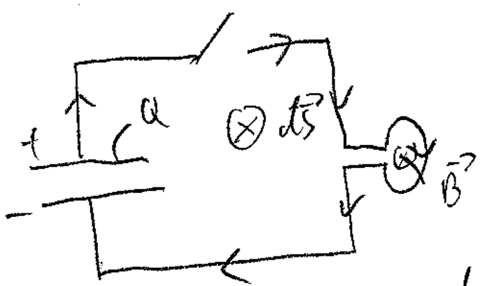
Solutions of problems chapt. 12

(1)



$$\oint \vec{E}\, d\vec{r} = -\frac{Q}{C} = -L\frac{di}{dt}$$

$$i = -\frac{dQ}{dt} \Rightarrow -L\frac{di}{dt} = -L\frac{d^2Q}{dt^2} = \frac{Q}{C}$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

$$Q(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\dot{Q}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \stackrel{!}{=} -i(t)$$

$$\ddot{Q}(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \stackrel{!}{=} -\frac{1}{LC}\left(A\cos(\omega t) + B\sin(\omega t)\right]$$

$$\Rightarrow \omega^2 = \frac{1}{LC} \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

Initial conditions:

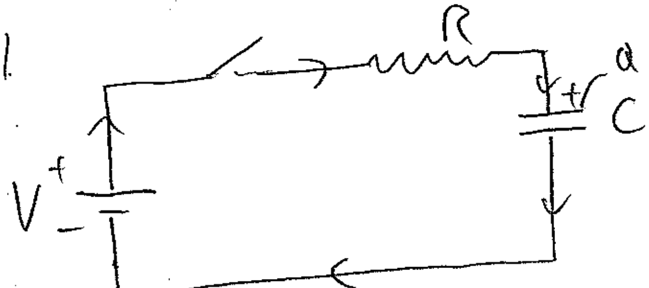
$$Q(t=0) = Q_0; \quad i(t=0) = 0$$

$$\Rightarrow A = Q_0 \qquad \Rightarrow B = 0$$

$$Q(t) = Q_0\cos(\omega t)$$

$$i(t) = -\dot{Q}(t) = Q_0\,\omega\sin(\omega t)$$

(2)



$$\oint d\vec{r}\,\vec{E} = -V + Ri + \frac{Q}{C} = 0; \quad i = \frac{dQ}{dt}$$

$$\Rightarrow R\frac{dQ}{dt} + \frac{Q}{C} = V$$

Homogeneous equation

$$R\dot{Q}_H + \frac{Q}{C} = 0 \Rightarrow \dot{Q}_H = -\frac{Q}{RC}$$

$$Q_H(t) = A\exp\left(-\frac{t}{RC}\right)$$

Particular solution of inhomogeneous eq.

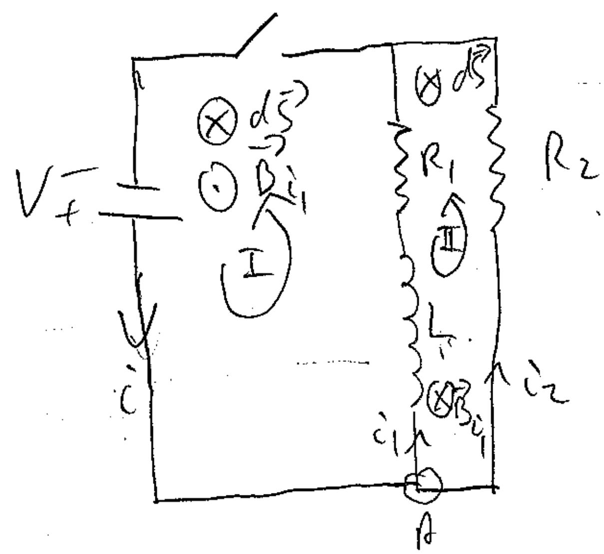
$$Q_I = \text{const} \Rightarrow \frac{Q_I}{C} = V \Rightarrow Q_I = CV$$

$$\Rightarrow Q(t) = Q_I + Q_H(t) = CV + A\exp\left(-\frac{t}{RC}\right)$$

$$Q(0) = Q_0 \Rightarrow CV + A = Q_0 \Rightarrow A = Q_0 - CV$$

$$\Rightarrow Q(t) = CV + (Q_0 - CV)\exp\left(-\frac{t}{RC}\right)$$

(3)



$$i = i_1 + i_2$$

$$①: \oint d\vec{r}\cdot\vec{E} = -V + i_1 R_1 = -L\frac{di_1}{dt}$$

$$②: \oint d\vec{r}\cdot\vec{E} = R_2 i_2 - R_1 i_1 = +L\frac{di_1}{dt}$$

Add ① and ②

$$-V + R_2 i_2 = 0 \Rightarrow i_2 = \frac{V}{R_2}$$

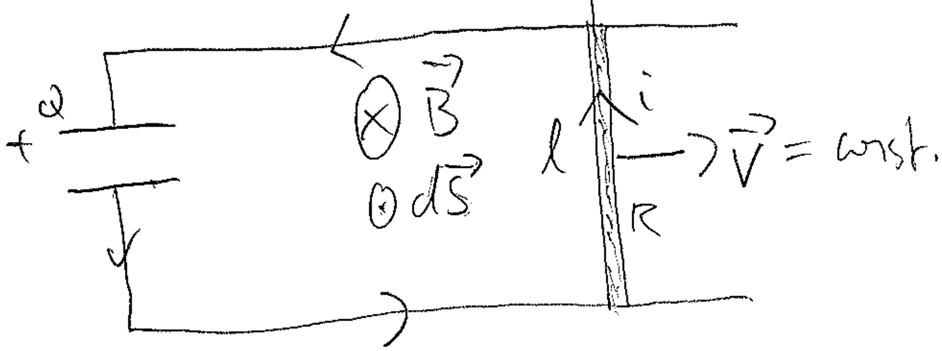
For i_1 we solve ①. Now we know the general solution for the RL circuit and can write right away

$$i_1(t) = \frac{V}{R_1} + A\exp\left(-\frac{R_1}{L}t\right)$$

Initial condition $i_1(t=0) = 0$

$$\Rightarrow i_1(t) = \frac{V}{R_1}\left[1 - \exp\left(-\frac{R_1}{L}t\right)\right]$$

(4)



No self-inductance $\Rightarrow$

$$\oint d\vec{r}\,\vec{E} = iR + \frac{Q}{C} = l v B$$

$$i = \frac{dQ}{dt} = \dot{Q}$$

$$\Rightarrow \quad \dot{Q} + \frac{Q}{RC} = \frac{lvB}{R}$$

$$Q(t) = A\exp\left(-\frac{t}{RC}\right) + lvBC$$

$$Q(t=0) = 0$$

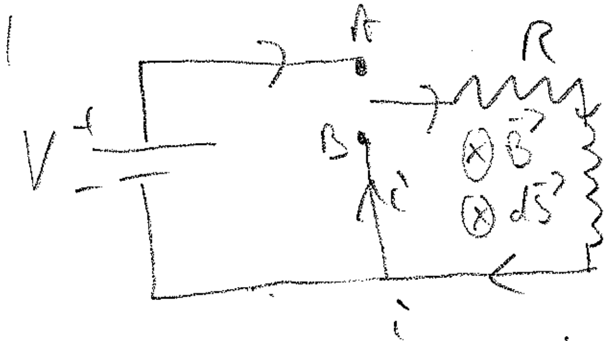
$$\Rightarrow \quad Q(t) = lvBC\left[1 - \exp\left(-\frac{t}{RC}\right)\right]$$

(5) $$i\dot{Q} + \frac{Q}{C} = lvB - \frac{d}{dt}(Li)$$

Here L changes with time either, because the geometry of the loop changes. We cannot calculate it with our means.

Exercises - Chapter 12

(1)



If switch is in position B, we have

$$\oint d\vec{r}\,\vec{E} = R i = -L \frac{di}{dt}$$

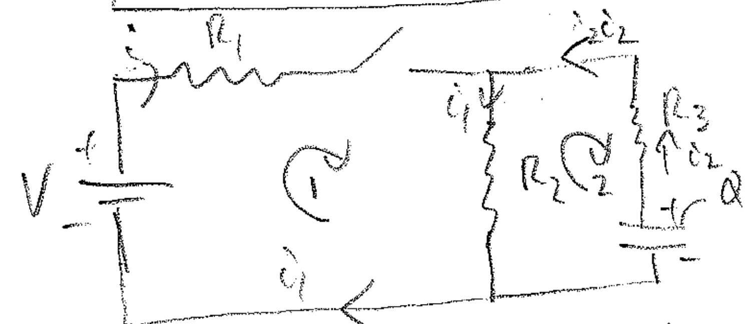
$$\Rightarrow i(t) = i_0 \exp\left(-\frac{R}{L}t\right)$$

The initial condition is determined by the steady state when the switch is in position A for a long time:

$$i_0 = \frac{V}{R}$$

$$\Rightarrow \boxed{i(t) = \frac{V}{R} \exp\left(-\frac{R}{L}t\right)}$$

(2)



First we solve the steady state problem when the switch is closed for a long time. Then $i_2 = 0$ and $i = i_1$

Then

$$① \Rightarrow \int \vec{E}\, d\vec{r} = -V + R_1 i_1 + R_2 i_1 = 0$$

$$\Rightarrow i_1 = \frac{V}{R_1 + R_2}$$

(2) $\Rightarrow \oint d\vec{r}\,\vec{E} = \frac{Q}{C} - R_3 i_2 - R_2 i_1 = 0$

because of $i_2 = 0 \Rightarrow Q = C R_2 i_1 = C R_2 \frac{V}{R_1 + R_2}$

If the switch is opened, $i = 0 = i_1 - i_2 \Rightarrow i_1 = i_2$

and loop (2) gives

$$\frac{Q}{C} = (R_2 + R_3)\, i_2$$

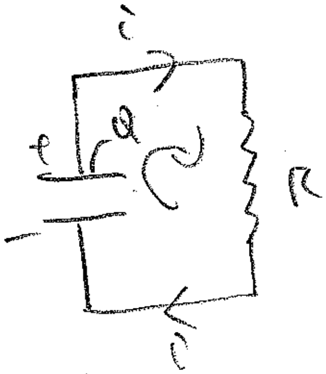
but a our direction of i_2 : $i_2 = -\frac{dQ}{dt} = -\dot{Q}$

$$\Rightarrow \dot{Q} = -\frac{Q}{C(R_2 + R_3)}$$

$$\Rightarrow Q(t) = Q_0 \exp\left(-\frac{t}{C(R_2 + R_3)}\right)$$

$$Q(t) = \frac{C R_2 V}{R_1 + R_2} \exp\left[-\frac{t}{C(R_2 + R_3)}\right]$$

(3)



$$\oint d\vec{r}\,\vec{E} = iR - \frac{Q}{C} = 0$$

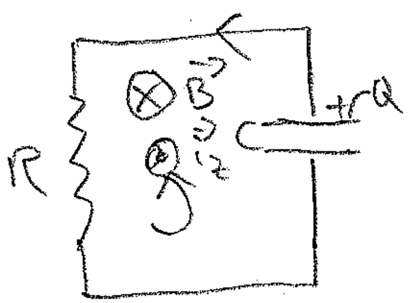
$$i = -\frac{dQ}{dt} = -\dot{Q}$$

$$\Rightarrow \dot{Q} = -\frac{Q}{RC}$$

$$Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right)$$

$$i(t) = -\dot{Q}(t) = \frac{Q_0}{RC} \exp\left(-\frac{t}{RC}\right)$$

(4)



$\vec{B} = -B_0 \sin(\omega t)\,\vec{i}_z$

$$\oint \vec{E}\,d\vec{r} = -\frac{Q}{C} + iR = \frac{d}{dt}\left[A B_0 \sin(\omega t)\right]$$

$$= A B_0 \omega \cos(\omega t)$$

$$i = -\frac{dQ}{dt}$$

$$\Rightarrow R\dot{Q} + \frac{Q}{C} = -A B_0 \omega \cos(\omega t)$$

Homogeneous equation

$$\dot{Q}_H = -\frac{Q_H}{RC} \Rightarrow Q_H(t) = a \exp\left(-\frac{t}{RC}\right)$$

Inhomogeneous equation

Ansatz: $Q_I(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$

$$\dot{Q}_I(t) = -\alpha\omega \sin(\omega t) + \beta\omega \cos(\omega t)$$

$$R\left[-\alpha\omega \sin(\omega t) + \beta\omega \cos(\omega t)\right] + \frac{1}{C}\left[\alpha \cos(\omega t) + \beta \sin(\omega t)\right]$$

$$= -A B_0 \omega \cos(\omega t)$$

$$\Rightarrow -R\alpha\omega + \frac{\beta}{C} = 0 \Rightarrow \beta = R\omega C\alpha$$

$$\frac{\alpha}{C} + \beta\omega R = -A B_0 \omega$$

$$\alpha\left(\frac{1}{C} + R^2\omega^2 C\right) = -A B_0 \omega$$

$$\alpha = -\frac{A B_0 \omega C}{1 + (R\omega C)^2}$$

$$\beta = -\frac{R A \omega^2 C^2 B_0}{1 + (R\omega C)^2}$$

$$Q(t) = \alpha \cos(\omega t) + \beta \sin(\omega t) - \alpha \exp\left(-\frac{t}{RC}\right)$$

(from initial condition $Q(t=0)=0$).

The full story is as follows

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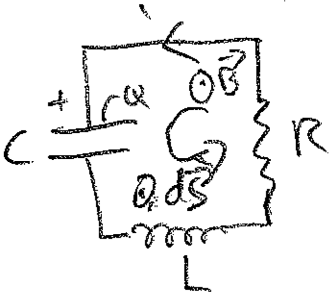

$$\oint d\vec{r}\,\vec{E} = \frac{Q}{C} + Ri = -L\frac{di}{dt}$$

$$i = +\frac{dQ}{dt} = \dot{Q}$$

$$\Rightarrow L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = 0$$

We expect a damped harmonic behaviour. Thus the right ansatz is

$$Q(t) = \exp(-\beta t)\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

We must have two linearly independent functions and two integration constants, because we have a 2^{nd} order linear ODE.

Now:

$$\dot{Q} = -\beta \exp(-\beta t)\left[A\cos(\omega t) + B\sin(\omega t)\right]$$
$$+ \exp(-\beta t)\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right]$$
$$= \exp(-\beta t)\left[(B\omega - \beta A)\cos(\omega t) - (B\beta + A\omega)\sin(\omega t)\right]$$

$$\ddot{Q} = \exp(-\beta t)\left[-\omega(B\omega - \beta A)\sin(\omega t) + \omega(B\beta + A\omega)\cos(\omega t)\right]$$

$$Q(t) = \exp(-\beta t)\, q(t)$$

$$\dot{Q}(t) = -\beta \exp(-\beta t)\, q + \exp(-\beta t)\, \dot{q}$$

$$= \exp(-\beta t)\, (\dot{q} - \beta q)$$

$$\ddot{Q}(t) = -\beta \exp(-\beta t)\, (\dot{q} - \beta q) + \exp(-\beta t)\, (\ddot{q} - \beta \dot{q})$$

$$= \exp(-\beta t)\, [\ddot{q} - 2\beta \dot{q} + \beta^2 q]$$

$$\Rightarrow \exp(-\beta t) \left[L(\ddot{q} - 2\beta\dot{q} + \beta^2 q) + R(\dot{q} - \beta q) + \frac{q}{C} \right] = 0$$

This is simplifying, if the coefficient in front of $\dot{q}$ is made to vanish:

$$-2\beta L + R = 0 \Rightarrow \beta = \frac{R}{2L}$$

Then the remaining ODE for q reads

$$L\ddot{q} + \left(L\beta^2 - \beta R + \frac{1}{C}\right) q = 0$$

$$L\ddot{q} + \left(\frac{R^2}{4L} - \frac{R^2}{2L} + \frac{1}{C}\right) q = 0$$

$$L\ddot{q} + \left(\frac{1}{C} - \frac{R^2}{4L}\right) q = 0$$

$$\Rightarrow \ddot{q} = -\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right) q$$

Now we have to consider three cases

(a) $\frac{1}{LC} - \frac{R^2}{4L^2} > 0.$

Then we have

$$\ddot{q} = -\omega^2 q \quad \text{with} \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

The general solution in this case is

$$q(t) = A\cos(\omega t) + B\sin(\omega t)$$

(b) $\frac{1}{LC} - \frac{R^2}{4L^2} = 0$

$\Rightarrow \ddot{q} = 0 \Rightarrow q(t) = At + B$

(c) $\frac{1}{LC} - \frac{R^2}{4L^2} < 0$

$\Rightarrow \ddot{q} = \lambda^2 q$ with $\lambda = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

Then the general solution is

$q(t) = A\exp(+\lambda t) + B\exp(-\lambda t)$

Solution for original equation

case (a)

$Q(t) = \exp\left(-\frac{Rt}{2L}\right)\left[A\cos(\omega t) + B\sin(\omega t)\right]$

with $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

Initial condition

$Q(t=0) = Q_0$

$\dot{Q}(t=0) = 0$

$Q(t=0) = A = Q_0$

$\dot{Q}(t) = -\frac{R}{2L}\exp\left(-\frac{Rt}{2L}\right)\left[A\cos(\omega t) + B\sin(\omega t)\right]$

$\exp\left(-\frac{Rt}{2L}\right)\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right]$

$\dot{Q}(t=0) = -\frac{R}{2L}A + B\omega = 0 \Rightarrow B = \frac{R}{2L\omega}$

$Q(t) = \exp\left(-\frac{Rt}{2L}\right)\left[Q_0\cos(\omega t) + \frac{R}{2\omega L}\sin(\omega t)\right]$

case (b)

$$Q(t) = \exp\left(-\frac{Rt}{2L}\right)(At + B)$$

$$Q(t=0) = Q_0 = B$$

$$\dot{Q}(t) = \exp\left(-\frac{Rt}{2L}\right)\left[-\frac{R}{2L}(At+B) + A\right]$$

$$\dot{Q}(t=0) = A - \frac{R}{2L}B = 0 \Rightarrow A = \frac{R}{2L}B = \frac{R}{2L}Q_0$$

$$Q(t) = Q_0 \exp\left(-\frac{Rt}{2L}\right)\left[\frac{R}{2L}t + 1\right]$$

case (c)

$$Q(t) = \exp\left(-\frac{Rt}{2L}\right)\left[A\exp(\lambda t) + B\exp(-\lambda t)\right],$$

with $\lambda = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

$$Q(t) = A\exp\left[\left(\lambda - \frac{R}{2L}\right)t\right] + B\exp\left[-\left(\lambda + \frac{R}{2L}\right)t\right]$$

Note that this is always damped, because

$$-\frac{R}{2L} \pm \lambda < 0$$

$$\dot{Q}(t) = A\left(\lambda - \frac{R}{2L}\right)\exp\left[\left(\lambda - \frac{R}{2L}\right)t\right] - B\left(\lambda + \frac{R}{2L}\right)\exp\left[-\left(\lambda + \frac{R}{2L}\right)t\right]$$

$$Q(t=0) = A + B = Q_0$$

$$\dot{Q}(t=0) = \lambda(A - B) - \frac{R}{2L}(A+B) = 0$$

$$A - B = \frac{R}{2L\lambda}Q_0$$

$$A = \frac{1}{2} Q_0 \left(1 + \frac{R}{2L\lambda}\right)$$

$$B = \frac{1}{2} Q_0 \left(1 - \frac{R}{2L\lambda}\right)$$