpha}} &= \frac{\mathrm{i}}{(4\pi)^{\omega}\Gamma(\alpha)} \frac{1}{(q^2 + m^2)^{\alpha - \omega}} \times \\ &\times \left[q_\mu q_\nu \Gamma(\alpha - \omega) - \frac{1}{2} g_{\mu\nu} (q^2 + m^2) \Gamma(\alpha - \omega - 1) \right]. \end{aligned} \tag{C.10}$$

$$\begin{aligned} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{p^2}{(m^2 - p^2 - 2pq - \mathrm{i}\eta)^{\alpha}} &= \frac{\mathrm{i}}{(4\pi)^{\omega}\Gamma(\alpha)} \frac{1}{(q^2 + m^2)^{\alpha - \omega}} \times \\ &\times [q^2 \Gamma(\alpha - \omega) - \omega (q^2 + m^2) \Gamma(\alpha - \omega - 1)]. \end{aligned} \tag{C.11}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^2}{(m^2 - l^2 - \mathrm{i}\eta)^{\alpha}} = \frac{\mathrm{i}}{(4\pi)^{\omega}} \omega(\omega + 1) \frac{\Gamma(\alpha - 2 - \omega)}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha - 2 - \omega}} \tag{C.12}$$

$$\begin{aligned} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(m^2 - l^2 - \mathrm{i}\eta)^{\alpha}} &= \frac{\mathrm{i}}{(4\pi)^{\omega}} \frac{\Gamma(\alpha - 2 - \omega)}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha - 2 - \omega}} \times \\ &\times \frac{1}{4} \left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right) \end{aligned} \tag{C.13}$$

C.3 Laurent expansion of the Γ-Function

$$\forall n \in \mathbb{N} : \ \Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \Psi_1(n + 1) + O(\epsilon) \right]. \tag{C.14}$$

$$\Psi_1(1) = -\gamma, \ \forall n \geq 1 : \ \Psi_1(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \tag{C.15}$$

Herein $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant.

C.4 Feynman's Parameterisation

$$\begin{aligned}
\frac{1}{ab} &= \int_0^1 \frac{\mathrm{d}x}{[ax+b(1-x)]^2} \\
\frac{1}{abc} &= 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{[a(1-x-y)+bx+cy]^3} \\
\frac{1}{\prod_{k=1}^m b_k^{\alpha_k}} &= \frac{\Gamma(\sum_{k=1}^m \alpha_k)}{\prod_{k=1}^m \Gamma(\alpha_m)} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{m-2}} \mathrm{d}x_{m-1} \times \\
&\quad \times \frac{x_{m-1}^{\alpha_1-1}(x_{m-2}-x_{m-1})^{\alpha_2-1}\cdots(1-x_1)^{\alpha_m-1}}{[b_1 x_{m-1} + b_2(x_{m-2}-x_{m-1}) + \cdots + b_m(1-x_1)]^{\sum_{k=1}^m \alpha_k}}
\end{aligned} \tag{C.16}$$

Bibliography

[Abb81] L. F. Abbott, The background field method beyond one loop, Nucl. Phys. B **185** (1981) 189.
URL https://www-lib.kek.jp/cgi-bin/img_index?198012137

[Abb82] L. F. Abbott, Introduction to the background field method, Acta Phys. Polon. B **13** (1982) 33.
URL https://www-lib.kek.jp/cgi-bin/img_index?198107383

[Adl69] S. L. Adler, Axial vector vertex in spinor electrodynamics, Phys. Rev. **177** (1969) 2426.
URL https://doi.org/10.1103/PhysRev.177.2426

[AGS83] L. F. Abbott, M. T. Grisaru, R. K. Schaefer, The background field method and the S Matrix, Nucl. Phys. B **229** (1983) 372.
URL https://www-lib.kek.jp/cgi-bin/img_index?8308278

[AL73] E. S. Abers, B. W. Lee, Gauge Theories, Phys. Rept. **9** (1973) 1.

[Bel39] F. J. Belinfante, On the spin angular momentum of mesons, Physica **6** (1939) 887.
URL https://doi.org/10.1016/S0031-8914(39)90090-X

[BJ69] J. S. Bell, R. Jackiw, A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the sigma model, Nuovo Cim. A **60** (1969) 47.
URL https://doi.org/10.1007/BF02823296

[BL86] D. Bailin, A. Love, Introduction to Gauge Field Theory, Adam Hilger, Bristol and Boston (1986).

[BN37] F. Bloch, A. Nordsieck, Note on the radiation field of the electron, Phys. Rev. **52** (1937) 54.
URL https://link.aps.org/doi/10.1103/PhysRev.52.54

[BP57] N. N. Bogoliubov, O. S. Parasiuk, On the Multiplication of the causal function in the quantum theory of fields, Acta Math. **97** (1957) 227.

[CM74] J. C. Collins, A. J. Macfarlane, New methods for the renormalization group, Phys. Rev. D **10** (1974) 1201.
URL https://doi.org/10.1103/PhysRevD.10.1201

[Col74] J. C. Collins, Scaling behavior of ϕ^4 theory and dimensional regularization, Phys. Rev. D **10** (1974) 1213.
URL https://doi.org/10.1103/PhysRevD.10.1213

[Col86] J. C. Collins, Renormalization, Cambridge University Press, Cambridge, New York, Melbourne (1986).

[Dys49a] F. J. Dyson, The Radiation theories of Tomonaga, Schwinger, and Feynman, Phys. Rev. **75** (1949) 486.
URL https://doi.org/10.1103/PhysRev.75.486

[Dys49b] F. J. Dyson, The S matrix in quantum electrodynamics, Phys. Rev. **75** (1949) 1736.
URL https://doi.org/10.1103/PhysRev.75.1736

[Fey49] R. Feynman, The Theory of positrons, Phys. Rev. **76** (1949) 749.
URL https://doi.org/10.1103/PhysRev.76.749

[FLS72] K. Fujikawa, B. W. Lee, A. I. Sanda, Generalized Renormalizable Gauge Formulation of Spontaneously Broken Gauge Theories, Phys. Rev. D **6** (1972) 2923.
URL https://link.aps.org/abstract/PRD/v6/i10/p2923

[Fuj79] K. Fujikawa, Path-Integral Measure for Gauge-Invariant Fermion Theories, Phys. Rev. Lett. **42** (1979) 1195.
URL https://doi.org/10.1103/PhysRevLett.42.1195

[Fuj80] K. Fujikawa, Path Integral for Gauge Theories with Fermions, Phys. Rev. D **21** (1980) 2848, [Erratum: Phys. Rev.D **22**,1499(1980)].
URL https://doi.org/10.1103/PhysRevD.21.2848,10.1103/PhysRevD.22.1499

[Gol61] J. Goldstone, Field theories with 'superconductor' solutions, Nuovo Cim. **19** (1961) 154.
URL https://doi.org/10.1007/BF02812722

[GW73] D. Gross, F. Wilczek, Ultraviolet Behavior of Nonabelian Gauge Theories, Phys. Rev. Lett. **30** (1973) 1343.
URL https://doi.org/10.1103/PhysRevLett.30.1343

[JL64] G. Jona-Lasinio, Relativistic field theories with symmetry breaking solutions, Nuovo Cim. **34** (1964) 1790.
URL https://doi.org/10.1007/BF03156476

[Kak93] M. Kaku, Quantum Field Theory, Oxford University Press (1993).

[Koc97] V. Koch, Aspects of chiral symmetry, Int. J. Mod. Phys. E **6** (1997) 203.
URL https://doi.org/10.1142/S0218301397000147

[Kug77] T. Kugo, Symmetric and mass-independent renormalization, Prog. Theor. Phys. **57** (1977) 593.
URL https://doi.org/10.1143/PTP.57.593

[Kug97] T. Kugo, Eichtheorie, Springer-Verlag, Heidelberg (1997).

[LSZ55] H. Lehmann, K. Symanzik, W. Zimmermann, On the formulation of quantized field theories, Nuovo Cim. **1** (1955) 205.
URL https://doi.org/10.1007/BF02731765

[LSZ57] H. Lehmann, K. Symanzik, W. Zimmermann, On the formulation of quantized field theories. II, Nuovo Cim. **6** (1957) 319.
URL https://doi.org/10.1007/BF02832508

[LZJ72a] B. W. Lee, J. Zinn-Justin, Spontaneously Broken Gauge Symmetries I. Preliminaries, Phys. Rev. D **5** (1972) 3121.
URL https://link.aps.org/abstract/PRD/v5/i12/p3121

[LZJ72b] B. W. Lee, J. Zinn-Justin, Spontaneously Broken Gauge Symmetries II. Perturbation Theory and Renormalization, Phys. Rev. D **5** (1972) 3137.
URL https://link.aps.org/abstract/PRD/v5/i12/p3137

[LZJ72c] B. W. Lee, J. Zinn-Justin, Spontaneously Broken Gauge Symmetries III. Equivalence, Phys. Rev. D **5** (1972) 3155.
URL https://link.aps.org/abstract/PRD/v5/i12/p3155

[Mut87] T. Muta, Foundations of Quantum Chromodynamics, World Scientific (1987).

[NJL61] Y. Nambu, G. Jona-Lasinio, Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I, Phys. Rev. **122** (1961) 345.
URL https://doi.org/10.1103/PhysRev.122.345

[O+14] K. Olive, et al., Review of Particle Physics, Chin. Phys. C **38** (2014) 090001.
URL https://doi.org/10.1088/1674-1137/38/9/090001

[Pol73] H. D. Politzer, Reliable Perturbative Results for Strong Interactions?, Phys. Rev. Lett. **30** (1973) 1346.
URL https://doi.org/10.1103/PhysRevLett.30.1346

[PS95] M. Peskin, D. V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley Publ. Comp., Reading, Massachusetts (1995).

[Ram89] P. Ramond, Field Theory: A Modern Primer, Addison-Wesley, Redwood City, Calif., 2 ed. (1989).

[Sch14] M. D. Schwartz, Quantum field theory and the Standard Model, Cambridge University Press, Cambridge, New York (2014).

[Ste49] J. Steinberger, On the Use of subtraction fields and the lifetimes of some types of meson decay, Phys. Rev. **76** (1949) 1180.
URL https://doi.org/10.1103/PhysRev.76.1180

[Tay76] J. C. Taylor, Gauge Theories of Weak Interactions, Cambridge University Press (1976).

[Wei60] S. Weinberg, High-Energy Behavior in Quantum Field Theory, Phys. Rev. **118** (1960) 838.
URL https://link.aps.org/abstract/PR/v118/i3/p838

[Wei95] S. Weinberg, The Quantum Theory of Fields, vol. 1, Cambridge University Press (1995).

[Wei96] S. Weinberg, The Quantum Theory of Fields, vol. 2, Cambridge University Press (1996).

[Wic54] G. C. Wick, Properties of Bethe-Salpeter Wave Functions, Phys. Rev. **96** (1954) 1124.
URL `https://doi.org/10.1103/PhysRev.96.1124`

[Zim68] W. Zimmermann, The Power Counting Theorem for Minkowski Metric, Comm. Math. Phys. **11** (1968) 1.
URL `https://doi.org/10.1007/BF01654298`

[Zim69] W. Zimmermann, Convergence of Bogoliubov's Method of Renormalization in Momentum Space, Commun. Math. Phys. **15** (1969) 208.
URL `https://doi.org/10.1007/BF01645676`

[Zim70] W. Zimmermann, Local Operator Products and Renormalization in Quantum Field Theory, Lectures on Elementary Particles and Quantum Field Theory. M.I.T. Press (1970) 397.