Problems Chapter IV (Lecture Feb/9/)

(1)

$$\Phi_S^{(1)} = \int_S d\vec{S}\,\vec{E}$$

A plane has a constant surface element. Proof: A finite plane surface can always be described by

$$S: \vec{r}(\lambda_1,\lambda_2) = \vec{r}_0 + \lambda_1\vec{i}_1 + \lambda_2\vec{i}_2$$

where $\vec{i}_1$ and $\vec{i}_2$ are arbitrary vectors which are not parallel to each other

$$\Rightarrow d\vec{S} = \frac{\partial\vec{r}}{\partial\lambda_1}\times\frac{\partial\vec{r}}{\partial\lambda_2}\,d\lambda_1\,d\lambda_2 = \vec{i}_1\times\vec{i}_2\,d\lambda_1\,d\lambda_2$$

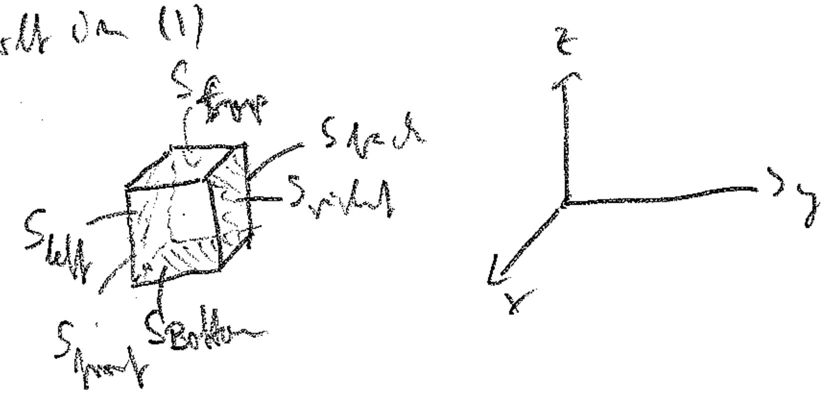
and this is constant along the surface (the λ_1, λ_2 run over a finite area of the plane)

With $\vec{E}$ = const, you obtain

$$\Phi_S^{1} = \int_S d\vec{S}\,\vec{E} = \int_A d\lambda_1\int d\lambda_2\,\vec{E}\cdot(\vec{i}_1\times\vec{i}_2) = \vec{E}\cdot\vec{S}$$

where $\vec{S}\perp S$ and $|\vec{S}| = A_S$, the area of the plane.

(2) Use results from (1)

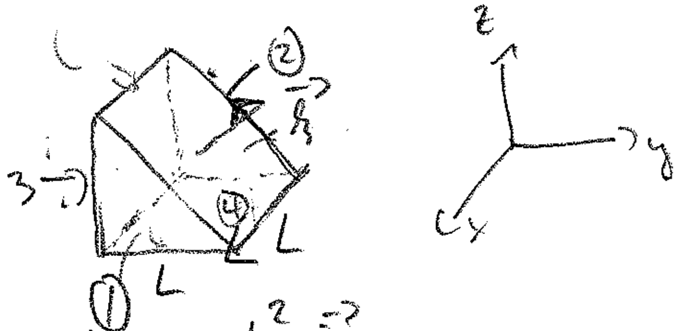


$-\vec{S}_{left} = \vec{S}_{right} = \vec{i}_y$ ...

$\Phi_{right} = -\Phi_{left} = E_x L^2$ ...

$\Rightarrow \Phi^{(2)}_{cube} = 0$

(3)



$$\vec{S}_1 = -\vec{S}_2 = \frac{L^2}{2}\vec{i}_x$$

$$\vec{S}_3 = -L^2\vec{i}_y$$

$$\vec{S}_4 = -L^2\vec{i}_z$$

$$\vec{S}_5 = L^2\sqrt{2}\;\vec{n}\times\vec{i}_x$$

$$\vec{n} = \frac{1}{\sqrt{2}}\left(-\vec{i}_y + \vec{i}_z\right)$$

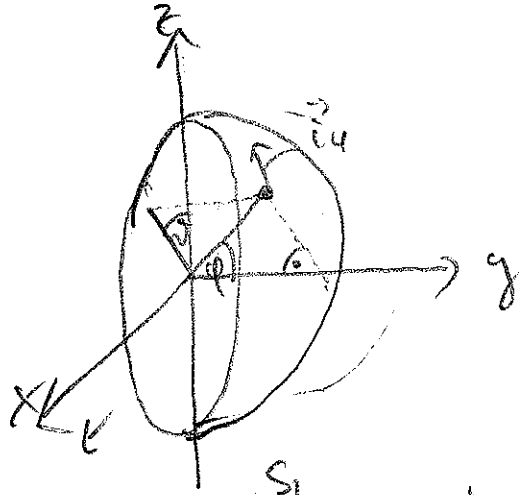
$$\vec{n}\times\vec{i}_x = \frac{1}{\sqrt{2}}\left(-\vec{i}_y + \vec{i}_z\right)\times\vec{i}_x$$

$$= \frac{1}{\sqrt{2}}\left(\vec{i}_z + \vec{i}_y\right)$$

$$\vec{S}_5 = L^2\left(\vec{i}_z + \vec{i}_y\right)$$

$$\Phi^{(1)} = \vec{E}\cdot\left(\vec{S}_1 + \cdots + \vec{S}_5\right) = 0$$

(4)



$\varphi \in (0, \pi/2)$
$\vartheta \in (0, 2\pi)$

The hemispherical S_1 is most easily solved by brute force. The only real problem is to read off all the distances correctly

$$S_1 : \vec{r}(\varphi, \vartheta) = R \begin{pmatrix} \sin\varphi \sin\vartheta \\ \cos\varphi \\ \sin\varphi \cos\vartheta \end{pmatrix} = R\, \vec{i}_r$$

Surface vector

$$\pm d\vec{S} = \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \vartheta} R^2 \, d\varphi \, d\vartheta$$

$$= R^2 \begin{pmatrix} \cos\varphi \sin\vartheta \\ -\sin\varphi \\ \cos\varphi \cos\vartheta \end{pmatrix} \times \begin{pmatrix} \sin\varphi \cos\vartheta \\ 0 \\ -\sin\varphi \sin\vartheta \end{pmatrix} d\varphi \, d\vartheta$$

$$= R^2 \begin{pmatrix} \sin^2\varphi \sin\vartheta \\ \sin\varphi \cos\varphi \cos^2\vartheta + \sin\varphi \cos\varphi \sin^2\vartheta \\ \sin^2\varphi \cos\vartheta \end{pmatrix} d\varphi \, d\vartheta$$

$$= R^2 \sin\varphi \begin{pmatrix} \sin\varphi \sin\vartheta \\ \cos\varphi \\ \sin\varphi \cos\vartheta \end{pmatrix} d\varphi \, d\vartheta$$

$$= R^2 \sin\varphi \, \hat{i}_r$$

need upper sign since $d\vec{S}$ should point such that $dS_y > 0$ (4)

$$\Rightarrow \Phi_1^{(4)} = \int_{S_1} d\vec{S}\cdot\vec{E} = ER^2\int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta \, \sin\varphi \, \vec{i}_r \, \vec{i}_y$$

$$= ER^2\int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta \, \sin\varphi \cos\varphi$$

$$= 2\pi R^2 \int_0^{\pi/2} d\varphi \, \sin\varphi \cos\varphi$$

$$= \pi E R^2 \int_0^{\pi/2} d\varphi \sin(2\varphi)$$

$$= \pi E R^2 \left[-\frac{1}{2}\cos(2\varphi)\right]_0^{\pi/2}$$

$$= \pi R^2 E$$

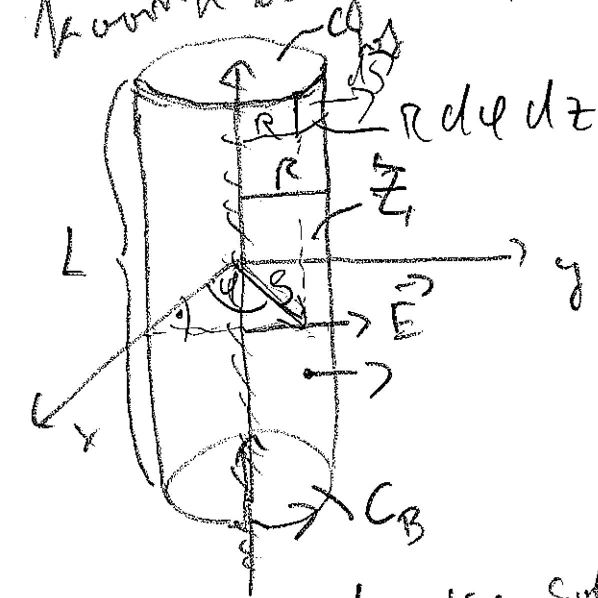
Since the circular disk is plane, we can use (1) again

$$\Phi_2^{(4)} = \vec{S}_2\cdot\vec{E} = (-\pi R^2 \vec{i}_y)\,\vec{E} = -\pi R^2 E$$

$$\Phi_{tot}^{(4)} = \Phi_1^{(4)} + \Phi_2^{(4)} = 0$$

<u>One other example</u>

Take my favorite "long-wire problem"



The electrical field is given by (see solution to quiz 2):

$$\vec E = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{x\vec i_x + y\vec i_y}{x^2+y^2}$$

What's the electrical flux through the surface of the above given cylinder

$\Phi_{C_B} = \Phi_{C_T} = 0$, because $\vec S_T = -\vec S_B = \pi R^2 \vec i_z$

$\Rightarrow \vec S_T \vec E = -\vec S_B \vec E \equiv 0$ everywhere on these surfaces.

Need the flux through cylinder Z

<u>Parametrization</u>

$Z: \vec r(\varphi, z) = R(\cos\varphi\,\vec i_x + \sin\varphi\,\vec i_y) + z\,\vec i_z$

$\varphi \in (0, 2\pi)$; $z \in (-L/2, L/2)$

$$d\vec S = \frac{\partial \vec r}{\partial \varphi} \times \frac{\partial \vec r}{\partial z}\,d\varphi\,dz = R(-\sin\varphi\,\vec i_x + \cos\varphi\,\vec i_y) \times \vec i_z\,d\varphi\,dz$$

$$= R(\cos\varphi\,\vec i_x + \sin\varphi\,\vec i_y)\,d\varphi\,dz$$

$$\int_{\vec r \in Z_1} d\vec S \; E(\vec r) = \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi \; R(\cos\varphi \, \vec i_x + \sin\varphi \, \vec i_y) \cdot \frac{\lambda}{2\pi\varepsilon_0} \frac{R(\cos\varphi \, \vec i_x + \sin\varphi \, \vec i_y)}{R^2}$$

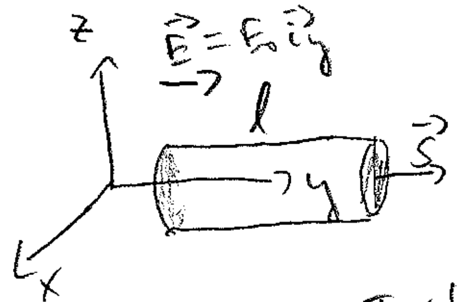
$$= \frac{\lambda}{2\pi\varepsilon_0} \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi \cdot 1$$

$$= \frac{\lambda}{\varepsilon_0} \cdot L \quad (2)$$

That we could have told without calculation, because of Gauß's law. The total charge inside Z_1 is λL!

Exercises Chapter IV (Review 02/10)

(1)



Constant $\vec E$-field. For the endcaps we can use solution of problem 1.

(a) $\vec S_1 = \pi R^2 \vec i_y \Rightarrow \Phi_1^{(1)} = \pi R^2 E_0$

(b) $\vec S_2 = -\pi R^2 \vec i_y \Rightarrow \Phi_2^{(1)} = -\pi R^2 E_0$

(c) $d\vec S_3 = (\cos\vartheta\, \vec i_x + \sin\vartheta\, \vec i_z)\,\frac{1}{2}\,\ell R\, d\vartheta$

$\Rightarrow \Phi_3^{(1)} = \int d\vec S_3\, \vec E = 0$

(d) $\Phi^{(1)} = \Phi_1 + \Phi_2 + \Phi_3 = 0$

(2) (a) and (b): $\Phi_1 = \Phi_2 = 0$, because $\vec S_1 = \pi R^2 \vec i_z = \vec S_2$.

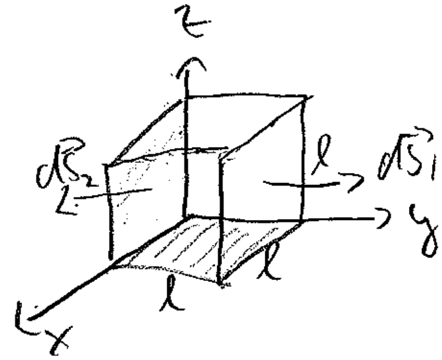
(c) $d\vec S = \frac{1}{2} R\, \ell\, d\vartheta\, (\cos\vartheta\, \vec i_x + \sin\vartheta\, \vec i_y)$

$$\Phi_3 = \int_0^{2\pi} d\vartheta\, \vec E\, \frac{1}{2} R l\, (\cos\vartheta\, \vec i_x + \sin\vartheta\, \vec i_y)$$

$$= \int_0^{2\pi} d\vartheta\, \frac{Rl}{2} E_0 \sin\vartheta = 0$$

(d) $\Phi_{tot} = 0$.

(3)



$d\vec{S}_1 = dx\,dz\,\vec{i}_y$

$\vec{S}_1: \vec{r} = x\vec{i}_x + l\vec{i}_y + z\vec{i}_z;\quad x,z \in [0,l]$

$$\Rightarrow \Phi_1 = \int_{C_1} d\vec{S}_1\, \vec{E}(\vec{r}) = \int_0^l dx \int_0^l dz\ \beta l$$

$$= \beta l^3$$

$d\vec{S}_2 = -dx\,dz\,\vec{i}_y \quad \Rightarrow \Phi_2 = \int_{C_2} d\vec{S}_1 \cdot \vec{E}(\vec{r}) = 0$

$\vec{S}_2: \vec{r} = x\vec{i}_x + z\vec{i}_z$

The same goes through with the other 4 surfaces

$$\Rightarrow \Phi_{tot} = (\alpha+\beta+\gamma)\, l^3$$

[Check with local form etc:

$\operatorname{div}\vec{E} = \alpha+\beta+\gamma = \text{const}$

volume of cube is l^3 $\Rightarrow$ passt

$$\int_{\partial V} d\vec{S}\cdot\vec{E} = \int_V dV \operatorname{div}\vec{E} = (\alpha+\beta+\gamma)\, l^3$$]