Name: .... Section: ....

Physics 208 Quiz 5

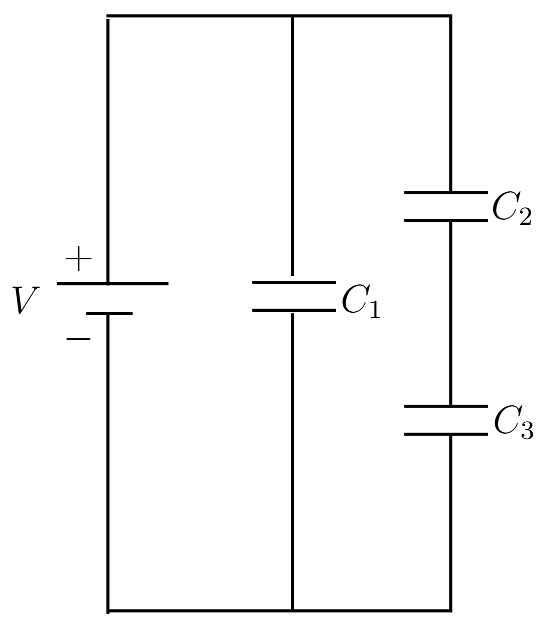
March 03, 2008 (due March 07, 2008)

Problem 1

(a) Three capacitors are hooked to an ideal battery with voltage V as shown in the figure. Calculate the total capacitance of this circuit.

(b) How much charge is stored on the (positively charged) upper plates of each capacitor and how much in total?

(c) How much energy is stored in each capacitor and how much in total?



Solutions

(a) First take capacitors C_2 and C_3 which are obviously connected in series. They can thus be substituted by a capacitor with capacitance

$$C_{23} = \frac{C_2 C_3}{C_2 + C_3}. \tag{1}$$

Then this capacitor is hooked up to the battery parallel to C_1. Thus, the total capacitance of the three capacitors is

$$C = C_1 + C_{23} = C_1 + \frac{C_2 C_3}{C_2 + C_3}. \tag{2}$$

(b) On the total capacitor we have a charge

$$Q = CV. \tag{3}$$

On C_1 and C_2 sits the same charge (I always give the charge on the positively charged plates):

$$Q_2 = Q_3 = Q_{23} = C_{23} V = \frac{C_2 C_3}{C_2 + C_3} V. \tag{4}$$

On C_1 we have

$$Q_1 = C_1 V. \tag{5}$$

Note that the total charge moved from the plus terminal to the positively charged plates is

$$Q = Q_1 + Q_{23} = CV \tag{6}$$

since the (negative) charge on the lower plate of C_2, which is $-Q_{23}$ compensates the (positive) charge on the upper plate of C_3.

(c) The total energy stored in the electric fields between the plates of the capacitors is

$$W = \frac{1}{2} C V^2, \tag{7}$$

where C is given by Eq. (2).

In C_1 we have

$$W_1 = \frac{1}{2} C_1 V^2. \tag{8}$$

To get the energy stored in C_2 and C_3 we need the voltages on these capacitors which can be calculated with help of (4):

$$V_2 = \frac{Q_2}{C_2} = \frac{C_3}{C_2 + C_3} V, \quad V_3 = \frac{Q_3}{C_3} = \frac{C_2}{C_2 + C_3} V. \tag{9}$$

Of course, we have $V_2 + V_3 = V$ as it should be.

Now we can calculate the energy contents:

$$\begin{aligned} W_2 &= \frac{1}{2} C_2 V_2^2 = \frac{C_2}{2} \frac{C_3^2}{(C_2 + C_3)^2} V^2 = \frac{C_{23}^2}{2C_2} V^2, \\ W_3 &= \frac{1}{2} C_3 V_3^2 = \frac{C_3}{2} \frac{C_2^2}{(C_2 + C_3)^2} V^2 = \frac{C_{23}^2}{2C_3} V^2. \end{aligned} \tag{10}$$

We have used Eq. (1) for the final expressions, because then we can easily check the consistency of these calculations. Let's see, how much energy is stored on capacitors C_2 and C_3 together:

$$W_2 + W_3 = \frac{C_{23}^2}{2} \left(\frac{1}{C_2} + \frac{1}{C_3} \right) = \frac{1}{2} C_{23} V^2, \tag{11}$$

as it should be. From this it's also easy to check that the total energy comes out right:

$$W = W_1 + W_2 + W_3 = \frac{1}{2}(C_1 + C_{23})V^2 = \frac{1}{2}CV^2. \tag{12}$$

Of course this total energy came from the battery.

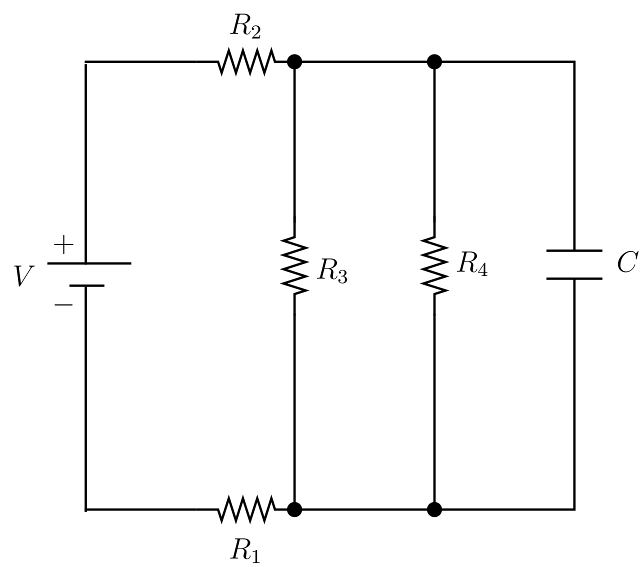
Problem 2

(a) For the network given in the circuit diagram below, calculate the currents through each resistor and the voltage at the capacitor. [Hint: Express all results in terms of the quantities given in the circuit diagram!]

(b) Suppose, you only need to know the voltage at the capacitor. How can you simplify the task to find it, compared to the complete network analysis, you have done in part (a)? Compare the results!

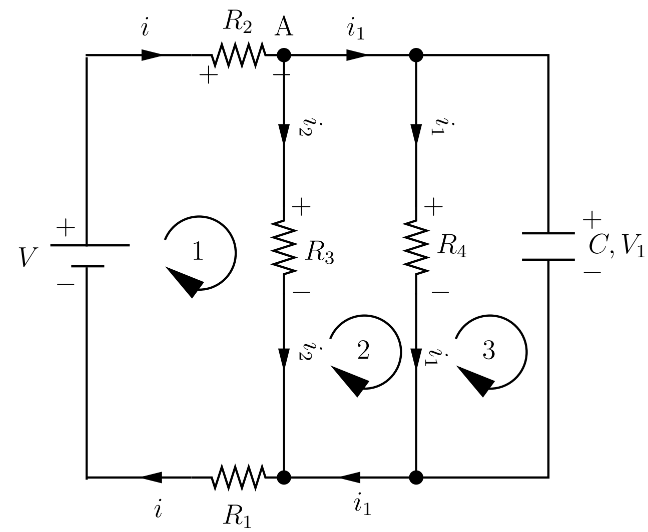
(c) What is the total power used by this circuit?

(d) How much energy is stored in the capacitor?



Solutions

(a) We assign currents through the resistors with directions of the surface vectors along the wire and ± assignments at the terminals of the batteries, resistors, and the capacitor. We have also already used some obvious continuity equations to relate the currents running along the upper horizontal line of wires and resistors and the lower one. We also have already figured in the fact that (in the steady state considered here) there is no current through the capacitor:



Now we use the circuital law (Kirchhoff's 2$^{\text{nd}}$ law) to the three loops shown in the figure:

$$
\begin{aligned}
1: &\quad V - R_1 i - R_3 i_2 - R_2 i = 0, && (13)\\
2: &\quad R_3 i_2 - R_4 i_1 \qquad\qquad = 0, && (14)\\
3: &\quad R_4 i_1 - V_1 \qquad\qquad\quad = 0. && (15)
\end{aligned}
$$

Finally we use the continuity equation (Kirchhoff's 1st law) at the knot, labeled A, to find a fourth equation

$$A: \quad -i + i_1 + i_2 = 0. \tag{16}$$

The next step is to sort the unknowns of the system of linear equations on the left-hand side and the inhomogeneous terms on the right side. To use Gauss's elimination algorithm efficiently, we write the linear equations in matrix-vector notation:

$$\begin{pmatrix} R_1 + R_2 & 0 & R_3 & 0 \\ 0 & -R_4 & R_3 & 0 \\ 0 & R_4 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ i_1 \\ i_2 \\ V_1 \end{pmatrix} = \begin{pmatrix} V \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{17}$$

Now one can systematically multiply and add/subtract rows of the matrix, given by the matrix on the left-hand side of the equation, augmented by the column given on the right-hand side to bring the matrix in "upper triangle form". This procedure (Gauss's elimination algorithm) gives the equivalent system of linear equations:

$$\begin{pmatrix} R_1 + R_2 & 0 & R_3 & 0 \\ 0 & -R_4 & R_3 & 0 \\ 0 & 0 & -R_3 & 1 \\ 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} i \\ i_1 \\ i_2 \\ V_1 \end{pmatrix} = \begin{pmatrix} V \\ 0 \\ 0 \\ R_3 R_4 V \end{pmatrix}, \tag{18}$$

where

$$c = (R_1 + R_2 + R_3) R_4 + R_3 (R_1 + R_2) = (R_1 + R_2)(R_3 + R_4) + R_3 R_4. \tag{19}$$

Now one can easily solve for i, i_1, i_2, and V_1 "from the bottom to the top":

$$\begin{aligned} V_1 &= \frac{R_3 R_4}{c} V, \\ i_2 &= \frac{V_1}{R_3} = \frac{R_4}{c} V \\ i_1 &= \frac{R_3 i_2}{R_4} = \frac{R_3}{c} V \\ i &= \frac{V - R_3 i_2}{R_1 + R_2} = \frac{R_3 + R_4}{c} V, \end{aligned} \tag{20}$$

where c is given by Eq. (19).

(b) If we do not need to know all the currents through the resistors, but only the voltage at the capacitor, we can first combine R_3 and R_4 to an auxiliary resistor, R_5. Since R_3 and R_4 are in parallel, we have

$$R_5 = \frac{R_3 R_4}{R_3 + R_4}. \tag{21}$$

Drawing a simplified circuit diagram with R_5 substituting R_3 and R_4 gives a single-loop circuit which is much easier to analyze than the more complicated detailed network. We have three resistors in series, and thus the total resistance of the circuit is

$$R = R_1 + R_2 + R_5 = \frac{c}{R_3 + R_4}, \tag{22}$$

where c is again given by (19).

The current through R_5 is identical with the one labeled i above. In our simplified scheme it's easily calculated to be

$$i = \frac{V}{R} = \frac{R_3 + R_4}{c} V, \tag{23}$$

which of course coincides with our finding in (20).

The voltage on the capacitor, V_1, is the one on R_5 and thus

$$V_1 = R_5 i = \frac{R_3 R_4}{c} V, \tag{24}$$

again in agreement with (20).

(c) Here we have two possibilities. The most easy one is to use our simplified circuit, we figured out in (2): The the power (energy per unit time, converted to heat in the resistors) is

$$P = R i^2 = \frac{V^2}{R} = \frac{R_3 + R_4}{c} V^2. \tag{25}$$

The more cumbersome method is to add up all the powers of the single resistors,

$$
\begin{aligned}
P_1 &= R_1 i^2 = \frac{R_1(R_3+R_4)^2}{c^2} V^2, \\
P_2 &= R_2 i^2 = \frac{R_2(R_3+R_4)^2}{c^2} V^2, \\
P_3 &= R_3 i_2^2 = \frac{R_3 R_4^2}{c^2} V^2, \\
P_4 &= R_4 i_1^2 = \frac{R_4 R_3^2}{c^2} V^2,
\end{aligned} \tag{26}
$$

and then add them up to the total power

$$
P = P_1 + P_2 + P_3 + P_4 = \frac{R_3+R_4}{c^2}[(R_1+R_2)(R_3+R_4)+R_3R_4]V^2 = \frac{R_3+R_4}{c} V^2 \tag{27}
$$

which is, of course the same as (25).

(d) The energy stored in the capacitor is given by

$$
W = \frac{C}{2} V_1^2 = \frac{C}{2}\left(\frac{R_3R_4}{c}\right)^2 V^2. \tag{28}
$$

Remark

In our DC context, capacitors are quite “inactive”, because there runs no current through them. To analyze a DC network, you can thus simply omit the capacitors and calculate all the currents through the resistors in the network. The voltages on capacitors are then given by the voltages on the resistors (which can be composed of several resistors in series and parallel) between the points in the circuit to which each capacitor is attached.

In our case, this would simplify matters slightly, because it reduces the set of linear equations with four unknowns to one with only three, and then you can use the method detailed in problem (2) to find the voltage at the capacitor.