Renormalized Φ-Derivable Approximations to Theory with Spontaneously Broken $O(N)$ Symmetry

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Abstract

The renormalization of a gapless Φ-derivable Hartree–Fock approximation to the $O(N)$-symmetric $\lambda\phi^4$ theory is considered in the spontaneously broken phase. This kind of approach was proposed in our previous paper [1] in order to preserve all the desirable features of Φ-derivable Dyson-Schwinger resummation schemes (i.e., validity of conservation laws and thermodynamic consistency) while simultaneously restoring the Nambu–Goldstone theorem in the broken phase. It is shown that unlike for the conventional Hartree–Fock approximation this approach allows for a scale-independent renormalization in the vacuum. However, the scale dependence still persists at finite temperatures. Various branches of the solution are studied. The occurrence of a limiting temperature inherent in the renormalized Hartree–Fock approximation at fixed renormalization scale μ is discussed.

I. INTRODUCTION

Self-consistent Φ-derivable approximations were introduced long ago in the context of the nonrelativistic many-body problem [2, 3] and then extended to relativistic quantum field theory [4, 5]. Recently the interest in this method has been revived in view of its fruitful applications to calculations of the thermodynamic properties of the quark–gluon plasma [6] and to non-equilibrium quantum-field dynamics [7–9], in particular in terms of the off-shell kinetic equation [10].

Φ-derivable approximations are preferable for the dynamical treatment of a system, since they fulfill the conservation laws of energy, momentum, and charge [3, 7, 10]. Moreover, the Φ-derivable scheme also guarantees the thermodynamic consistency of an approximation [3], which makes it advantageous also for thermodynamic calculations. However, the Φ-derivable

scheme has its generic problems which were also realized long ago [5, 11].

The first problem is related to the fact that Φ-derivable Dyson-Schwinger resummation schemes violate Ward–Takahashi identities beyond the one-point level. This, in particular, results in the violation of the Nambu–Goldstone (NG) theorem [5, 11–13] in the phase of spontaneously broken symmetry. On the other hand, so called "gapless" approximations [11] respect the NG theorem (which is referred to as the Hugenholtz–Pines theorem in physics of Bose–Einstein condensed systems), though violate conservation laws and thermodynamic consistency. In Ref. [13] it was shown that any Φ-derivable approximation can be corrected in such a way that it respects the NG theorem and becomes gapless. However, such modifications again violate conservation laws and thermodynamic consistency and, hence, leads back to the problems of the gapless scheme. Recently we have proposed a phenomenological way to construct a "gapless Φ-derivable" Hartree–Fock (gHF) approximation to the $\lambda\phi^4$ theory in the phase of spontaneously broken $O(N)$ symmetry [1]. This approximation simultaneously pre-

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serves all the desirable features of Φ-derivable schemes and respects the Nambu–Goldstone theorem in the broken phase. The treatment of Ref. [1] was based on a naive renormalization, where all divergent terms were simply omitted. This was done in order to avoid possible confusions between effects of restoring the NG theorem and those related to renormalization. In the present paper we return to the issue of renormalization.

The renormalization of the Φ-derivable approximations is precisely the second main problem. Following Baym and Grinstein [5] it was believed that renormalization of Φ-derivable approximations is possible only with medium (e.g., temperature) dependent counter terms, which is inconsistent with the goal of renormalization. Great progress in the proper renormalization of such schemes was recently achieved in Refs. [13, 14, 16–18]. As the main result it was shown that partial resummation schemes can indeed be renormalized with medium-independent counter terms provided the scheme is generated from a two-particle irreducible (2PI) functional, i.e., a Φ-functional. Still, as we are going to demonstrate below, certain problems remain in the case of spontaneously broken symmetry.

As an example case we investigate the $O(N)$ model in the spontaneously broken phase which is a traditional touchstone for new theoretical approaches, well applied to a variety of physical phenomena, such as the chiral phase transition in nuclear matter. Thus we continue the discussion of the "gapless Hartree-Fock approximation started in our previous paper [1] and investigate its features towards renormalization in comparison to the standard HF-approximation.

II. GAPLESS HARTREE–FOCK (gHF) APPROXIMATION

We consider the $O(N)$-model Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_a)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4N}(\phi^2)^2 + H \cdot \phi, \tag{1}$$

where $\phi = (\phi_1, \phi_2, ..., \phi_N)$ is an N-component scalar field, $\phi^2 = \phi_a\phi_a$, with summation over a

implied. For $H = 0$ this Lagrangian is invariant under $O(N)$ rotations of the fields. If $H = 0$ and $m^2 < 0$, the symmetry of the ground state is spontaneously broken down to $O(N-1)$, with $N - 1$ Goldstone bosons (pions). The external field $H \cdot \phi = H_a\phi_a$ is a term which explicitly breaks the $O(N)$ symmetry. It is introduced to give the physical value of 140 MeV to the pion mass.

The effective action Γ for this Lagrangian in the real-time formalism is defined as (cf. Ref. [7])

$$\Gamma\{\phi, G\} = I_0(\phi) + \frac{\mathrm{i}}{2}\mathrm{Tr}\left(\ln G^{-1}\right) + \frac{\mathrm{i}}{2}\mathrm{Tr}\left(D^{-1}G - 1\right) + \Phi_{\text{real-time}}\{\phi, G\}, \tag{2}$$

where ϕ is the expectation value of the field, G is the Green's function, D is the *free* Green's function, $I_0(\phi)$ is the *free* classical action of the ϕ field, Tr implies space–time integration and summation over field indices $a, b, ...$ All the considerations below are performed in terms of the thermodynamic Φ functional which differs from $\Phi_{\text{real-time}}$ in the factor of $\mathrm{i}\beta$, where $\beta = 1/T$ is the inverse temperature. In the case of a spatially homogeneous thermodynamic system, an additional factor appears: the volume V of the system. Thus, the thermodynamic Φ is

$$\Phi = (-\mathrm{i}T/V)\Phi_{\text{real-time}}.$$

In terms of the CJT formalism [4], the same effective action Γ is given as (e.g., cf. Ref. [8])

$$\Gamma\{\phi, G\} = I(\phi) + \frac{\mathrm{i}}{2}\mathrm{Tr}\left(\ln G^{-1}\right) + \frac{\mathrm{i}}{2}\mathrm{Tr}\left(D_\phi^{-1}G - 1\right) + \Gamma_2\{\phi, G\}, \tag{3}$$

i.e., already in terms of the *tree-level* Green's function, $D_\phi(1, 2) = \delta^2 I/\delta\phi_2\delta\phi_1$, and the *full* classical action of the ϕ field, $I(\phi)$. In the thermodynamic limit, the effective potential, V_{CJT}, and its interaction part, V_2, are defined as

$$V_{\text{CJT}} = (-\mathrm{i}T/V)\Gamma, \quad V_2 = (-\mathrm{i}T/V)\Gamma_2.$$

Naturally, V_2 is similar to Φ but is not quite the same. Contrary to V_2, the Φ functional includes all 2PI interaction terms, i.e. also those of zero

and first loop order which result from interactions with the classical field (first two graphs in (4)).

The gHF approximation to the $O(N)$ theory is defined by the Φ functional [1]

$$\Phi_{\mathrm{gHF}} = \text{} + \text{} + \text{} + \Delta\Phi, \quad (4)$$

where the diagrams on the r.h.s. constitutes the conventional HF approximation, while *the phenomenological NG-theorem-restoring correction* $\Delta\Phi$ is specified below, see Eq. (9). Here the crossed pins denote the classical fields ϕ_a, and loops are tadpoles

$$\text{} = Q_{ab} = \int_\beta d^4k\, G_{ab}(k) \qquad (5)$$

in terms of G_{ab} Green's functions, where the Matsubara summation

$$\int_\beta d^4q\, f(q) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3q}{(2\pi)^3} f(2\pi \mathrm{i} nT, \vec{q}) \quad (6)$$

is implied with T being a finite temperature.

Within the Φ-derivable scheme the r.h.s. of the equations of motion for the classical field (J) and the Green's function (self-energy Σ) follow from the functional variation of Φ_{gHF} with respect to the classical field ϕ and Green's function G, respectively

$$\Box\phi + m^2\phi = J = \frac{\delta\Phi_{\mathrm{gHF}}}{\delta\phi}$$
$$= \text{} + \text{}, \quad (7)$$

$$G^{-1} - D^{-1} = \Sigma = 2\frac{\delta\Phi_{\mathrm{gHF}}}{\delta G}$$
$$= \text{} + \text{} + 2\frac{\delta\Delta\Phi}{\delta G}, \quad (8)$$

where D is the free propagator.

The $\Delta\Phi$ correction, introduced in Ref. [1], is unambiguously determined proceeding from the following requirements: (i) it restores the NG theorem in the broken-symmetry phase, (ii) it does not change results in the phase of restored $O(N)$ symmetry, because there is no need for it, (iii) it does not change the HF equation for the classical field, since the conventional Φ-derivable and gapless schemes [5, 11] provide the same classical-field equation already without any modifications. In particular, due to this latter requirement the $\Delta\Phi$ correction does not contribute to the classical-field equation (7). Proceeding from these requirements, this $\Delta\Phi$ can be presented in manifestly $O(N)$ symmetric form

$$\Delta\Phi = -\frac{\lambda}{2N}\left[N(Q_{ab})^2 - (Q_{aa})^2\right]. \quad (9)$$

Here and below, summation over repeated indices $a, b, c, ...$ is implied, if it is not pointed out otherwise.

The nature of this correction can be understood as follows. For the full theory, i.e., when all diagrams in the Φ functional are taken into account, the gapless and Φ-derivable schemes are identical and both respect the NG theorem. The conventional Φ-derivable HF approximation omits an infinite set of diagrams which is necessary to restore its equivalence with the gapless scheme. The $\Delta\Phi$ correction to the HF approximation takes into account a part of those omitted diagrams (at the level of the actual approximation), and thus restores this equivalence in the pion sector. For the further discussion we switch to the notation in terms of the CJT effective potential, see e.g. [4, 19, 20], in order to comply with previous considerations in the literature.

The manifestly symmetric form of the CJT effective potential in the gHF approximation reads

$$V_{\text{gHF}}(\phi, G) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4N}(\phi^2)^2 - H \cdot \phi + \frac{1}{2}\int_\beta d^4k \ln \det G^{-1}(k)$$

$$+ \frac{1}{2}\int_\beta d^4k \left\{ \left[(k^2 + m^2)\,\delta_{ab} + \frac{\lambda}{N}\left(\phi^2\delta_{ab} + 2\phi_a\phi_b\right) \right] G_{ba}(k) - 1 \right\}$$

$$+ \frac{\lambda}{4N}(Q_{aa}Q_{bb} + 2Q_{ab}Q_{ba}) + \Delta\Phi, \tag{10}$$

e.g., cf. [20]. All quantities are symmetric with respect to permutations of indices.

Since the self-energy (8) is momentum independent, the general form of the Green's function can be written as follows

$$G_{ab}^{-1}(k) = k^2\delta_{ab} + M_{ab}^2, \tag{11}$$

where M_{ab}^2 is a constant mass matrix. The equations for G_{cd}, i.e., for the corresponding tadpoles Q_{ab}, and the fields ϕ_c result from variations of V_{gHF} over G_{cd} and ϕ_c, respectively,

$$M_{cd}^2 = m^2\delta_{cd} \tag{12}$$
$$+ \frac{\lambda}{N}\left[\phi^2\delta_{cd} + 2\phi_c\phi_d + 3Q_{aa}\delta_{cd} + 2(1-N)Q_{cd}\right],$$

$$H_c = m^2\phi_c + \frac{\lambda}{N}\left[\phi^2\phi_c + Q_{aa}\phi_c + 2Q_{cd}\phi_d\right]. \tag{13}$$

These are equations in a general nondiagonal representation. Applying projectors

$$\Pi_{cd}^\pi = \frac{1}{N-1}\left(\delta_{cd} - \phi_c\phi_d/\phi^2\right), \tag{14}$$

$$\Pi_{cd}^\sigma = \phi_c\phi_d/\phi^2, \tag{15}$$

to Eq. (12), we project it on π and σ states. In order to project the mean-field equation (13) on the σ-direction, we just multiply it by ϕ_c.

In the diagonal representation ($\phi_\sigma \neq 0$, $H_\sigma = H$ and $H_\pi = \phi_\pi = 0$) these equations take the following form

$$M_\sigma^2 = m^2 + \frac{\lambda}{N}\left[3\phi^2 + (5 - 2N)Q_\sigma + 3(N-1)Q_\pi\right]$$
$$= M_\pi^2 + \frac{\lambda}{N}\left[2\phi^2 + 2(N-1)(Q_\pi - Q_\sigma)\right], \tag{16}$$

$$M_\pi^2 = m^2 + \frac{\lambda}{N}\left[\phi^2 + 3Q_\sigma + (N-1)Q_\pi\right], \tag{17}$$

$$H = \phi\left[m^2 + \frac{\lambda}{N}\left(\phi^2 + 3Q_\sigma + (N-1)Q_\pi\right)\right], \tag{18}$$

where $M_\pi^2 = \Pi_{dc}^\pi M_{cd}^2$ and $M_\sigma^2 = \Pi_{dc}^\sigma M_{cd}^2$. Here we used $Q_\sigma = Q_{\sigma\sigma}$ and $Q_\pi = Q_{\pi\pi}$ in terms of definition (5). From these equations it is evident that the NG theorem is fulfilled. Indeed, in the phase of spontaneously broken symmetry ($H = 0$) the square-bracketed term of the field equation (18) equals zero, which is precisely the pion mass, cf. Eq. (17). At the same time, as it has been demonstrated in numerous papers (see, e.g., Refs. [5, 11, 12, 19]), the solution of the conventional HF set of equations (7)–(8), i.e., without $\Delta\Phi$, violates the NG theorem. A detailed analysis of the conventional HF equations in a notation similar to ours has been given in Ref. [19].

III. RENORMALIZATION OF THE gHF APPROXIMATION

Significant progress in proper renormalization of Φ-derivable approximations was recently achieved in Refs. [13, 14, 16]. Here we follow the renormalization scheme of Ref. [13], i.e., that constructed precisely for the conventional HF approximation to $\lambda\phi^4$ theory in the $O(N)$ broken phase. This renormalization is based on the BPHZ formalism.

A. Equations of Motion

The equations of motion (16)–(18) involve tadpole terms which, based on the explicit form of the Green's function (11), can be written as

$$Q_a = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon_a(\vec{k})}\left[n\left(\epsilon_a(\vec{k})\right) + \frac{1}{2}\right], \tag{19}$$

where $\epsilon_a(\vec{k}) = (\vec{k}^2 + M_a^2)^{1/2}$ and

$$n(\epsilon) = \frac{1}{\exp(\epsilon/T) - 1} \qquad (20)$$

is the thermal occupation number. Evidently, the Q-function consists of two parts

$$Q_a = Q_a^T + Q_a^{(\text{div})}, \qquad (21)$$

where

$$Q_a^T = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_a(\vec{k})} n\left(\epsilon_a(\vec{k})\right) \qquad (22)$$

is the convergent thermal part of the tadpole, which is finite, and the divergent part

$$\begin{aligned} Q_a^{(\text{div})} &= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_a(\vec{k})} \\ &= \frac{M_a^2}{(4\pi)^2} \left(-\frac{1}{\epsilon} + \ln\frac{M_a^2}{\mu^2} - 1\right) \end{aligned} \quad (23)$$

is regularized within dimensional regularization. Here $\epsilon \to 0$, and μ is a regularization scale. We apply the same mass independent renormalization conditions as in Ref. [13], i.e., that in the *symmetric vacuum* the self-energies vanish [21]

$$\Sigma_a(T = 0, \phi = 0, m^2 = \mu^2 > 0) = 0, \quad (24)$$
$$\partial_{m^2}\Sigma_a(T = 0, \phi = 0, m^2 = \mu^2 > 0) = 0, \quad (25)$$

for all a. Such a renormalization scheme preserves the O(N) symmetry of the model [21]. Since Σ_a is momentum independent in the approximation under consideration, additional momentum-derivative conditions are not required. Upon application of this scheme, Eqs. (16)–(18) keep their form with the Q_a quantities substituted by the renormalized tadpoles.

$$\begin{aligned} Q_a^{(\text{ren})} &= Q_a^T \\ &+ \frac{1}{(4\pi)^2}\left[M_a^2\left(\ln\frac{M_a^2}{\mu^2} - 1\right) + \mu^2\right]. \end{aligned} \quad (26)$$

As it was shown in Ref. [13], this renormalization description requires only vacuum (temperature independent) counter terms. For the sake of further discussion, note that according to Ref. [13] the renormalization of the conventional HF approximation results in precisely the same equations as those in the CT scheme of Ref. [19], in spite of the different approaches used. The renormalized conventional HF approximation was thoroughly studied in [19]. Therefore, those results are very useful for comparison with the present treatment.

B. Effective Potential

The thermodynamic potential (10) is renormalized following the procedure outlined in Ref. [14]. In the gHF approximation complications arise only in the $\ln\det G^{-1}(k)$ term. Because of the topology of the diagrams used in the gHF approximation, for all other contributions to the effective action, we only have to insert the already renormalized self-energies, i.e., $Q_a^{(\text{ren})}$ tadpoles of Eq. (26), to renormalize them. This is legitimate, because we do not encounter any additional contributions from subdivergences, cf. Ref. [14]. In the diagonal representation the remaining part to be renormalized takes the form

$$\frac{1}{2}\int_\beta d^4k \ln\det G^{-1}(k) = L_\sigma + (N-1)L_\pi, \quad (27)$$

where we have introduced the brief notation

$$\begin{aligned} L_a &= \frac{1}{2}\int_\beta d^4k \ln G_a^{-1}(k) \\ &= \int \frac{d^3k}{(2\pi)^3}\left\{\frac{\epsilon_a}{2} + T\ln\left[1 - \exp\left(-\frac{\epsilon_a}{T}\right)\right]\right\} \end{aligned} \quad (28)$$

The L_a also consists of two parts: the convergent thermal part

$$L_a^T = T\int \frac{d^3k}{(2\pi)^3}\ln\left[1 - \exp\left(-\frac{\epsilon_a(\vec{k})}{T}\right)\right], \quad (29)$$

which is finite, and the divergent integral

$$L_a^{(\text{div})} = \int \frac{d^3k}{(2\pi)^3}\frac{\epsilon_a(\vec{k})}{2}. \qquad (30)$$

This expression implicitly depends on temperature through the mass M_a. We regularize it by means of a momentum cut-off Λ:

$$L_a^{(\mathrm{reg})}(M_a) = \int \frac{d^3k}{(2\pi)^3} \frac{\epsilon_a(\vec{k})}{2} \Theta\left(\Lambda^2 - \epsilon_a^2(\vec{k})\right) = \frac{1}{3(2\pi)^2}\left(\left(\Lambda^2 - M_a^2\right)^{3/2}\Lambda\right.$$
$$\left. - \frac{1}{8}\left[2\Lambda^3\sqrt{\Lambda^2 - M^2} - 5\Lambda M^2\sqrt{\Lambda^2 - M^2} - 3M^4\ln M + 3M^4\ln\left(\Lambda + \sqrt{\Lambda^2 - M^2}\right)\right]\right) \quad (31)$$

To renormalize the effective potential, we use a mass-independent renormalization scheme in order to avoid effects of unphysical IR singularities. We impose the following renormalization conditions on the effective potential

$$V^{(\mathrm{ren})}\left(T=0, \phi=0, m^2\right)\big|_{m^2=\mu^2>0} = 0, \quad (32)$$

$$\partial_{m^2}V^{(\mathrm{ren})}\left(T=0, \phi=0, m^2\right)\big|_{m^2=\mu^2>0} = 0, \quad (33)$$

$$\partial_{m^2}^2 V^{(\mathrm{ren})}\left(T=0, \phi=0, m^2\right)\big|_{m^2=\mu^2>0} = 0. \quad (34)$$

We need precisely these three conditions to remove all the divergences. Imposing these conditions, we keep in mind that other parts of the effective potential, except for L_a, have already been renormalized such that they fulfill these conditions on their own. This leads to

$$L_a^{(\mathrm{ren})} = L_a^T + \lim_{\Lambda\to\infty}\left[L_a^{(\mathrm{reg})}(M_a) - L_a^{(\mathrm{reg})}(\mu) - (M_a^2 - \mu^2)\frac{\partial L_a^{(\mathrm{reg})}(\mu)}{\partial\mu^2} - \frac{1}{2}(M_a^2 - \mu^2)^2\frac{\partial^2 L_a^{(\mathrm{reg})}(\mu)}{\partial(\mu^2)^2}\right]$$
$$= L_a^T - \frac{1}{128\pi^2}\left(3M^4 - 4M^2\mu^2 + \mu^4 - 2M^4\ln\frac{M^2}{\mu^2}\right). \quad (35)$$

In terms of these $L_a^{(\mathrm{ren})}$ and $Q_a^{(\mathrm{ren})}$ the renormalized effective potential reads

$$V_{\mathrm{gHF}}^{(\mathrm{ren})}(\phi, T) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4N}\phi^4 - H\phi + L_\sigma^{(\mathrm{ren})} + (N-1)L_\pi^{(\mathrm{ren})}$$
$$+\frac{1}{2}\left[m^2\left(Q_\sigma^{(\mathrm{ren})} + (N-1)Q_\pi^{(\mathrm{ren})}\right) - M_\sigma^2 Q_\sigma^{(\mathrm{ren})} - (N-1)M_\pi^2 Q_\pi^{(\mathrm{ren})} + \frac{\lambda}{N}\phi^2\left(3Q_\sigma^{(\mathrm{ren})} + (N-1)Q_\pi^{(\mathrm{ren})}\right)\right]$$
$$+\frac{\lambda}{4N}\left[3\left(Q_\sigma^{(\mathrm{ren})} + (N-1)Q_\pi^{(\mathrm{ren})}\right)^2 - 2(N-1)\left(\left[Q_\sigma^{(\mathrm{ren})}\right]^2 + (N-1)\left[Q_\pi^{(\mathrm{ren})}\right]^2\right)\right]. \quad (36)$$

C. Vacuum ($T = 0$)

At $T = 0$, the quantities under investigation are "experimentally" known[1]: $M_\pi(T=0) = m_\pi = 139$ MeV, $M_\sigma(T=0) = m_\sigma = 600$ MeV, and the pion decay constant $\phi_0 = f_\pi = 93$ MeV. At $T = 0$ these known quantities should satisfy Eqs. (16)–(18) with renormalized tadpoles $Q_a^{(\mathrm{ren})}$, cf. Eq. (26),

$$m_\sigma^2 = m_\pi^2 + \frac{2\lambda}{N}f_\pi^2 + \frac{2(N-1)\lambda}{(4\pi)^2 N}\left[m_\pi^2\left(\ln\frac{m_\pi^2}{\mu^2} - 1\right) - m_\sigma^2\left(\ln\frac{m_\sigma^2}{\mu^2} - 1\right)\right], \quad (37)$$

$$m_\pi^2 = m^2 + \frac{\lambda}{N}f_\pi^2 + \frac{\lambda}{(4\pi)^2 N}\left[(N+2)\mu^2 + 3m_\sigma^2\left(\ln\frac{m_\sigma^2}{\mu^2} - 1\right) + (N-1)m_\pi^2\left(\ln\frac{m_\pi^2}{\mu^2} - 1\right)\right], \quad (38)$$

$$\frac{H}{f_\pi} = m^2 + \frac{\lambda}{N}f_\pi^2 + \frac{\lambda}{(4\pi)^2 N}\left[(N+2)\mu^2 + 3m_\sigma^2\left(\ln\frac{m_\sigma^2}{\mu^2} - 1\right) + (N-1)m_\pi^2\left(\ln\frac{m_\pi^2}{\mu^2} - 1\right)\right]. \quad (39)$$

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[1] These values are relevant for the case $N = 4$.

In order to make this set of equations consistent, we should put

$$H = m_\pi^2 f_\pi, \tag{40}$$

i.e., precisely the same as at the tree level. Then Eqs. (38) and (39) become identical. Now we can solve for the remaining equations, (37) and (38), and express the renormalized quantities m^2 and λ in terms of physical quantities m_π, m_σ, and f_π

$$\lambda = N \left(m_\sigma^2 - m_\pi^2\right) \left\{ 2f_\pi^2 + \frac{2(N-1)}{(4\pi)^2} \left[m_\pi^2 \left(\ln \frac{m_\pi^2}{\mu^2} - 1 \right) - m_\sigma^2 \left(\ln \frac{m_\sigma^2}{\mu^2} - 1 \right) \right] \right\}^{-1}, \tag{41}$$

$$-m^2 = -m_\pi^2 + \frac{\lambda}{N}f_\pi^2 + \frac{\lambda}{(4\pi)^2 N} \left[(N+2)\mu^2 + 3m_\sigma^2 \left(\ln \frac{m_\sigma^2}{\mu^2} - 1 \right) + (N-1)m_\pi^2 \left(\ln \frac{m_\pi^2}{\mu^2} - 1 \right) \right] \tag{42}$$

The expressions manifestly show that the performed renormalization is μ-scale independent in the vacuum. Indeed, m^2 and λ can take any values depending on the choice of the renormalization scale μ, while the same observables keep their physical values at any renormalization scale μ (within a certain range of μ). In this respect, the gHF approximation is similar to the leading-order $1/N$-approximation, where also both the NG theorem and scale-independent renormalization in the vacuum are fulfilled [19]. The restriction to a certain range is related to the conditions $\lambda > 0$ and $m^2 < 0$ which should be met. At the scale μ_0, determined by the equation

$$f_\pi^2 + \frac{(N-1)}{(4\pi)^2} \left(m_\pi^2 \ln \frac{m_\pi^2}{\mu_0^2} - m_\pi^2 \right.$$
$$\left. - m_\sigma^2 \ln \frac{m_\sigma^2}{\mu_0^2} + m_\sigma^2 \right) = 0, \tag{43}$$

λ and m^2 become singular, cf. Eqs. (41) and (42). Moreover, $\mu < \mu_0$ implies $m^2 > 0$ and $\lambda < 0$, which defers a spontaneously broken phase and makes the theory unstable because of uncompensated attraction ($\lambda\phi^4 < 0$). Therefore, the range of scale independent renormalization in the vacuum is restricted from below by

$$\mu > \mu_0. \tag{44}$$

The situation is completely different within the conventional HF approximation. In that case, physical observables do depend on the renormalization scale and can be reproduced only at a single value of the scale $\mu^2 = m_\sigma^2/e$, as demonstrated in [19]. The reason of this difference is that the conventional HF approximation of [19] violates the NG theorem at $H = 0$. This stems from the fact that the equations for the pion self-energy and the mean field are not identical in the broken phase and thus leave three nondegenerate equations for three quantities m^2, λ and μ, from which the μ value is unambiguously determined. In our case, this set of equations is degenerate, i.e., the equations for the pion self-energy and the mean field are identical as a consequence of the fulfilled NG theorem. This gives us freedom for an arbitrary choice of μ. Thus, we arrive at an important conclusion which concerns all partial resummation schemes applied to the case of spontaneously broken symmetry: a scale-independent renormalization in the vacuum is possible only if the scheme preserves the NG theorem. This is an important aspect with respect to possible renormalization-group considerations for this type of self consistent approximations.

However, the scale dependence still persists at finite temperature, which is already seen from the analysis of the symmetry-restoration points.

D. Symmetry Restoration Points at $H = 0$

Starting from the broken phase, where $M_\pi^2 = 0$ and $\phi^2 > 0$, there exists a temperature range $T_R \in [T_1, T_2]$, cf. Fig. 1 below, where the classical field vanishes together with the pion mass

$$M_\pi^2(T_R) = \phi^2(T_R) = 0. \tag{45}$$

This precisely occurs, when the two equations (17) and (18) reduce to a single one. Solving

for this single equation with renormalized $Q_a^{(\mathrm{ren})}$ tadpoles (26) gives

$$T_R^2 = \frac{12}{(N+2)}\left[\left(1 + \frac{3}{(N-1)}\frac{M_\sigma^2(T_R)}{m_\sigma^2}\right)f_\pi^2 + \frac{3m_\sigma^2}{(4\pi)^2}\left(\ln\frac{m_\sigma^2}{\mu^2} - 1\right)\left(1 - \frac{M_\sigma^2(T_R)}{m_\sigma^2}\right)\right], \quad (46)$$

where we have used Eqs. (41) and (42) to express m^2 and λ in terms of the vacuum mass m_σ. Strictly speaking, this is not a solution for T_R, since the r.h.s. of (46) still depends on T_R through $M_\sigma^2(T_R)$. Nevertheless, this expression is already quite simple to analyze.

The lower bound T_1 of condition (45) is of central importance for the phase transition in the conventional HF-approximation [1] but of minor relevance in the gHF-scheme, since it corresponds to the metastable solution, cf. Sect. IV below. It is determined if simultaneously also $M_\sigma^2 = 0$ occurs

$$T_1^2 = \frac{12}{(N+2)}\left[f_\pi^2 + \frac{3m_\sigma^2}{(4\pi)^2}\left(\ln\frac{m_\sigma^2}{\mu^2} - 1\right)\right]. \quad (47)$$

It is still μ-dependent, in spite of the scale-independent renormalization in the vacuum. At large μ, i.e., above some μ_1, T_1^2 can even become negative, which means that then this solution does not exist.

For the stable solution of the gHF approximation it numerically occurs that $M_\sigma(T_R) \approx m_\sigma$, cf. Ref. [1] and Sect. IV below. This almost removes the $M_\sigma(T_R)$ dependence from the r.h.s. of Eq. (46) and makes the corresponding temperature T_2 almost μ-independent

$$T_2^2 \simeq \frac{12}{(N-1)}f_\pi^2. \quad (48)$$

This value coincides with that of the naive renormalization [1]. The solution T_2 corresponds to a partial symmetry restoration, since here we still have $M_\sigma(T_2) \neq M_\pi(T_2)$ in spite of $\phi = 0$.

IV.   RESULTS FOR N = 4

For the numerical calculations we use the following parameters: $m_\sigma = 600$ MeV and $f_\pi = 93$ MeV. The pion mass is either zero, $m_\pi = 0$,

in the case of exact symmetry, or $m_\pi = 139$ MeV for the approximate symmetry. The general structure of the solutions to the renormalized Eqs. (16)–(18) is similar to that obtained with the naive renormalization [1] but with extra complications caused by the additional dependence on the renormalization scale μ. According to Eq. (43) physically reasonable solutions exits only for $\mu > \mu_0$, where μ_0 as the solution of Eq. (43) equals $\simeq 200$ MeV for the above specified parameters.

There are several different branches of the solution. Stable and physically meaningful are determined by the principle of maximum pressure, the pressure being given by the effective potential (36)

$$P = -V_{\mathrm{gHF}}^{(\mathrm{ren})}(\phi, T) + \mathrm{const}. \quad (49)$$

Here the constant is determined by the condition that the pressure should vanish for the physical vacuum, i.e., in the spontaneously broken phase, while our renormalization condition (32) determines $V_{\mathrm{gHF}}^{(\mathrm{ren})}$ to be zero in the unphysical, symmetric vacuum.

A.   Exact $O(4)$ Symmetry

The actual structure of the solution depends on the renormalization scale μ. We start with a moderate scale $\mu = 600$ MeV, i.e., of the order of m_σ. The results are presented in Figs. 1–5.

In the narrow temperature range, displayed in Fig. 1, the results are qualitatively similar to those obtained with the naive renormalization [1]. The stable branch starts at $T = 0$ from the physical vacuum values for the masses and the classical field and crosses the metastable branch at $T_{\mathrm{cross}} \approx 440$ MeV. In terms of the pressure, they are touching rather than crossing (see Fig. 2). Therefore, no transition from one branch to another occurs at T_{cross}. In the broken-symmetry phase, the pion mass equals zero. Then a phase transition of the second order occurs at $T_2 \simeq 180$ MeV, at which the field becomes zero (see Fig. 3). However, the π and σ masses still differ beyond this transition point. They become equal only after a second phase
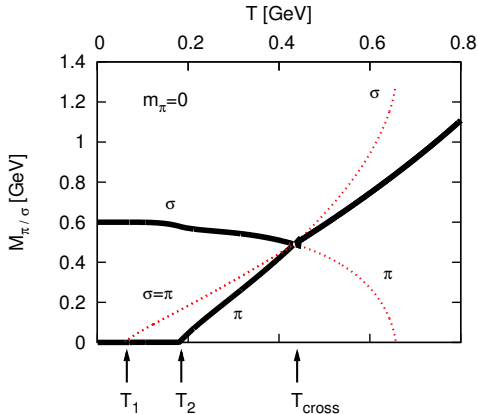
FIG. 1: Meson masses as functions of temperature for $\mu = 600$ MeV and $m_\pi = 0$ case. Stable branch is presented by solid lines, whereas the metastable one – by dashed lines.
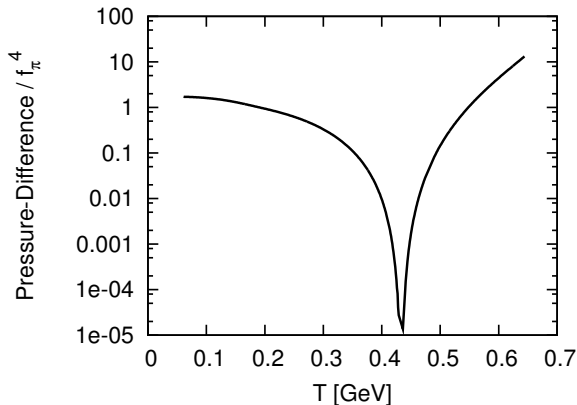

FIG. 2: The same as in Fig. 1 but for the pressure difference between stable and metastable branches.
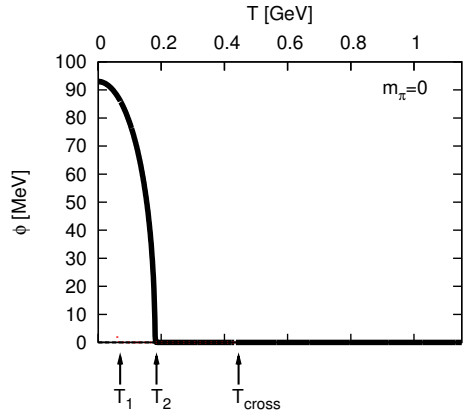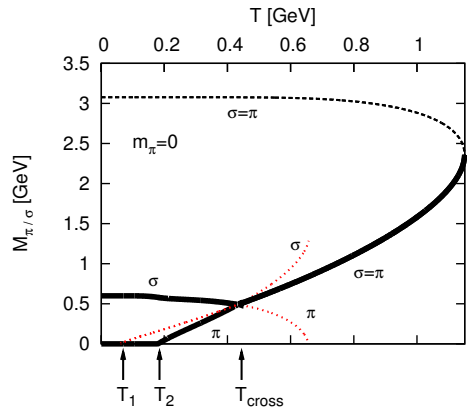

FIG. 3: The same as in Fig. 1 but for the field ϕ.

FIG. 4: Meson masses of Fig. 1 but in wider temperature region. The upper metastable branch is displayed by the long-dashed line.

transition, which is also of second order, at T_{cross}. Note that the equal-mass solution above T_{cross} is precisely the same as in the conventional HF approximation (cf. Refs. [13, 19]), since the gapless modification term (9) vanishes in this case. The T_1 point proves to be irrelevant for the stable branch. Rather it is the starting point for the metastable branch, which in the range of $T_1 < T < T_{\text{cross}}$ precisely coincides with the solution of the conventional HF approximation [13, 19]. The corresponding field is always zero for this branch. Contrary to the case of the naive renormalization [1], this metastable branch ends at some temperature (≈ 650 MeV).

However, in the wider temperature range, displayed in Fig. 4, we see a significant difference from the results obtained with the naive renor-

malization [1]. Neither stable nor metastable solutions exist above a certain temperature T_{end} which is ≈ 1.15 GeV for the considered $\mu = 600$ MeV. Let us remind that this solution above T_{cross} corresponds to the conventional HF approximation. Therefore, it is precisely the same as in previous conventional HF calculations with renormalization [13, 19]. The occurrence of such an end point in the HF approximation was first pointed out by Baym and Grinstein [5].

At T_{end}, the stable branch of the solution joins the upper branch. This upper branch at any temperature corresponds to equal masses and zero field, i.e., $M_\pi = M_\sigma$ and $\phi = 0$, and hence is also a solution to the conventional renormalized HF approximation. It starts with very high values of masses (≈ 3 GeV) in the vacuum and also ends at T_{end}. The vacuum pressure for
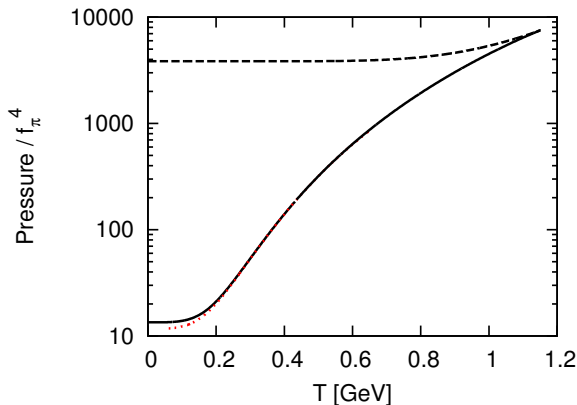
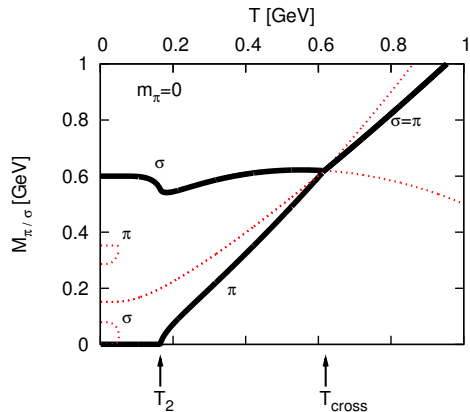FIG. 5: The same as in Fig. 4 but for the pressure.

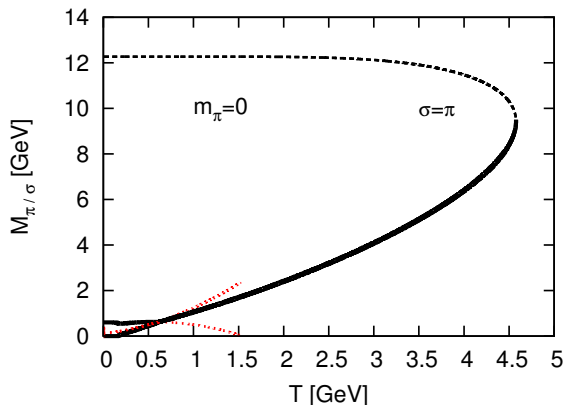
FIG. 7: Zoomed low-temperature region of Fig. 6.

FIG. 6: Meson masses as functions of temperature for $\mu = 1200$ MeV and $m_\pi = 0$ case. Stable branches are presented by solid lines, whereas metastable ones by dashed lines.

FIG. 8: The same as in Fig. 6 but for the field ϕ.

this branch, evaluated according to Eq. (49), is rather high (see Fig. 5).

It is important note that for the upper branch and at temperatures near the endpoint the logarithmic terms in the gap equation, $\propto \ln(M^2_{\pi/\sigma}/\mu^2)$, become large. This indicates that at such points the expansion of the Φ functional in powers of the renormalized coupling becomes unreliable, because the effective coupling becomes large. Therefore we consider the upper branch not a physically meaningful solution. The same holds true for temperatures close to the endpoint temperature. Such a behavior must be expected for any effective theory and was indeed also observed in Quantum Hadro Dynamics (QHD) in [15].

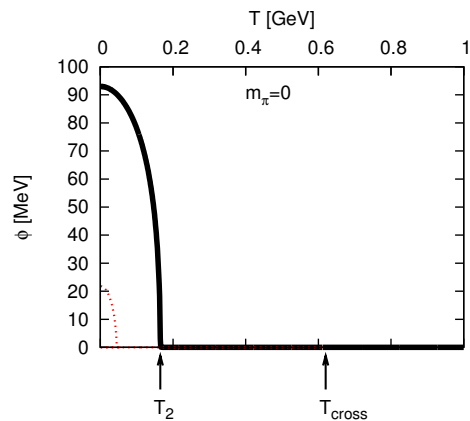At larger renormalization scales, μ, the global

pattern of the solution remains qualitatively similar, as seen from Fig. 6. Only T_{cross} and T_{end} move to higher temperatures. Inspecting the low-temperature region in more detail, cf. Fig. 7, we see that a new metastable solution, which ends already at rather low temperature, appears. This new solution has a nonzero field (Fig. 8) but violates the NG theorem. In addition the metastable solution, which before started at T_1, now begins at zero temperature, since at $\mu = 1.2$ GeV we have already $T_1^2 < 0$, cf. Eq. (47).

If we take a low value for the scale μ, the structure of the stable solution becomes more involved, see Figs. 9–11. In the broken-symmetry sector we still have a massless pion and a nonzero field, see Fig. 10. However, at higher temperatures the metastable branch, displayed by the dotted line, reveals back-bending. This back-bended part of the branch turns out to be the most stable one, see Fig. 11. As a result we
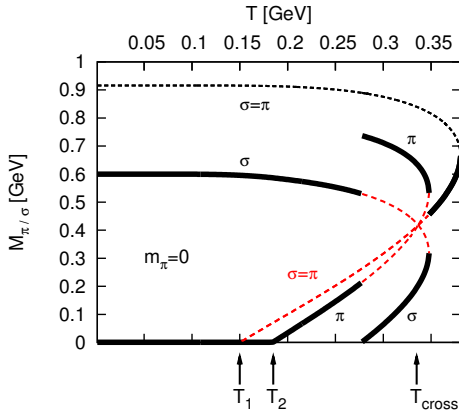
FIG. 9: Meson masses as functions of temperature for $\mu = 300$ MeV and $m_\pi = 0$ case. Stable branches are presented by solid lines, whereas metastable ones – by dashed lines.
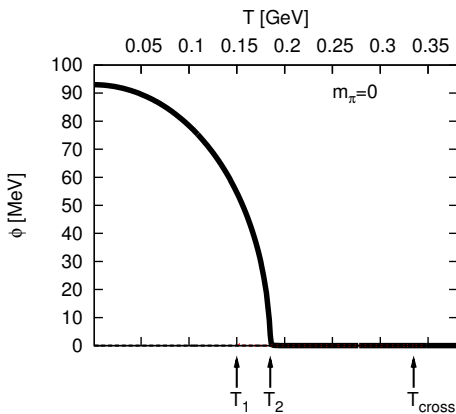

FIG. 10: The same as in Fig. 9 but for the field ϕ.

arrive at a complicated structure for the stable solution, where even the pressure turns out to be discontinuous.

A common feature for all scales is that the point of the first phase transition, $T_2 \simeq 180$ MeV, is approximately μ-independent and has about the same value as that in the naive renormalization scheme [1]. At the same time, the point of the second phase transition, T_{cross}, and the end point of the stable solution, T_{end}, are essentially μ-dependent.

B. Approximate $O(4)$ Symmetry

In the case of explicitly broken symmetry ($m_\pi = 139$ MeV), the structure of solutions at various μ scales is similar to that described
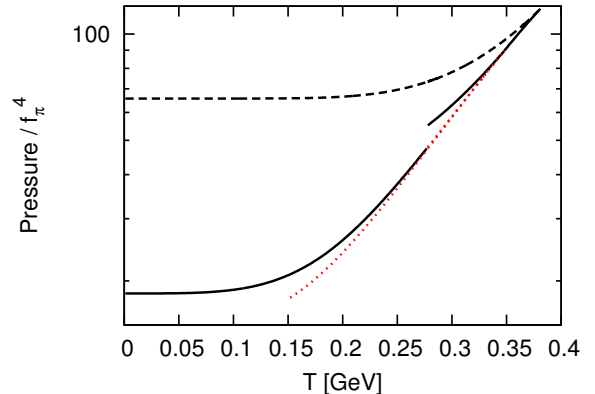

FIG. 11: The same as in Fig. 9 but for the pressure.

above for the chiral limit. Even the behavior of metastable branches remains similar. We illustrate the changes on the example of $\mu = 600$ MeV, see Figs. 12–14 which looks the most physically appealing and is close to results of the naive renormalization [1]. The main difference from the $m_\pi = 0$ case is that the sequence of two phase transitions is transformed here into a smooth cross-over transition.
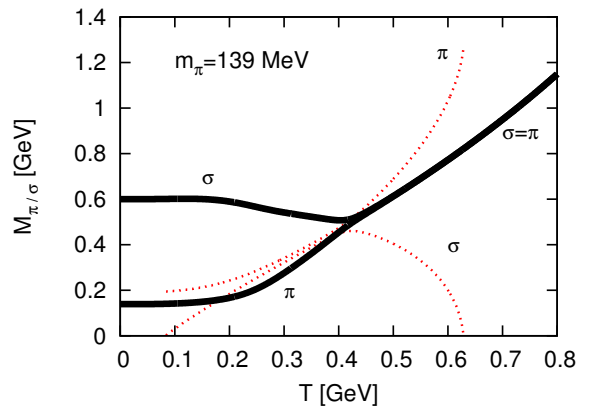

FIG. 12: Meson masses as functions of temperature for $\mu = 600$ MeV and $m_\pi = 139$ MeV case. Stable branches are presented by solid lines, whereas metastable ones – by dashed lines.

V. CONCLUSION

We have studied a renormalized version of the gapless Φ-derivable HF approximation to the $\lambda\phi^4$ theory with spontaneous breaking of the $O(N)$ symmetry, proposed in Ref. [1]. This gHF
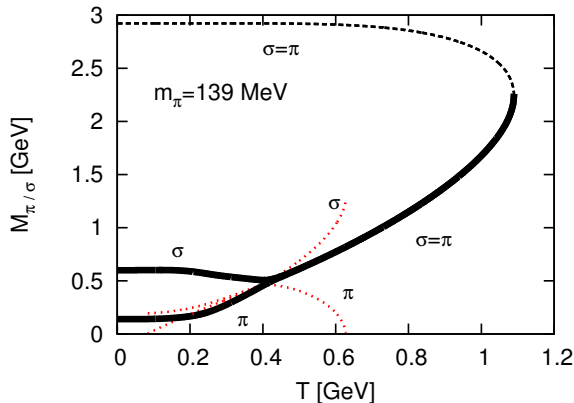
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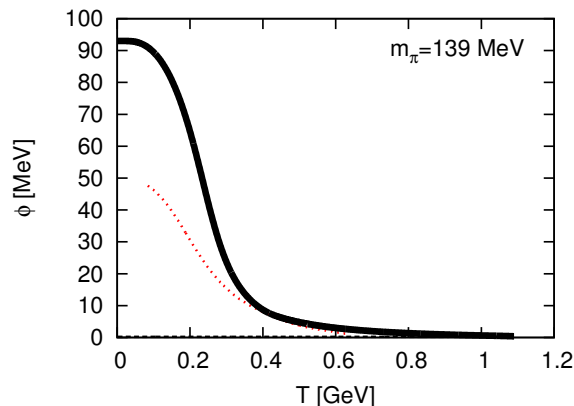FIG. 13: Meson masses of Fig. 12 but in wider temperature region.

FIG. 14: The same as in Fig. 12 but for the field ϕ.

approximation simultaneously preserves all the desirable features of Φ-derivable approximation schemes (i.e., the validity of conservation laws and thermodynamic consistency) and respects the NG theorem in the phase of spontaneously broken symmetry. This is achieved by adding a correction $\Delta\Phi$ to the conventional Φ functional. The nature of this correction can be understood as follows. The conventional Φ-derivable HF approximation cuts off an infinite series of diagrams, among which are those providing the NG theorem in the phase of spontaneously broken symmetry. By introducing the $\Delta\Phi$ correction to the HF approximation we take into account a part of those omitted diagrams (at the level of the actual approximation), which restores the NG theorem.

An advantage of the gHF approximation is that it allows for scale-independent renormaliza-

tion in the vacuum, unlike the conventional HF approximation [19]. The scale independence in the vacuum is a direct consequence of the NG theorem, which makes the equations for the pion self-energy and classical field degenerated. However, even in the gHF approximation, only renormalization scales higher than a certain value (μ_0, cf. Eq. (43)) are allowed, in order to ensure stability of the renormalized approximation.

Nevertheless, the scale dependence still persists at finite temperatures. The violation of renormalization-scale independence of Φ-derivable approximations was shown in [23, 24] from the point of view of the renormalization-group β function. There the β function, evaluated from the Φ-functional formalism, was shown to deviate from its perturbative expansion, beginning at orders in the expansion parameter, higher than that explicitly taken in the Φ functional. The reason is the violation of "crossing symmetry" in the sense of [14]: Solving the self-consistent equations of motion corresponds to a partial resummation of diagrams to any order in the expansion parameter (e.g., the coupling constant λ or $\hbar$, i.e., the order of loops in perturbative Feynman diagrams) which is necessarily incomplete for any truncation of the Φ functional.

Within our renormalization scheme, it becomes clear that the renormalization-scale dependence at finite temperatures originates from the subtraction of the "hidden subdivergence" of the four-point function inside the self-consistent tadpole loop. As shown in [14], this four-point function consists of a resummation in only one channel, and thus the β function of this resummed four-point function deviates from the correct one at orders higher than contained in the approximation of the Φ-functional, i.e., to $\mathcal{O}(\lambda^2)$.

At large scales ($\mu \gtrsim m_\sigma$) the chiral phase transition proceeds similar to that in the naive renormalization scheme [1]. In the case of the exact $O(N)$ symmetry, it proceeds through a sequence of two second-order phase transitions rather than a single one. In the first transition the mean field vanishes but the meson masses still remain different. The temperature of this phase transition, $T_2 \simeq 180$ MeV, is approx-

imately μ-independent and has approximately the same value as that in the naive renormalization scheme [1]. In the second transition also the masses become equal, and the $O(N)$ symmetry is completely restored. The corresponding temperature, T_{cross}, turns out to be essentially μ-dependent. When the $O(N)$ symmetry is explicitly violated, the sequence of two phase transitions is transformed into a smooth cross-over transition. Moreover, at $\mu \simeq m_\sigma$ the results are even qualitatively close to those obtained with the naive renormalization [1]. At small scales (say $\mu \lesssim m_\sigma/2$), the phase structure becomes very complicated, however still respecting the NG theorem in the phase of spontaneously broken symmetry.

Another result concerns both the conventional renormalized HF and gHF approximations, which in fact are identical in the phase of restored $O(N)$ symmetry. There exists an end point of the solution, i.e., a temperature T_{end} above which there are no solutions to the gap equations. The occurrence of such an end point in the HF approximation was first pointed out by Baym and Grinstein [5] and is caused by the dominant role of $\ln(M^2/\mu^2)$ terms in the gap equations, which originate from the renormalization procedure. The dominant behavior of the $\ln(M^2_{\pi/\sigma}/\mu^2)$ terms signals a breakdown of the HF approximation and the need to include higher-order corrections into the Φ functional [5]. Another interesting question is whether one can find a renormalization-group improved Φ-derivable approximation to cure this problem. We have found that at low scales (like $\mu \lesssim m_\sigma/2$), the end point appears at rather low temperatures, leaving almost no room for the HF (as well as gHF) approximation in the phase of restored $O(N)$ symmetry. At the same time, at ($\mu \gtrsim m_\sigma$), the end point moves to rather high energies $T_{\mathrm{end}} \gtrsim 1.2$ GeV, hence allowing this approximation at least at $T \lesssim 1$ GeV.

Summarizing, we have found that the gHF approximation is certainly advantageous as compared to the conventional HF one, since it respects the NG theorem and allows scale-independent renormalization in the vacuum. These properties are closely interrelated. They both require that the set of equations of motion are degenerate in the phase of spontaneously broken symmetry. Nevertheless, there still are serious problems with the renormalization of Φ-derivable approximations for a theory with a spontaneously broken symmetry. At finite temperatures the predictions of renormalized gHF approximations essentially depend on the renormalization scale, contrary to the case of the renormalized perturbation theory. In this respect, the gHF approximation becomes similar to the leading-order $1/N$-approximation, where also both the NG theorem and scale-independent renormalization in the vacuum hold true [19]. However, in view of the medium-independent renormalization performed in accordance with Refs. [13, 14, 16], this scale-dependence at finite temperatures cannot be already interpreted as an artifact of temperature-dependent counter terms. It may turn out that this scale dependence is a consequence of triviality of the $\lambda\phi^4$ theory [25] which, when it is renormalized, therefore requires an external scale to serve for a scale, below which it can be used as an effective field theory to describe the low-energy phenomenology.

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