Name: .... Section: ....

Physics 208 Quiz 7

Solutions

Problem 1 (20 Points)

Take the magnetic field of an infinitely long current-carrying wire, which reads in our standard cylinder coordinates

$$\vec{B} = \frac{\mu_0 i}{2\pi\rho} \vec{i}_\varphi(\varphi). \tag{1}$$

(a) Express the field in terms of Cartesian coordinates.

(b) Express the field in terms of spherical coordinates.

(c) Show that the $\vec{B}$ field fulfills the condition

$$\oint_{S_R} \mathrm{d}\vec{S} \cdot \vec{B} = 0, \tag{2}$$

where S_R is the sphere (surface) with radius R around the origin of the coordinate system.

Note: As explained in the lecture, Eq. (2) expresses the fact that there are no magnetic monopoles found in nature. This is one of the fundamental laws of electromagnetism which is always valid, not only for time-independent fields. It is one of Maxwell's equations (in integral form)!

Solution

(a) In Cartesian coordinates

$$\vec{B} = \frac{\mu_0 i}{2\pi\sqrt{x^2+y^2}} \left(\frac{-y\vec{i}_x + x\vec{i}_y}{\sqrt{x^2+y^2}} \right) \tag{3}$$

(b) In spherical coordinates we also have $\vec{i}_\varphi$ which is of course the same as for cylinder coordinates, because it is the same angle in both cases. Since $\rho = r\sin\vartheta$, we find

$$\vec{B} = \frac{\mu_0 i}{2\pi r\sin\vartheta} \vec{i}_\varphi(\varphi). \tag{4}$$

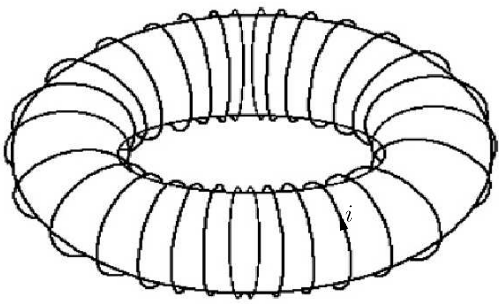
(c) In spherical coordinates the surface-element vector for the sphere with radius R around the origin is

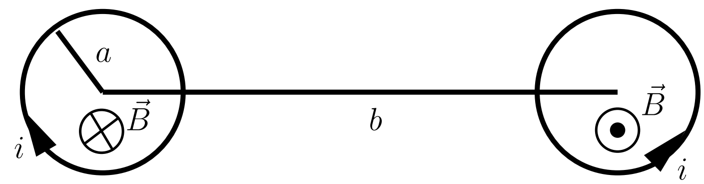
$$\mathrm{d}\vec{S} = R^2 \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \, \vec{i}_r(\varphi). \tag{5}$$

Since $\vec{i}_r(\varphi) \cdot \vec{i}_\varphi(\varphi) = 0$ for all φ, because the vectors are always perpendicular to each other, the surface integral over $\vec{B}$ vanishes, as it must be.

Problem 2 (40 Points)

A wire is wrapped N times around a torus ("donut") as shown in the figures, and a current is made running through the coil as indicated.

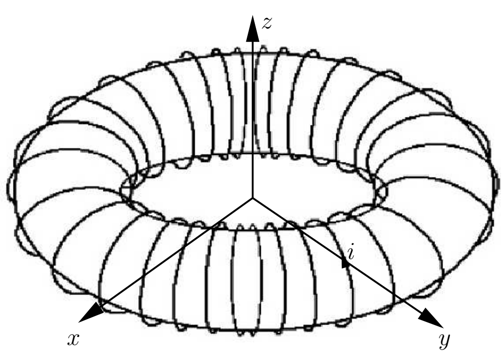




Use Ampère's Circuital Law to calculate the magnetic field, $\vec{B}$, everywhere!

Solution

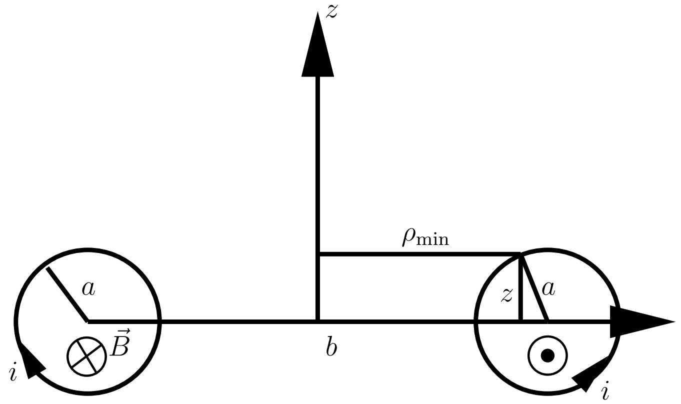
First we assign a coordinate system as shown:

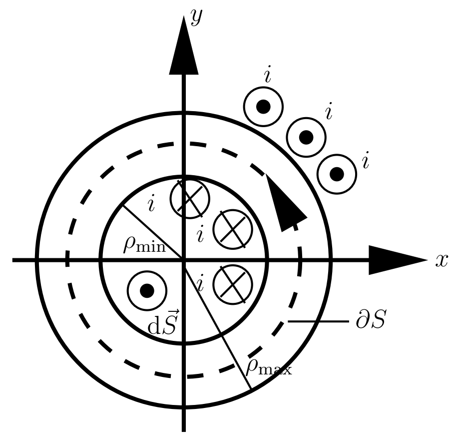


We use the usual cylinder coordinates coordinates (ρ, ϕ, z). Due to the symmetry of the problem we can assume that

$$\vec{B}(\vec{r}) = B_\phi(\rho, z)\vec{i}_\phi(\phi). \tag{6}$$

According to this symmetry, in Ampère's Circuital Law, we use circles parallel to the xy-plane (dashed line of the right drawing in the next figure). We orient them in the usual sense (counter-clockwise), so that $\mathrm{d}\vec{r}$ is pointing in direction of $\vec{i}_\phi$ everywhere.





It is clear that we can intersect the torus with a plane parallel to the xy-plane only if $-a < z < a$ (see left figure). In such a case this plane is drawn in the right figure. From the left figure we also read that for such a z a point $\vec{r}(\rho, \varphi, z)$ is inside the torus for $\rho_{\text{min}} < \rho < \rho_{\text{max}}$, and we read off from this figure that

$$\begin{aligned} \rho_{\text{min}} &= \frac{b}{2} - \sqrt{a^2 - z^2}, \\ \rho_{\text{max}} &= \frac{b}{2} + \sqrt{a^2 - z^2}. \end{aligned} \tag{7}$$

If now our dashed line is in the whole of the torus (i.e., $\rho < \rho_{\text{min}}$), then we encircle no currents and thus, according to Ampère's circuital Law,

$$\int_{\partial S} \mathrm{d}\vec{r} \cdot \vec{B}(\vec{r}) = 2\pi\rho B_{\varphi}(\rho) = \mu_0 i_{\text{inside}}, \tag{8}$$

we have $B_{\varphi} = 0$.

If our dashed line runs outside the torus (i.e., $\rho > \rho_{\text{max}}$) there are N currents of magnitude i going in and also N currents going out. Thus, the total current inside the line vanishes again, and thus also there $B_{\varphi} = 0$.

Finally, for a path inside the torus, we encircle N currents, running into the plane. According to the right-hand-rule convention to orient the boundary ∂S to its encircled surface, the surface-normal vector comes out of the plane, and thus $i_{\text{inside}} = -Ni$. So the complete solution is

$$B_{\varphi}(\rho) = \begin{cases} -\frac{\mu_0 N i}{2\pi\rho} & \text{for } \rho_{\text{min}} < \rho < \rho_{\text{max}} \text{ and } -a < z < a \\ 0 & \text{everywhere else.} \end{cases}, \tag{9}$$

where ρ_{min} and ρ_{max} are given in Eq. (7).

Note that the $-$ sign in Eq. (9) corresponds to the directions of $\vec{B}$ as indicated in the first figure above!

Problem 3 (40 Points)

Use the Biot-Savart Law to calculate the magnetic field, $\vec{B}(z)$, along the axis of a tightly wound cylindrical coil of radius, R, and finite length, L, carrying a current, i. Discuss the limiting cases $L \gg R$ and $L \ll R$.

For extra credit: Can you find the magnetic dipole moment by taking $z \gg L$?

Hints: If A is the cross-sectional area of the wire, the current-density vector per unit cylinder length (in standard-cylinder coordinates) is given by

$$\vec{j}\,' = \frac{Ni}{LA}\vec{i}_\varphi, \tag{10}$$

where N is the total number of loops. In the Biot-Savart Law, at each point along the cylinder axis you have to integrate over the volume of the wire looping around the cylinder there, but the volume element simplifies to $\mathrm{d}^3\vec{r}\,' = AR\mathrm{d}\varphi'$. Finally you have to integrate along the cylinder axis to sum up all loops. Thus, here the Biot-Savart Law takes the form

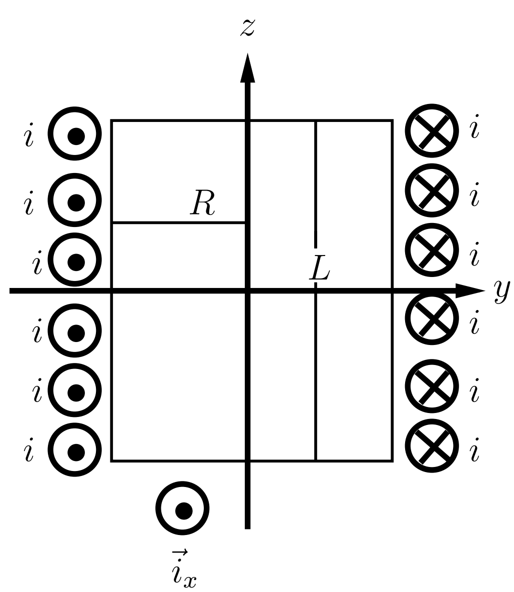
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{-L/2}^{L/2}\mathrm{d}z'\int_0^{2\pi}\mathrm{d}\varphi' AR\frac{\vec{j}\,'(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}, \tag{11}$$

where I have written $\vec{r}\,'$ in terms of cylinder coordinates ρ', φ', z'.

You can use the integrals

$$\int \mathrm{d}\varphi'\,\vec{i}_\rho(\varphi') = 0, \quad \int \mathrm{d}x\frac{1}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}. \tag{12}$$

Note also that you can solve this problem only along the symmetry axis of the coil, i.e., for $\vec{r} = z\vec{i}_z$. So do not struggle to calculate the field elsewhere!



Solution

According to the above described parameterization of the source point $\vec{r}'$ we have $\vec{r}' = R\vec{i}_\rho(\varphi') + z'\vec{i}_z$. Further $\vec{r} = z\vec{i}_z$ since we want the $\vec{B}$ field on the z axis only. Now we have

$$\vec{j}'(\vec{r}') \times (\vec{r} - \vec{r}') = \frac{Ni}{LA}\,\vec{i}_\varphi(\varphi') \times [(z - z')\vec{i}_z - R\,\vec{i}_\rho(\varphi')] = \frac{Ni}{LA}[(z - z')\vec{i}_\rho(\varphi') + R\vec{i}_z]. \tag{13}$$

The first term vanishes upon integration over φ' in the integral (11). Thus we are left with

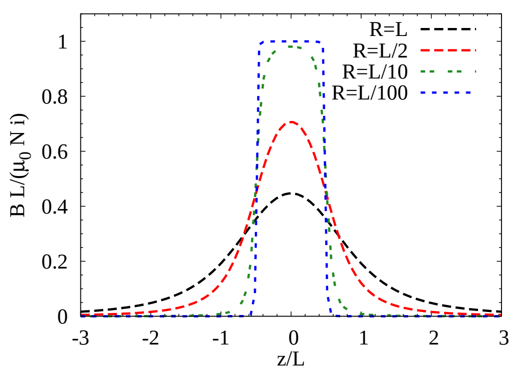
$$\vec{B}(z\vec{i}_z) = \frac{\mu_0 Ni}{4\pi LA} 2\pi \int_{-L/2}^{L/2} \mathrm{d}z' \frac{AR^2\vec{i}_z}{[(z - z')^2 + R^2]^{3/2}}, \tag{14}$$

where we have done the trivial integral over φ', giving a factor 2π, because the integrand is independent of φ' (as to be expected from the symmetry of the problem with respect to rotations around the z axis).

The remaining integral is given in the hint, Eq. (12), leading to the final result:

$$\vec{B}(z\vec{i}_z) = \frac{\mu_0 Ni}{2L}\left[\frac{z + L/2}{\sqrt{R^2 + (z + L/2)^2}} - \frac{z - L/2}{\sqrt{R^2 + (z - L/2)^2}}\right]\vec{i}_z. \tag{15}$$

We plot (15) for different ratios of R/L. This clearly shows that the simplified derivation for a long solenoidal coil, given in the lecture is indeed approximately valid, if z is not too close to the ends of the coil, provided that $L \gg R$, i.e., "long" is to be understood in comparison to R.



Finally we look at three limiting cases:

1. $L \gg R$ and $|z| \ll L$.

   Then we can take the limit $L \to \infty$ in the bracket, giving

$$\vec{B}(z\vec{i}_z) = \frac{\mu_0 Ni}{L}\vec{i}_z, \tag{16}$$

   which coincides with the result for a long coil, we have obtained in the lecture.

2. $|z| \gg L/2$ and $|z| \gg R$.

   Then we can use an expansion of (15) with respect to $1/z$, giving in leading order

$$\vec{B} = \frac{\mu_0 N i}{2} \frac{R^2}{z^3} \vec{i}_z, \tag{17}$$

   which is the field of a magnetic dipole with $\vec{r}$ pointing in direction of the dipole moment (compare our discussion in the lecture, when we calculated the $\vec{B}$-field of a single loop along the symmetry axis). Comparing the pertinent formula (see, e.g., Quiz 6!) with (17) gives

$$\vec{p}_m = \mu_0 \pi N i R^2 \vec{i}_z. \tag{18}$$

3. $L \ll R$ and $L \ll |z|$.

   Then we can expand (15) around $z = 0$, leading to

$$\vec{B} = \frac{\mu_0 N i R^2}{2(R^2 + z^2)^{3/2}} \vec{i}_z. \tag{19}$$

   For $N = 1$ this is the result we obtained for a single loop along the symmetry axis.