Renormalization of Conserving Selfconsistent Dyson equations

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Motivation

- Thermodynamics of strongly interacting systems
- Conservation laws, detailed balance, thermodynamical consistency
- Finite width effects (resonance, damping, $\cdots$)

Concepts

- Real time quantum field theory
- The Φ-derivable scheme (example $O(N)$)
- Renormalization
- Restoration of symmetries
- Gauge Symmetries and Vector Mesons

Schwinger-Keldysh Formalism

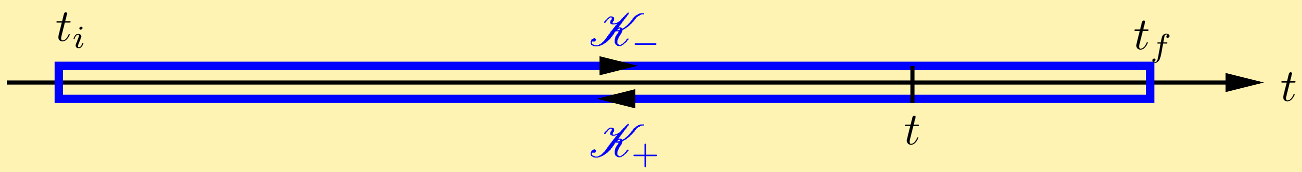
- Initial statistical operator $\boldsymbol{\rho}_i$ at $t = t_i$
- Time evolution

$$\langle O(t)\rangle = \mathrm{Tr}\left[\boldsymbol{\rho}(t_i)\,\underbrace{\mathcal{T}_a\left\{\exp\left[+\mathrm{i}\int_{t_i}^{t}\mathrm{d}t'\mathbf{H}_I(t')\right]\right\}}_{\text{anti time–orderd}}\right.$$

$$\mathbf{O}_I(t)$$

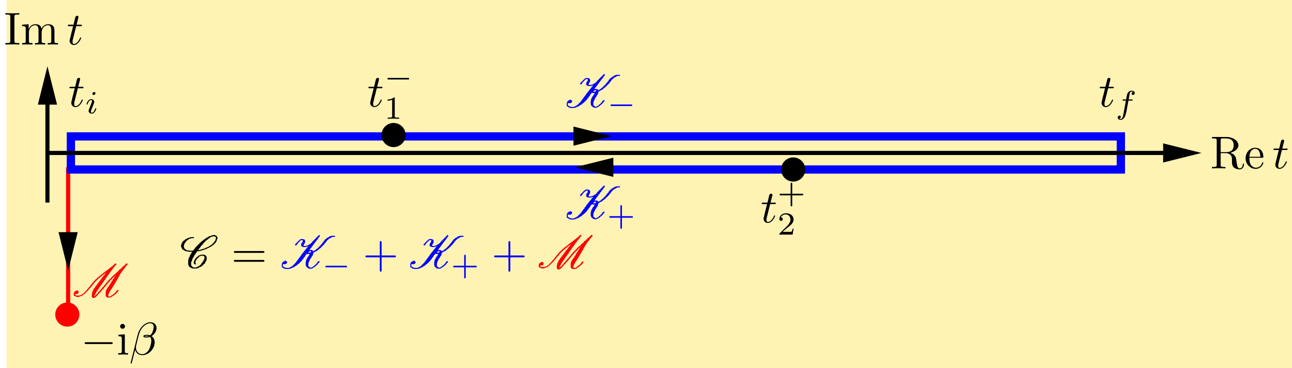
$$\left.\underbrace{\mathcal{T}_c\left\{\exp\left[-\mathrm{i}\int_{t_i}^{t}\mathrm{d}t'\mathbf{H}_I(t')\right]\right\}}_{\text{time–ordered}}\right].$$

- Difference to vacuum: Contour-ordered Green's functions



$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+$

- In equilibrium: $\boldsymbol{\rho} = \exp(-\beta\mathbf{H})/Z$ with $Z = \mathrm{Tr}\exp(-\beta\mathbf{H})$
- Imaginary part of the time contour



- Correlation functions with real times: $\mathrm{i}G_{\mathscr{C}}(x_1^-, x_2^+)$
- Fields periodic (bosons) or anti-periodic (fermions)
- Feynman rules $\Rightarrow$ time integrals $\rightarrow$ contour integrals

The Φ-Functional

- Introduce local and bilocal auxiliary sources
- Generating functional

$$Z[J,K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{J_1\phi_1\right\}_1 + \left\{\frac{\mathrm{i}}{2}K_{12}\phi_1\phi_2\right\}_{12}\right]$$

- Generating functional for connected diagrams

$$Z[J,K] = \exp(\mathrm{i}W[J,K])$$

- The mean field and the connected Green's function

$$\underbrace{\varphi_1 = \frac{\delta W}{\delta J_1},\ G_{12} = -\frac{\delta^2 W}{\delta J_1 \delta J_2}}_{\text{standard quantum field theory}} \Rightarrow \frac{\delta W}{\delta K_{12}} = \frac{1}{2}[\varphi_1\varphi_2 + \mathrm{i}G_{12}]$$

- Legendre transformation for φ and G:

$$\Gamma[\varphi,G] = W[J,K] - \{\varphi_1 J_1\}_1 - \frac{1}{2}\{(\varphi_1\varphi_2 + \mathrm{i}G_{12})K_{12}\}_{12}$$

- Exact closed form:

$$\begin{aligned}\Gamma[\varphi,G] = & S_0[\varphi] + \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(-\mathrm{i}G^{-1}) + \frac{\mathrm{i}}{2}\left\{D_{12}^{-1}(G_{12} - D_{12})\right\}_{12} \\ & + \Phi[\varphi,G] \Leftarrow \text{all closed 2PI interaction diagrams}\end{aligned}$$

$$D_{12} = \left(-\Box - m^2\right)^{-1}$$

Equations of Motion

- Physical solution defined by vanishing auxiliary sources:

$$\frac{\delta \Gamma}{\delta \varphi_1} = -J_1 - \{K_{12}\varphi_2\}_2 \stackrel{!}{=} 0$$

$$\frac{\delta \Gamma}{\delta G_{12}} = -\frac{\mathrm{i}}{2} K_{12} \stackrel{!}{=} 0$$

- Equation of motion for the mean field φ

$$-\Box\varphi - m^2\varphi := j = -\frac{\delta \Phi}{\delta \varphi}$$

- for the "full" propagator G ⇒ Dyson's equation:

$$-\mathrm{i}(D_{12}^{-1} - {G_{12}}^{-1}) := -\mathrm{i}\Sigma = 2\frac{\delta \Phi}{\delta G_{21}}$$

- Integral form of Dyson's equation:

$$G_{12} = D_{12} + \{D_{11'}\Sigma_{1'2'}G_{2'2}\}_{1'2'}$$

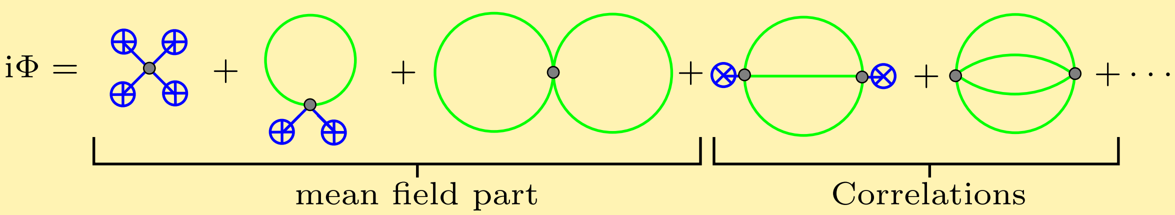
- Closed set of equations of for φ and G

"Diagrammar"

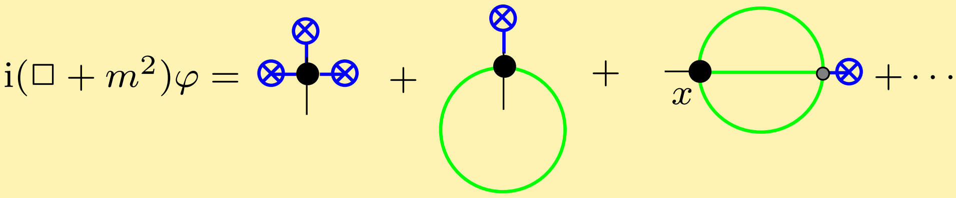
- $O(N)$-theory

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \frac{m^2}{2}\vec{\phi}^2 - \frac{\lambda}{4!}(\vec{\phi}^2)^2$$

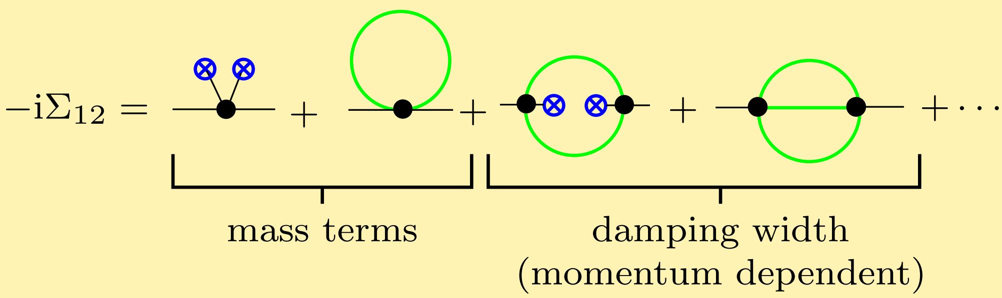
- 2PI Generating Functional



- Mean field equation of motion



- Self-energy



Properties of the Φ-derivable Approximations

Why using the Φ-functional?

- Truncation of the Series of diagrams for Φ

☞ Expectation values for currents are conserved
$\Rightarrow$ "Conserving Approximations"

- In equilibrium $\mathrm{i}\Gamma[\varphi, G] = \ln Z(\beta)$ (thermodynamical potential)
- consistent treatment of Dynamical quantities (real time formalism) and thermodynamical bulk properties (imaginary time formalism) like energy, pressure, entropy

- Real- and Imaginary-Time quantities "glued" together by Analytic properties from (anti-)periodicity conditions of the fields (KMS-condition)

- Self-consistent set of equations for self-energies and mean fields

Problem of Renormalization

Why renormalization?

- ☞ Diagrams UV-divergent
- ☞ Control the physical parameters in vacuum
- ☞ Temperature dependence from theory alone

How to renormalize self-consistent diagrams?

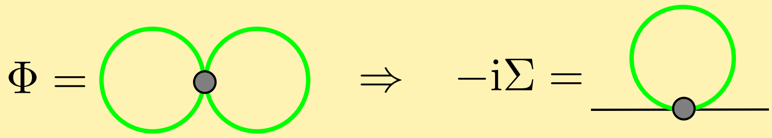
- ☞ In terms of perturbation theory: Resummation of all self-energy insertions in propagators
- ☞ Self-consistent diagrams with explicit nested and overlapping sub-divergences
- ☞ "Hidden" sub-divergences from self-consistency

How to manage it numerically?

- ☞ Power counting (Weinberg) valid for self-consistent diagrams
- ☞ At finite temperatures:
  Self-consistent scheme rendered finite with local counterterms independent of temperature
- ☞ Analytical properties ⇒ subtracted dispersion relations
- ☞ BPHZ-renormalization ⇒ Subtracting the integrands
- ☞ Advantage: Clear scheme how to subtract temperature independent sub-divergences
- ☞ Φ-functional ⇒ consistency of counterterms

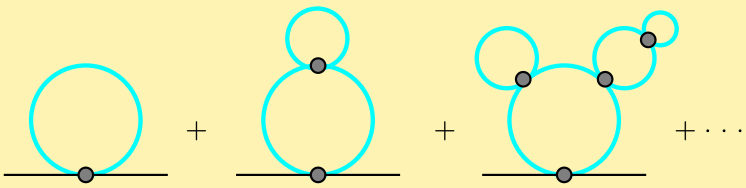
Self-consistent Renormalization

Example: Tadpole-Renormalization

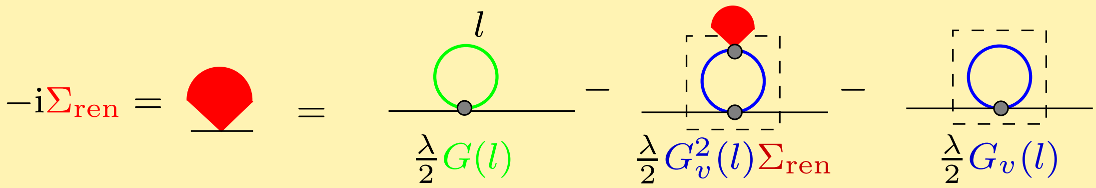


☞ Temperature dependent effective mass: $M^2 = m^2 + \Sigma_{\mathrm{ren}}$

- Resummation of the Dyson series



☞ Renormalized self-energy



- Renormalization: Subtraction of Vacuum–(sub)divergences

☞ Result: Finite "Gap equation"

$$M^2 = m^2 + \Sigma_{\mathrm{ren}} = m^2 + \frac{\lambda}{32\pi^2}\left(M^2 \ln\frac{M^2}{m^2} - \Sigma_{\mathrm{ren}}\right) + \underbrace{+\frac{\lambda}{2}\int\frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - M^2) n(p_0)}_{\to 0 \text{ for } T \to 0}$$

$n(p_0)$: Bose–Einstein distribution

Self-consistent Renormalization

First step: Vacuum

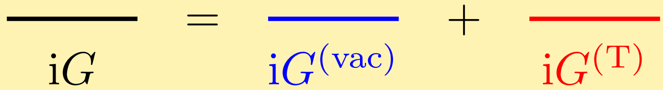
- Power-counting for self-consistent propagators as in perturbation theory: $\delta = 4 - E$
- Usual BPHZ-renormalization for wave function, mass and coupling constant renormalization
- In practice: Use Lehmann-representation and dimensional regularization

✔ Closed self-consistent finite Dyson-equations of motion

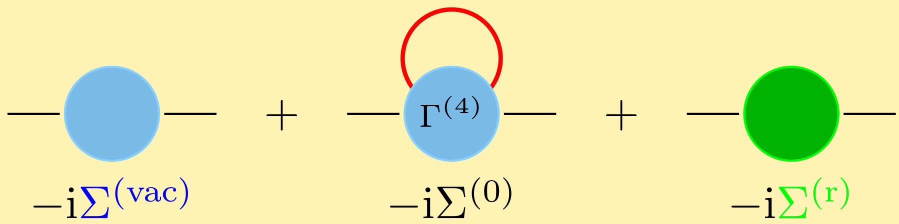
✔ Numerically treatable

Second step: Finite Temperature

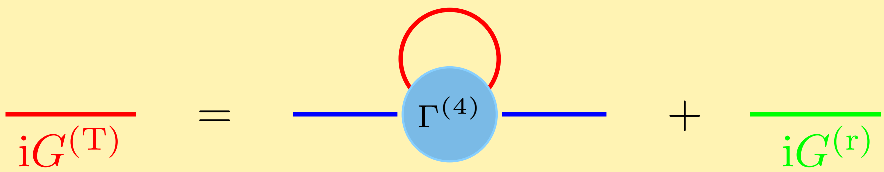
- Split propagator in vacuum and T-dependent part



- Expand self-energy around vacuum part



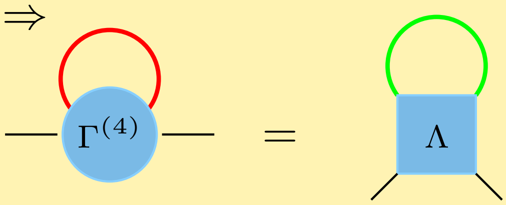
- Need further splitting of propagator



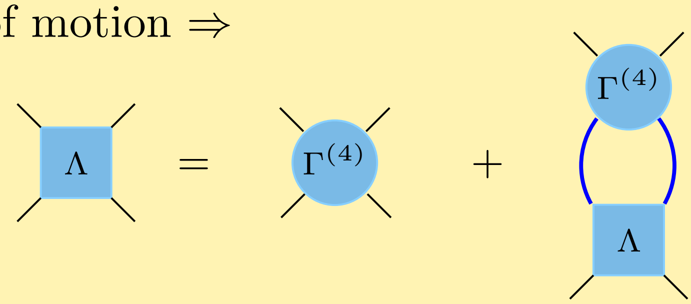
Self-consistent Renormalization

Third step: 4-point vertex renormalization

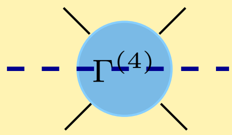
- Σ^0 linear in $G^{(\mathrm{r})}$ $\Rightarrow$



- Equation of motion $\Rightarrow$



☞ s-channel Bethe-Salpeter equation



cuts more than three lines!

$\Rightarrow$ "BPHZ Boxes" in ladder-diagrams do not cut inside $\Gamma^{(4)}$.

$\Rightarrow$ Asymptotics + BPHZ-formalism:

$$\Gamma^{(4)}(l,p) - \Gamma^{(4)}(l,0) \cong O(l^{-\alpha}) \text{ with } \alpha > 0$$

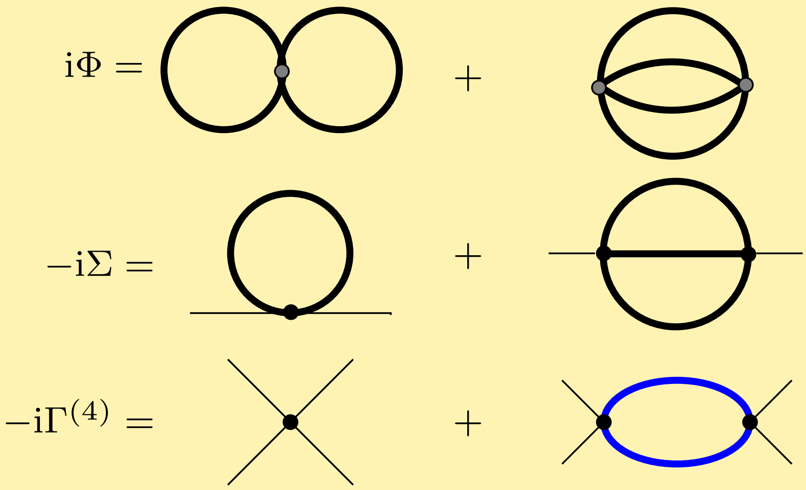
$\Rightarrow$ Renormalized eq. of motion for Λ:

$$\begin{aligned}\Lambda(p,q) =& \Lambda(0,0) + \Gamma^{(4)}(p,q) - \Gamma^{(4)}(0,0) \\ &+ \mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}[\Gamma^{(4)}(p,l) - \Gamma^{(4)}(0,l)][G^{\mathrm{vac}}]^2(l)\Lambda(l,q) \\ &+ \mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\Lambda(0,l)[G^{\mathrm{vac}}]^2(l)[\Gamma^{(4)}(l,q) - \Gamma^{(4)}(l,0)]\end{aligned}$$

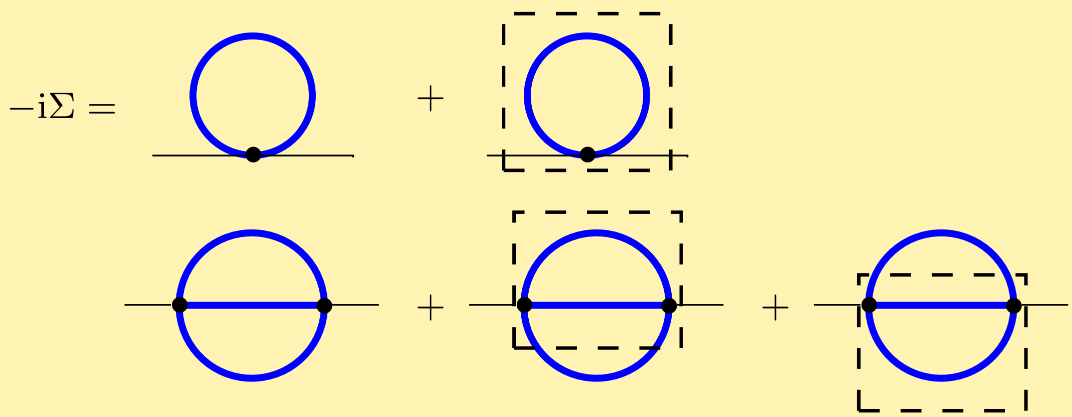
✔ Self-energy finite with vacuum counter terms

Example: Tadpole+Sunset

Approximation of the Φ-functional



Renormalization (vacuum)



+overall

- In practice: Use dispersionrelations for propagators

☞ Kernels, can be calculated analytically with standard formulae of dimensional regularization

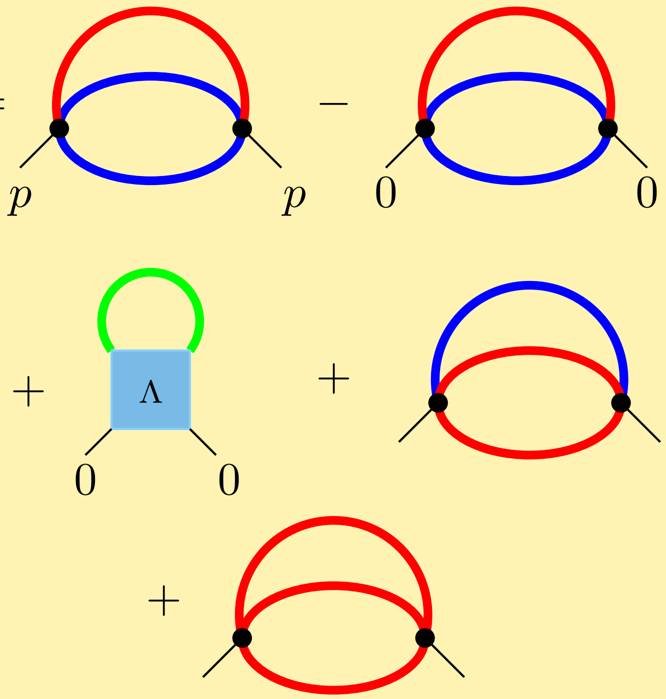
✔ Finite Self-consistent integral equations of motion ⇒ Solved iteratively

- Calculate also $\Gamma^{(4)}$ and $\Lambda(0,q)$

Example: Tadpole+Sunset

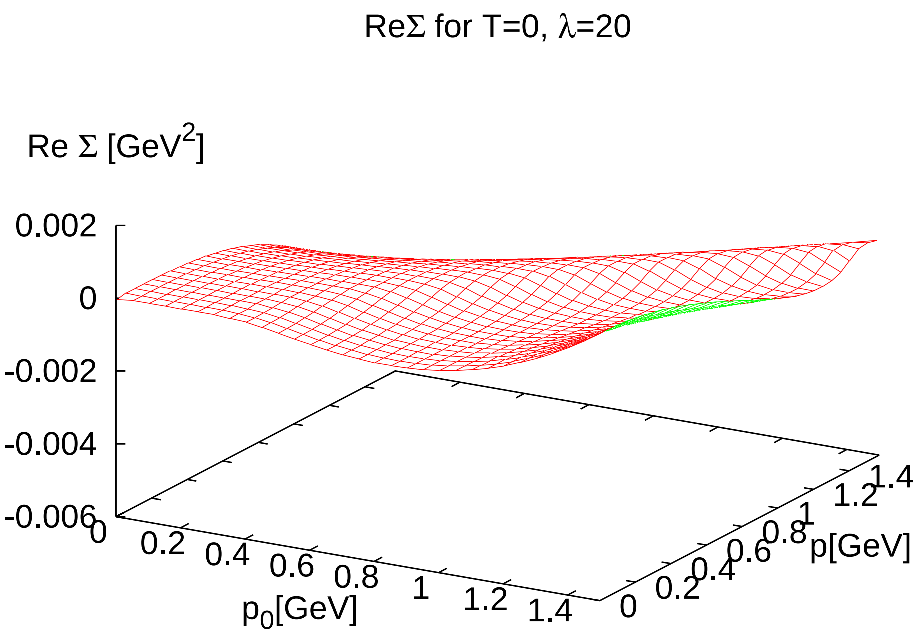
Renormalization (Finite Temperature)

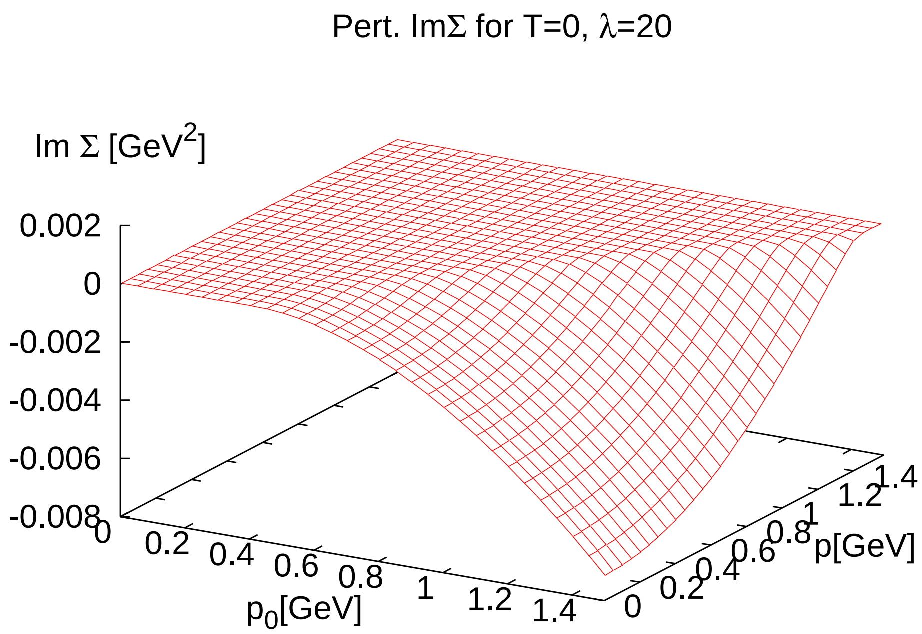
$-\mathrm{i}\Sigma^{(\mathrm{T})}(p) =$



- Only finite integrals
- ✔ Numerics for three-dim integrals on a lattice in p_0 and $|\vec{p}|$
- ✔ Equations of motion solved iteratively

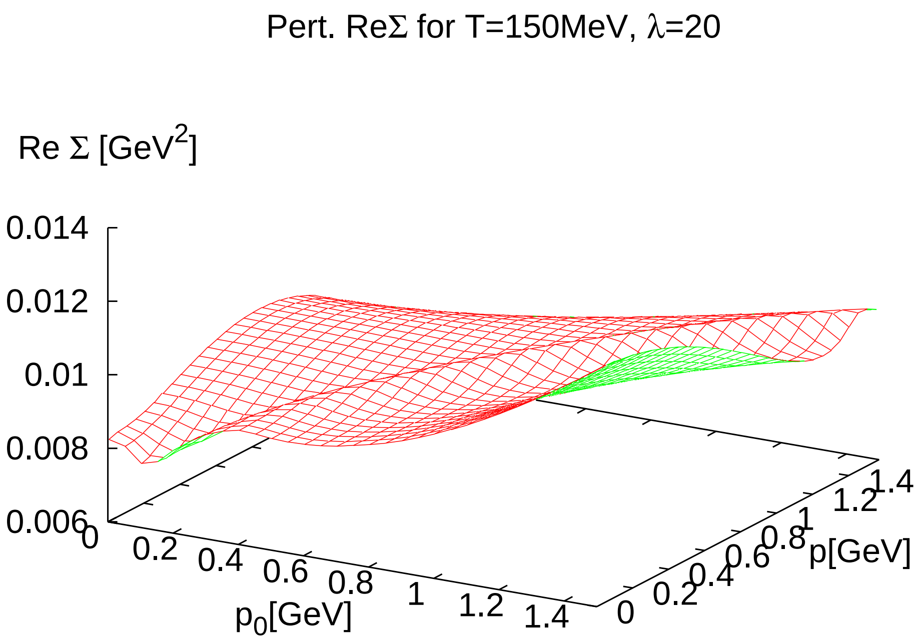
Results for the Vacuum Sunset Diagram

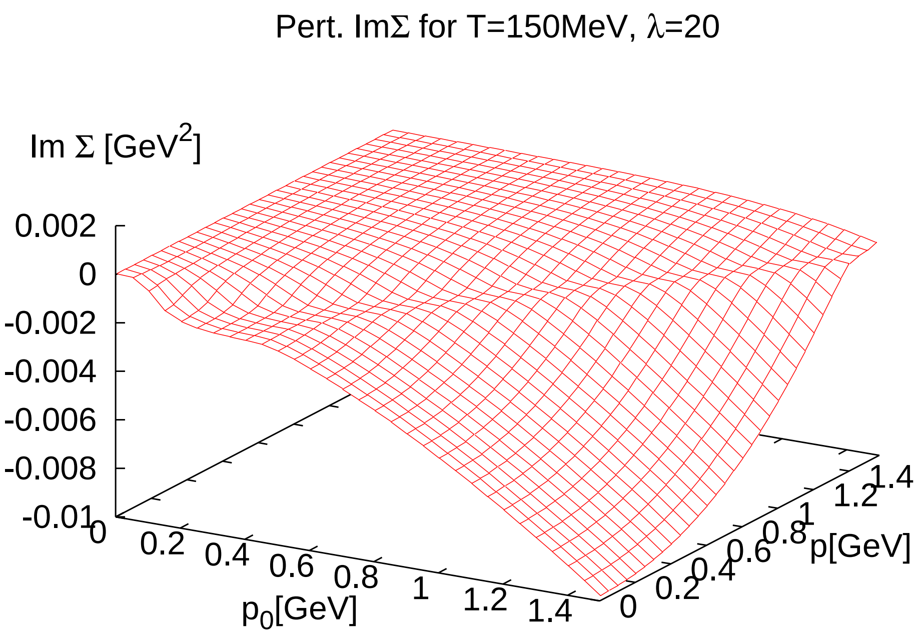




Vacuum: $m = 140\text{MeV}$, $\lambda = 20$

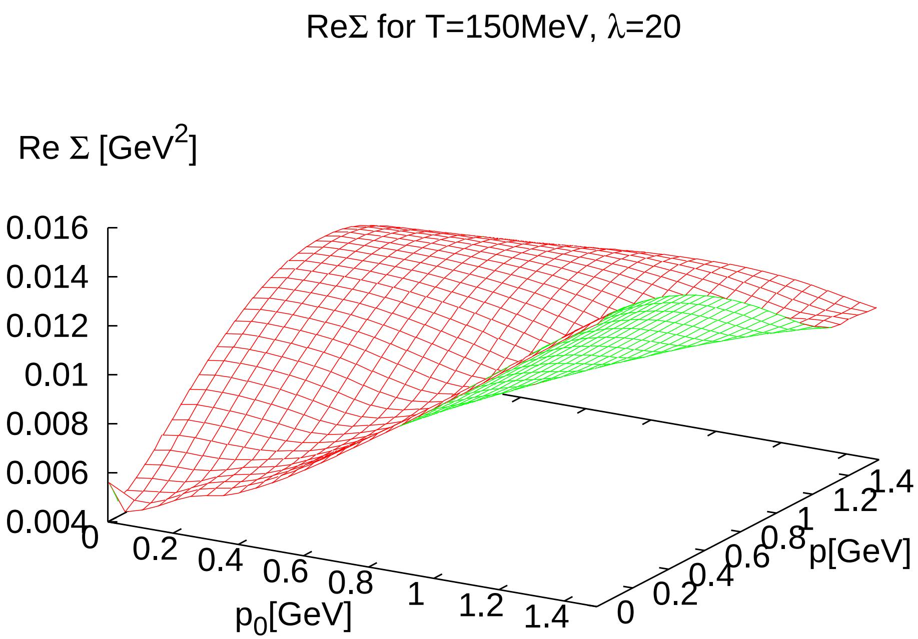
Perturbative Result for “Sunset + Tadpole” at $T > 0$

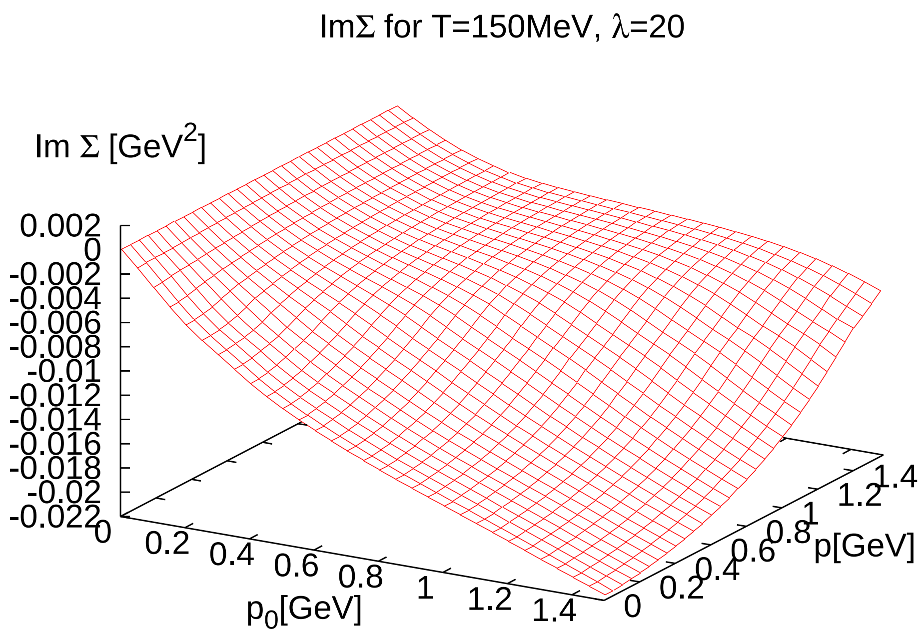




$T = 150\mathrm{MeV}$

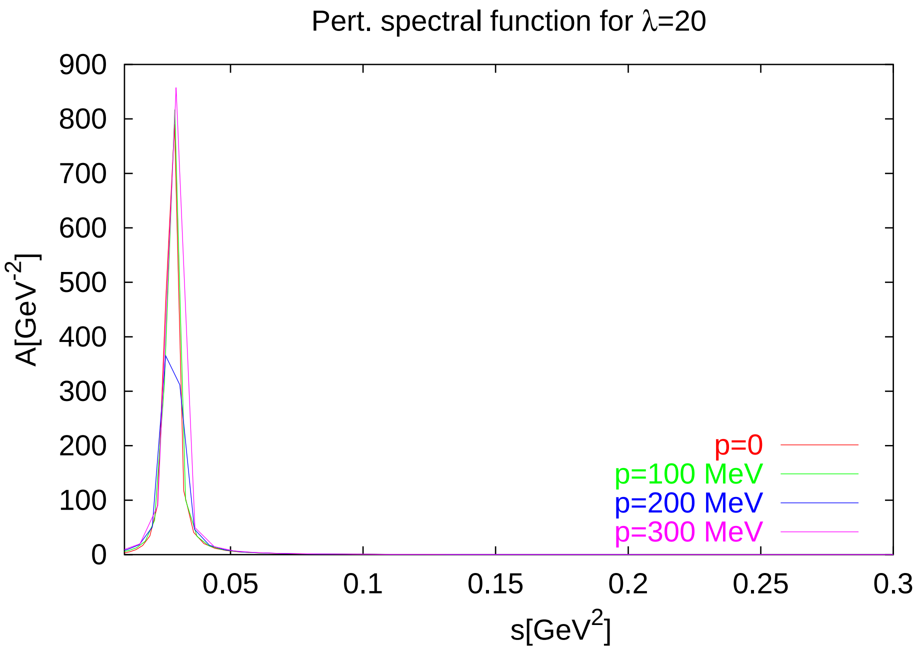
Self-consistent Result for "Sunset + Tadpole" at $T > 0$

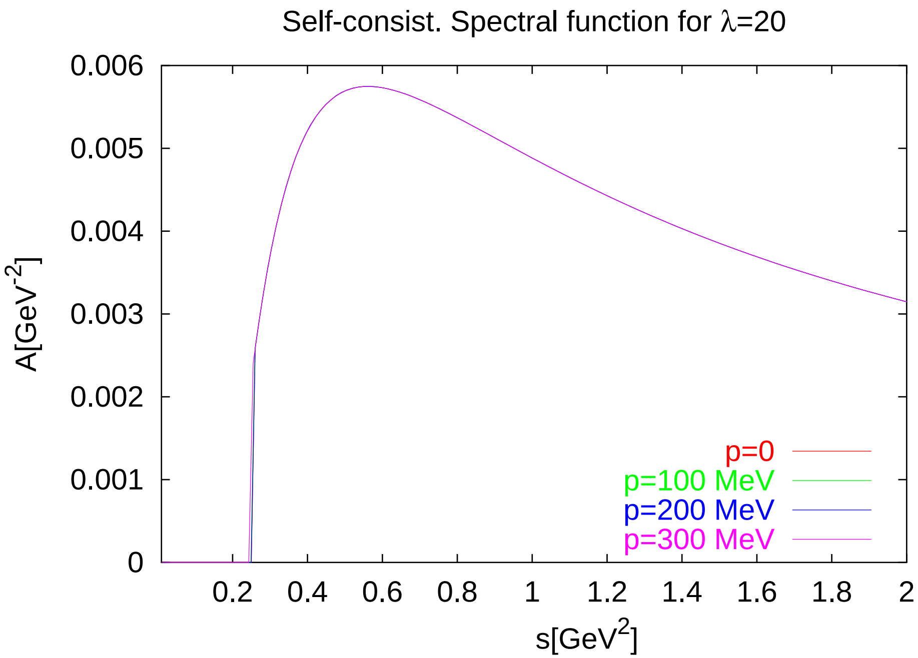




$T = 150\mathrm{MeV}$

Spectral function of the "Meson"





$T = 150\text{MeV}$:Perturbative (left) and self-consistent (right) calculation

Symmetries at the correlator level

Symmetry restoration

- Problem with Φ–Functional: Most approximations break symmetry!
- Reason: Only conserving for Expectation values for currents, not for correlation functions
- Dyson's equation as functional of φ:

$$\left.\frac{\delta \mathbb{\Gamma}[\varphi, G]}{\delta G}\right|_{G=G_{\text{eff}}[\varphi]} \equiv 0$$

- Define new effective action functional

$$\Gamma_{\text{eff}}[\varphi] = \mathbb{\Gamma}[\varphi, G_{\text{eff}}[\varphi]]$$

- Symmetry analysis ⇒ $\Gamma_{\text{eff}}[\varphi]$ symmetric functional in φ
- Stationary point

$$\left.\frac{\delta \Gamma_{\text{eff}}}{\delta \phi}\right|_{\varphi=\varphi_0} = 0$$

☞ φ_0 and $G = G_{\text{eff}}[\varphi_0]$: self–consistent Φ–Functional solutions!

☞ Γ_{eff} generates external vertex functions fulfilling Ward–Takahashi identities of symmetries

☞ External Propagator

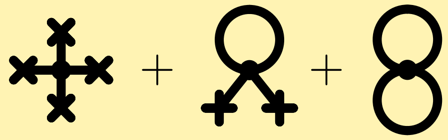
$$(G^{-1}_{\text{ext}})_{12} = \left.\frac{\delta^2 \Gamma_{\text{eff}}[\varphi]}{\delta\varphi_1 \delta\varphi_2}\right|_{\varphi=\varphi_0}$$

☞ G_{ext} generally not identical with Dyson resummed propagator

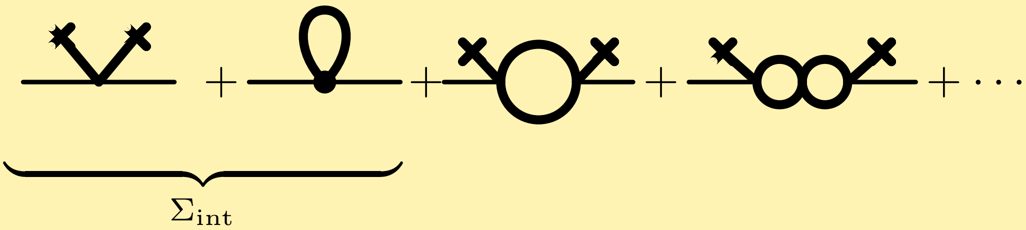
Example: Hartree approximation

External self–energy

- Hatree approximation:

$i\Phi =$ 

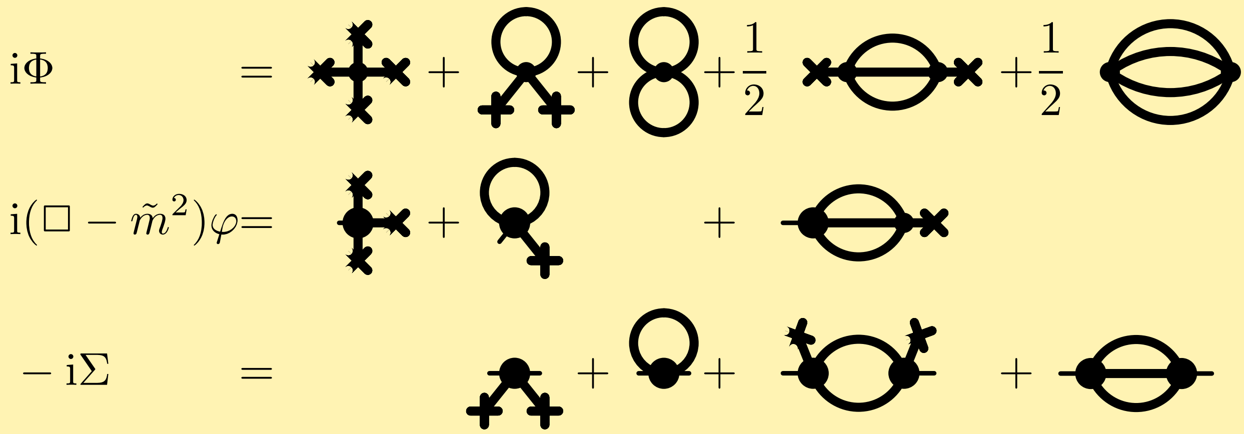
- External self–energy defined on top of Hartree approximation

$-i\Sigma_{ext} =$ 

☞ RPA–Resummation restores symmetry

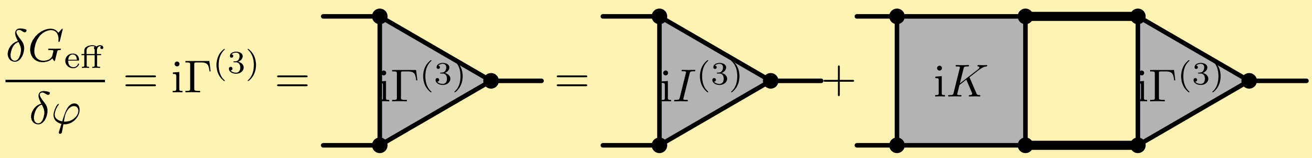
Diagrammar for external vertices I

1st step: define Φ and internal propagator

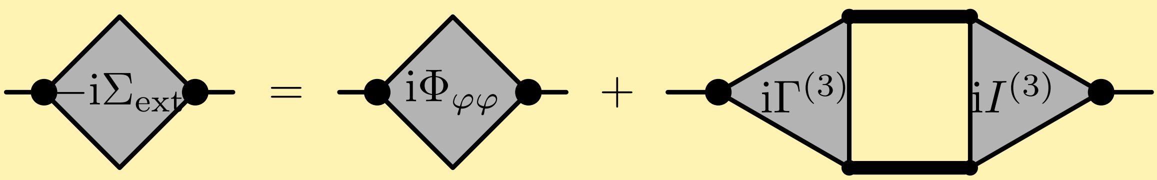


☞ Defines mean field and Dyson resummend internal propagator

2nd step: Derivatives



External self–energy

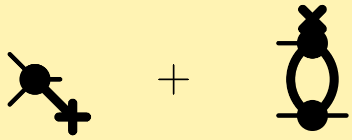


Diagrammar for external vertices II

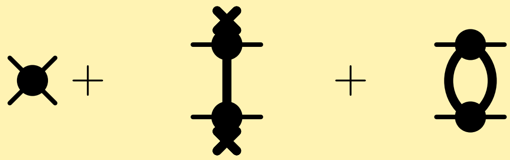
Definition of Bethe–Salpeter equation elements

$$\mathrm{i}\Phi_{\varphi,\varphi} =$$

$$\mathrm{i}I^{(3)} = \mathrm{i}\Phi_{\mathrm{i}G,\varphi} =$$



$$\mathrm{i}K = \mathrm{i}\Phi_{\mathrm{i}G,\mathrm{i}G} =$$



- Here: Green's function lines and mean fields: fixed from self–consistent Φ–Functional solution

Application to the π–ρ–System

The free vector meson

- Gauge invariant classical Lagrangian:

$$\mathscr{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} m^2 V_\mu V^\mu + \frac{1}{2} (\partial^\mu \varphi)(\partial_\mu \varphi) + m \varphi \partial_\mu V^\mu$$

- Gauge invariance:

$$\delta V_\mu(x) = \partial_\mu \chi(x), \; \delta\varphi = m\chi(x)$$

- Quantisation: Gauge fixing and ghosts

$$\begin{aligned} \mathscr{L}_V = & -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{m}{2} V_\mu V^\mu - \frac{1}{2\xi} (\partial_\mu V^\mu)^2 + \\ & + \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{\xi m^2}{2} \varphi^2 + \\ & + (\partial_\mu \eta^*)(\partial_\mu \eta) - \xi m^2 \eta^* \eta. \end{aligned}$$

- Free vacuum propagators

$$\begin{aligned} \Delta_V^{\mu\nu}(p) &= -\frac{g^{\mu\nu}}{p^2 - m^2 + \mathrm{i}\eta} + \frac{(1-\xi) p^\mu p^\nu}{(p^2 - m^2 + \mathrm{i}\eta)(p^2 - \xi m^2 + \mathrm{i}\eta)} \\ \Delta_\varphi(p) &= \frac{1}{p^2 - \xi m^2 + \mathrm{i}\eta} \\ \Delta_\eta(p) &= \frac{1}{p^2 - \xi m^2 + \mathrm{i}\eta}. \end{aligned}$$

☞Usual power counting ⇒ renormalisable
☞Partition sum: Three bosonic degrees of freedom!

Application to the π–ρ–System

Adding $\pi^{\pm}$ and γ

- Gauge–covariant derivative

$$\mathrm{D}_\mu \pi = \partial_\mu \pi + \mathrm{i} g V_\mu \pi + \mathrm{i} e A_\mu$$

☞Quantisation of free photon as usual

- Minimal coupling:

$$\mathscr{L}_{\pi V} = \mathscr{L}_V + (\mathrm{D}_\mu \pi)^* (\mathrm{D}^\mu \pi) - m_\pi^2 \pi^* \pi - \frac{\lambda}{8} (\pi^* \pi)^2 - \frac{e}{2 g_{\rho\gamma}} A_{\mu\nu} V^{\mu\nu}$$

☞Eqs. of motion: Vector meson dominance (Kroll, Lee, Zumino)

- Adding Leptons like in QED:

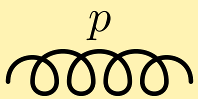
$$\mathscr{L}_{e\gamma} = \bar{\psi}(\mathrm{i}\not{D} - m_e)\psi$$

with

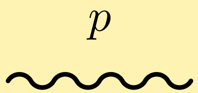
$$\mathrm{D}_\mu \psi = \partial_\mu \psi + \mathrm{i} e \psi \tag{1}$$

Application to the π–ρ–System

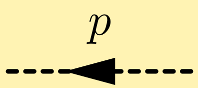
The Propagators



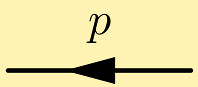
$$\mu \quad \nu = -\frac{\mathrm{i}g^{\mu\nu}}{p^2 - m_\rho^2 + \mathrm{i}\eta} + \frac{\mathrm{i}(1-\xi_\rho)p^\mu p^\nu}{(p^2 - m_\rho^2 + \mathrm{i}\eta)(p^2 - \xi m_\rho^2 + \mathrm{i}\eta)}$$



$$\mu \quad \nu = -\frac{\mathrm{i}g^{\mu\nu}}{p^2 + \mathrm{i}\eta} + \frac{\mathrm{i}(1-\xi_\gamma)p^\mu p^\nu}{(p^2 + \mathrm{i}\eta)^2}$$



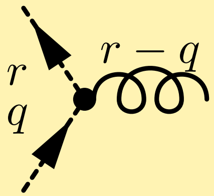
$$= \frac{\mathrm{i}}{p^2 - m_\pi^2 + \mathrm{i}\eta}$$



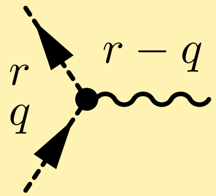
$$= \frac{\mathrm{i}(\not{p} + m)}{p^2 - m_l^2 + \mathrm{i}\eta}$$

Application to the π–ρ–System

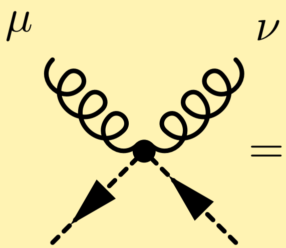
The Vertices



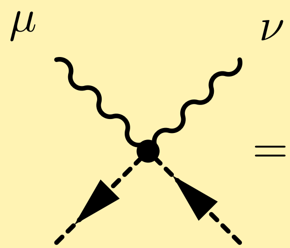
$$= -\mathrm{i}g_{\rho\pi}(q^\mu + r^\mu)$$



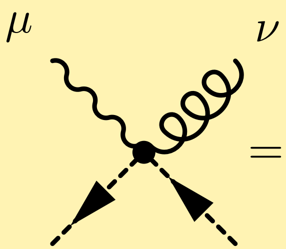
$$= -\mathrm{i}e(q^\mu + r^\mu)$$



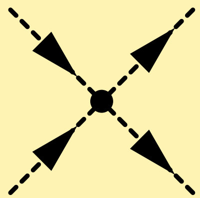
$$= \mathrm{i}g_{\rho\pi}^2 g^{\mu\nu}$$



$$= \mathrm{i}e^2 g^{\mu\nu}$$



$$= 2\mathrm{i}e g_{\rho\pi} g^{\mu\nu}$$



$$= -\frac{\mathrm{i}\lambda}{8}$$

$$= \mathrm{i}\frac{e}{g_{\rho\gamma}} p^2 \Theta^{\mu\nu}(p)$$

Application to Vector bosons

- Kroll–Lee–Zumino interaction: Coupling of massive vector bosons to conserved currents ⇒ gauge theory
- Symmetry breaking at correlator level

Problems:

☞ Internal propagators contain spurious degrees of freedom

☞ Negative norm states

☞ Numerically instable due to light cone singularities

- Classical picture (Fokker–Planck–equation):

$$\Pi^{\mu\nu}(\tau, \vec{p} = 0) \propto \langle v^\mu(\tau) v^\nu(0) \rangle$$

- „One–loop" approximation in the classical limit

$$\Pi^{\mu\nu}(\tau, \vec{p} = 0) \propto \exp(-\Gamma\tau)$$

☞ $1/\Gamma$: Relaxation time scale due to scattering

- Exact behaviour:

$$\Pi^{00}(\tau, \vec{p} = 0) \propto \langle 1 \cdot 1 \rangle = \text{const}$$

$$\Pi^{jk}(\tau, \vec{p} = 0) \propto \left\langle v^j v^k \right\rangle \propto \exp(-\Gamma_x \tau)$$

☞ For Π^{jk}: If $\Gamma \approx \Gamma_x$ ⇒ 1–loop approximation justified

☞ Classical limit also shows:
Π^{jk} only slightly modified by ladder resummation

- In self–consistent approximations:
Use only $p_j p_k \Pi^{jk}$ and $g_{jk} \Pi^{jk}$

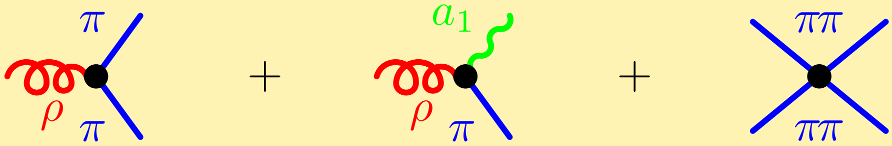
☞ Construct Π_T and Π_L

The interacting π-ρ-a_1 system

The self–consistent approximation

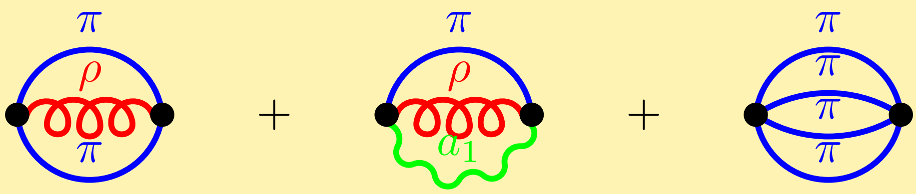
Lagrangian:

$\mathscr{L}_{\mathrm{int}} =$



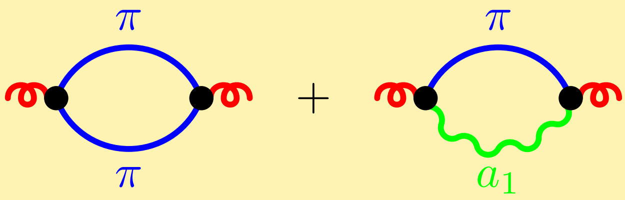
Φ–Funktional:

$\Phi =$

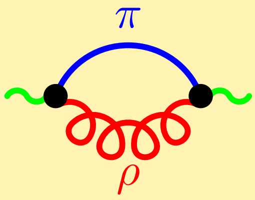


Self–energies:

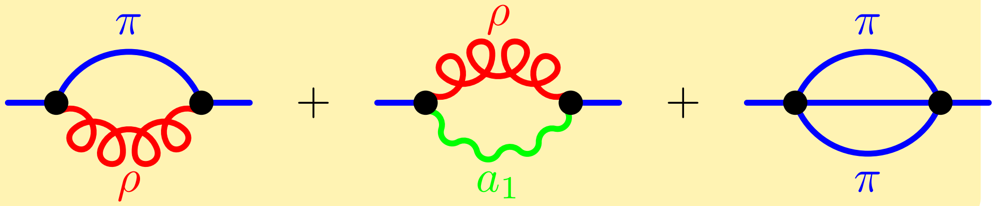
$\Pi_\rho =$



$\Pi_{a_1} =$

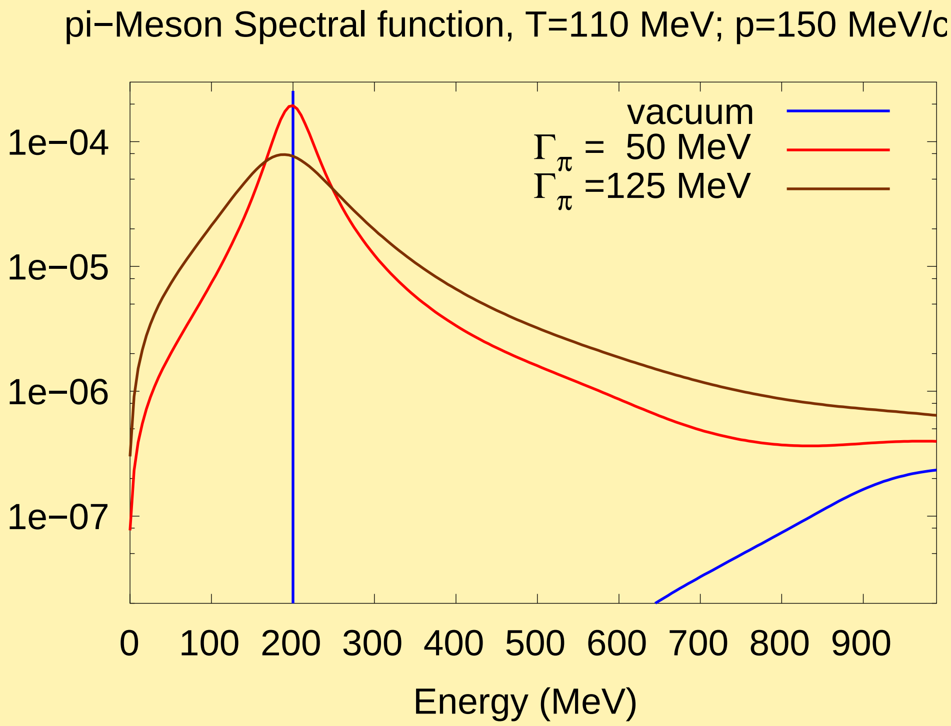


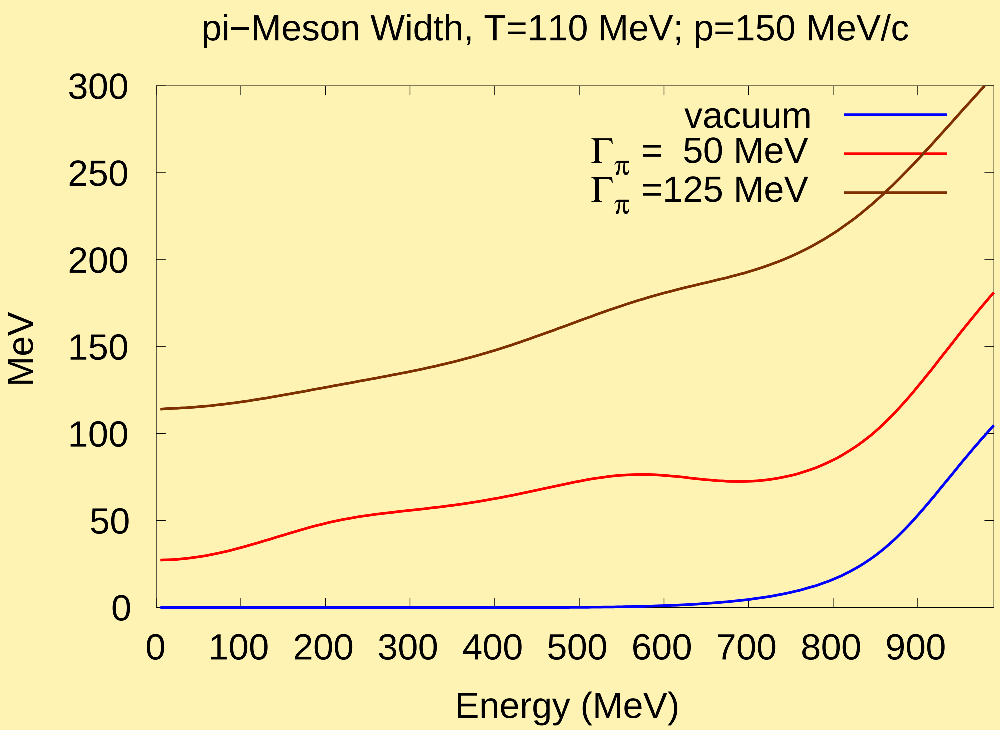
$\Sigma_\pi =$



Results for the $\pi\rho a_1$–System

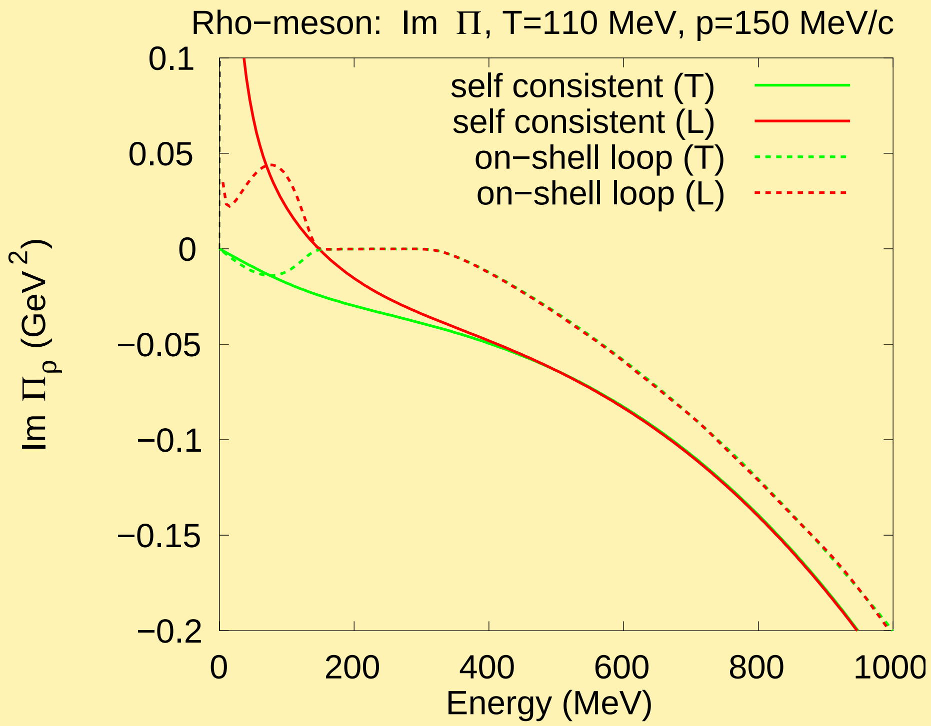
Broad pions in the medium

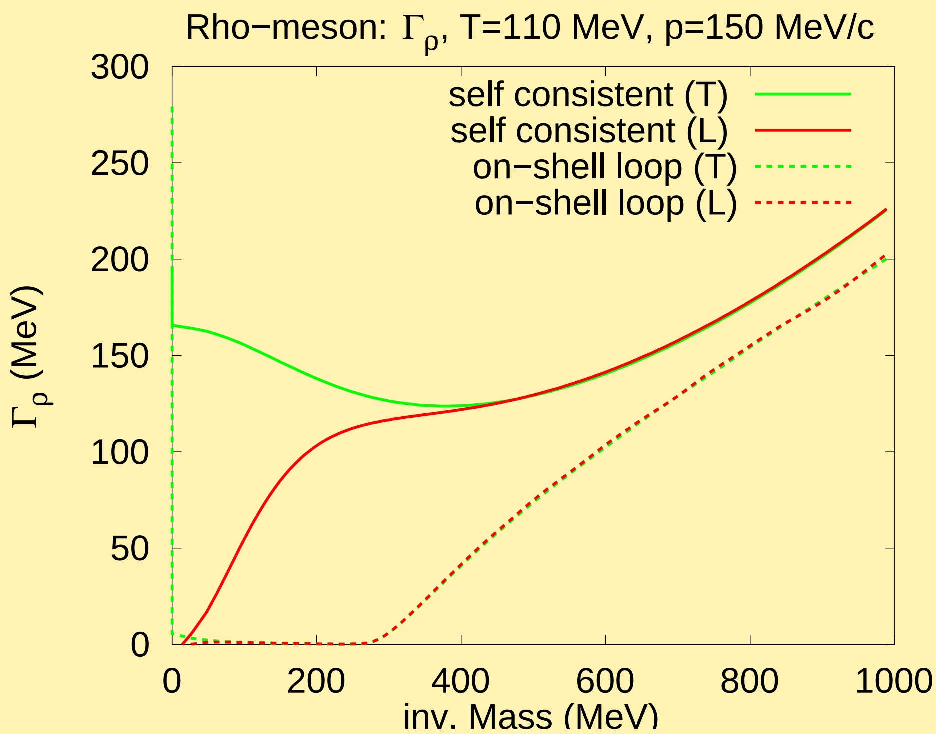




Results for the $\pi\rho a_1$–System

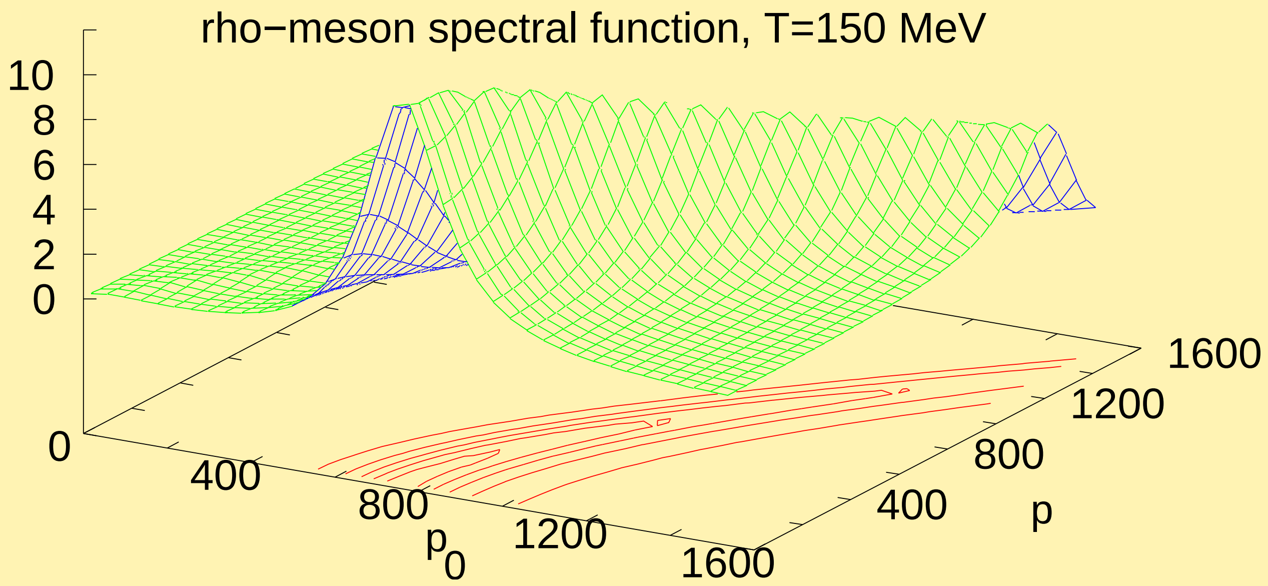
ρ–meson Polarisation tensor

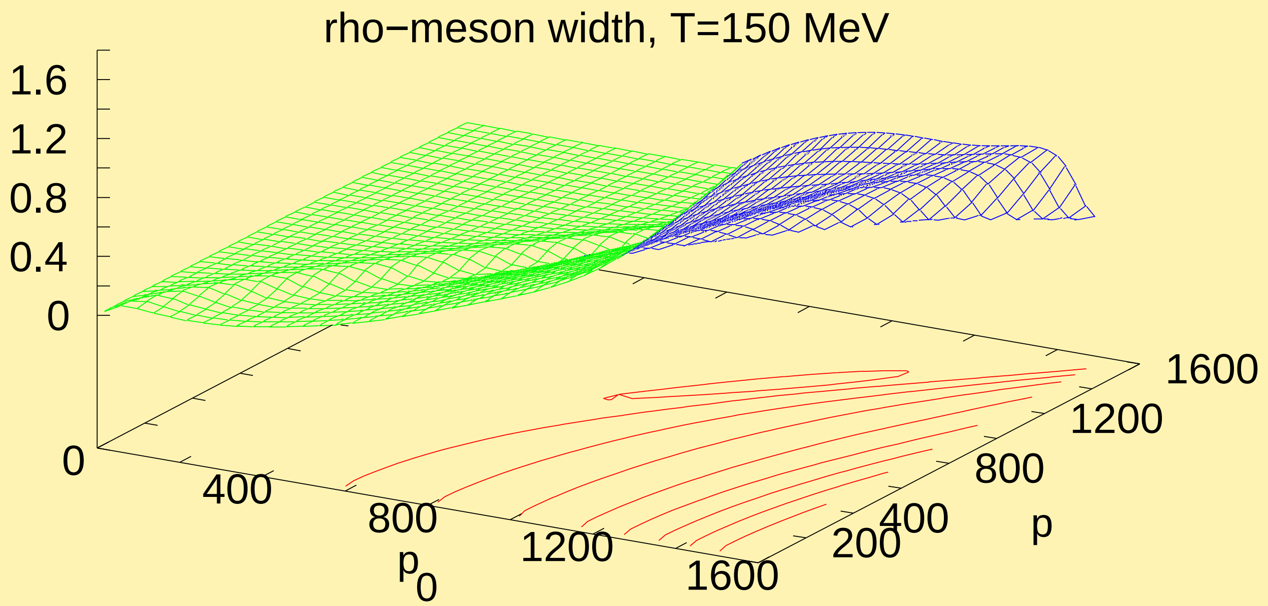




Results for the $\pi\rho a_1$–System

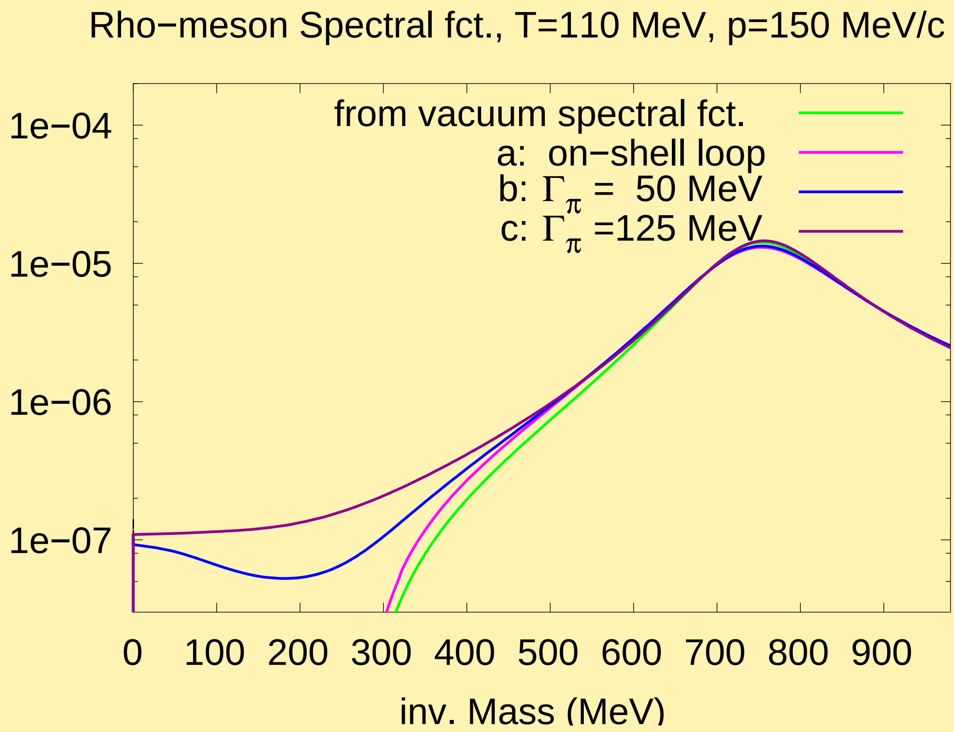
ρ–meson properties I

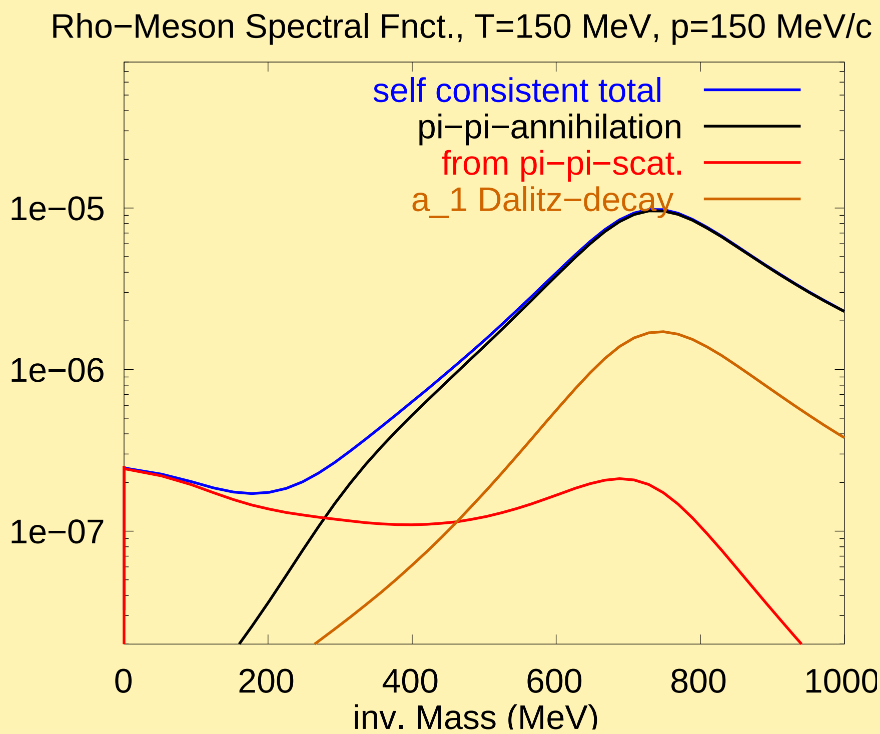




Results for the $\pi\rho a_1$–System

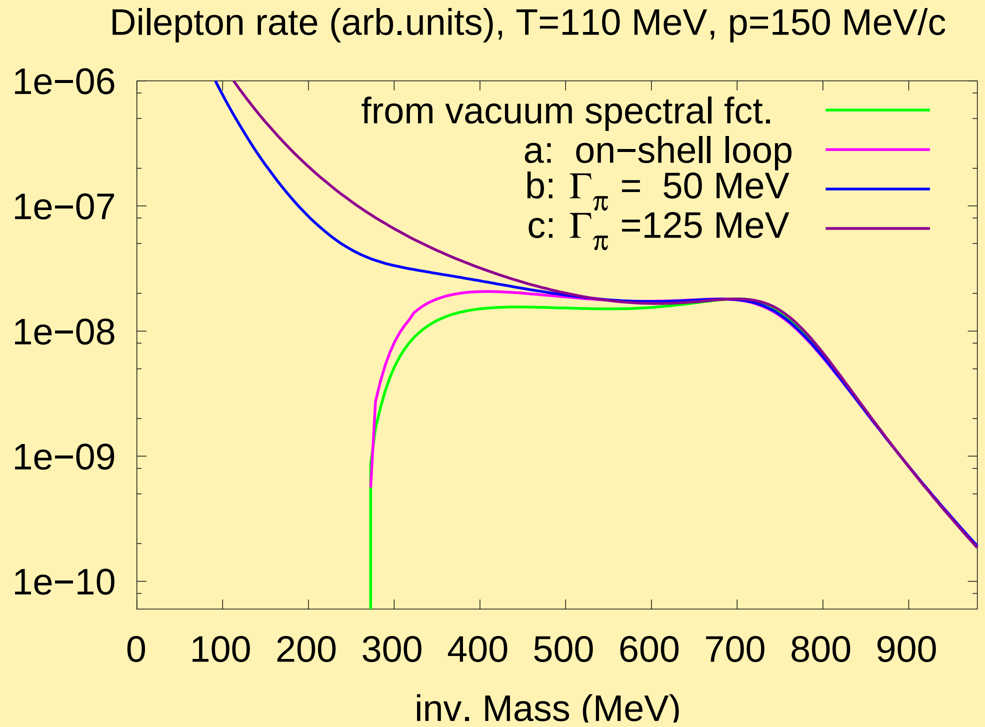
ρ–meson properties II

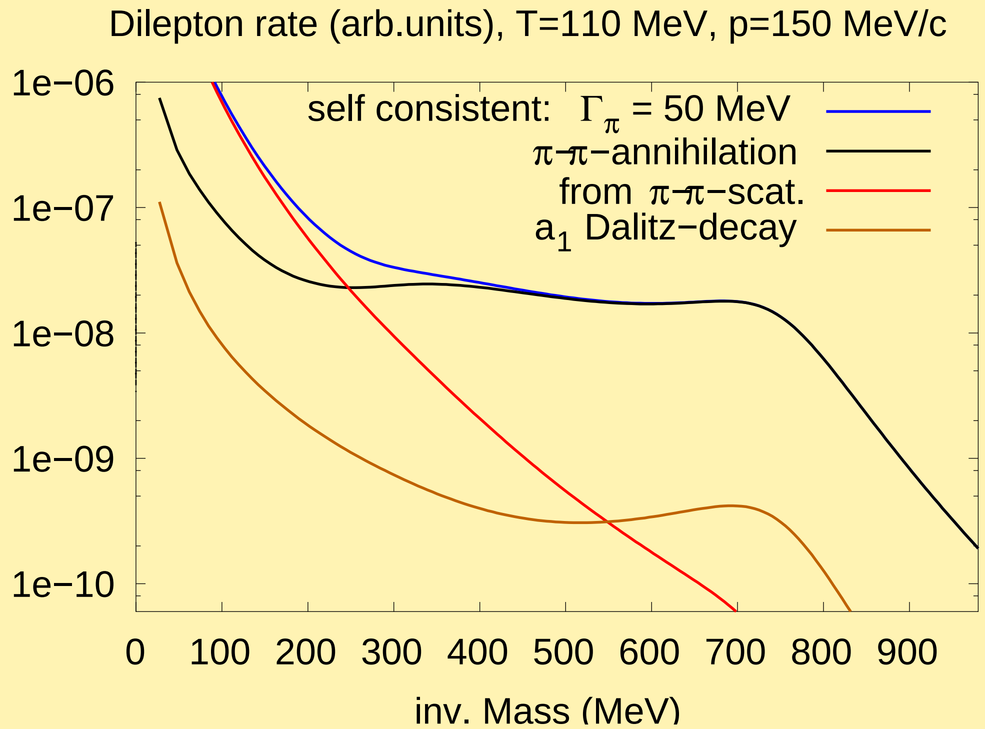




Results for the $\pi\rho a_1$–System

Dilepton rate





Conclusions

Summary

- Self–consistent Φ–derivable schemes
- Renormalization
- Symmetry analysis
- Scheme for vector particles
- Numerical treatment

Outlook

- "Toolbox" for application to realistic models
- Perspectives for self–consistent treatment of gauge theories
- QCD e.g. beyond HTL?
- Transport equations for particles with finite width