Name: .... Section:.................

Physics 208 Quiz 4

Solutions

February 13, 2008 (due: February 20, 2008)

Problem 1 (20 points)

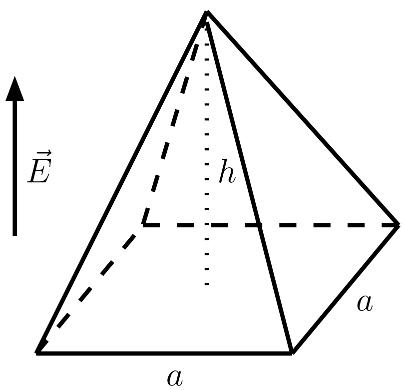
A circular loop with a diameter $d = 40$ cm is rotated in a uniform electric field until the position of maximal electric flux is found. The flux in this position is $\Phi = 5.2 \cdot 10^5$ Nm^2/C. What is the magnitude of the electric field?

The maximum flux is reached when the surface-normal vector of the circle enclosed by the loop is parallel to the electric field. Since the electric field is constant, we have

$$\Phi_E = AE \Rightarrow E = \frac{\Phi}{A} = \frac{4\Phi}{\pi d^2} \simeq 4.14 \cdot 10^6 \frac{\mathrm{N}}{\mathrm{C}} = 4.14 \cdot 10^6 \frac{\mathrm{V}}{\mathrm{m}}.$$

Problem 2 (20 points)

A pyramid with horizontal square base, $a = 6$ m on each side, and a height, $h = 4$ m is placed in an upward vertical electric field of magnitude $E = 52$ N/C. Calculate the electric flux through the pyramids four slanted surfaces.

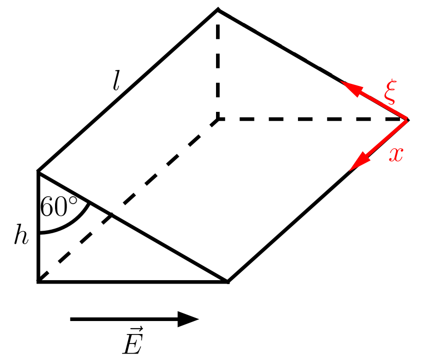


Hint: Think about the total flux through the pyramid first, before you do a lot of unnecessary work!

The total electric flux through the pyramid vanishes since the electric field is constant. Thus the flux through the four slanted surfaces is, up to the sign, the same as through the square base:

$$\Phi_E = EA \simeq 1.872 \cdot 10^3 \frac{\mathrm{Nm}^2}{\mathrm{C}} = 1.872 \cdot 10^3 \mathrm{Vm}.$$

Problem 3 (30 points)



Consider a closed triangular box resting within a horizontal electric field $\vec{E} = 7.8 \cdot 10^4$ N/C$\vec{i}_y$ (see figure). Calculate the electric flux, Φ, through

(a) the vertical rectangular surface,

The electric field is constant, and the surface-normal vector points in direction $-\vec{i}_y$. Thus we have

$$\Phi_{E,V} = -lhE.$$

(b) the slanted surface,

I choose parameters $x \in (0, l)$ and $\xi \in (0, d) = (0, 2h)$ as indicated in the figure.

$$\vec{r}(\xi, x) = x\vec{i}_x + \xi\vec{i}_\xi, \text{ with } \vec{i}_\xi = -\frac{\sqrt{3}}{2}\vec{i}_y + \frac{1}{2}\vec{i}_z$$

The surface-normal vector is

$$\mathrm{d}\vec{A} = \frac{\partial\vec{r}}{\partial\xi} \times \frac{\partial\vec{r}}{\partial x}\mathrm{d}\xi\,\mathrm{d}x = \vec{i}_\xi \times \vec{i}_x = \left(\frac{1}{2}\vec{i}_y + \frac{\sqrt{3}}{2}\vec{i}_z\right)\mathrm{d}\xi\,\mathrm{d}x. \tag{1}$$

Note that the order of the parameters, ξ and x, has been chosen such that we get the right orientation of the surface vector, namely pointing upwards! The flux is

$$\Phi_{E,S} = \int \mathrm{d}\vec{S} \cdot \vec{E} = \int_0^l \mathrm{d}x \int_0^{2h} \mathrm{d}\xi\, \frac{E}{2} = lhE.$$

(c) the entire surface of the box.

The total flux through the box is given by the sum of (a) and (b) since the normal vectors of the other surfaces are perpendicular to $\vec{E}$. The total flux is thus

$$\Phi_{E,\mathrm{tot}} = \Phi_{E,V} + \Phi_{E,S} = 0,$$

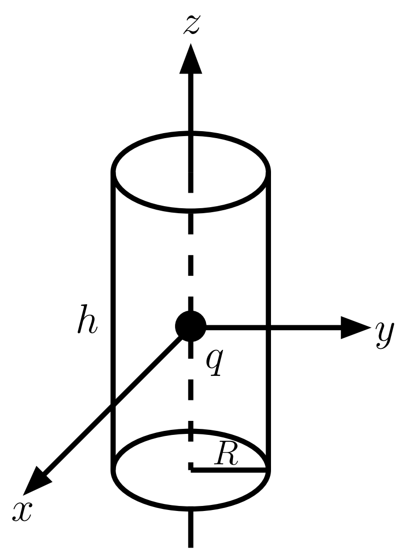
as to be expected for a constant electric field.

Problem 4 (30 Points)

A charge, q, is located in the center of a coordinate system. What is the total electric flux of its electric field through the entire surface of a cylinder with its axis along the z-axis (reaching from $z = -h/2$ to $z = +h/2$ and radius $\rho = R$ (see figure)?

(a) Use Gauss's Law to find the answer!

(b) Do the integrals to verify it!



Hint: Use cylinder coordinates, (ρ, φ, z)! The following integrals are needed to solve the problem:

$$\begin{aligned} \int \mathrm{d}\rho \frac{\rho}{(\rho^2 + z^2)^{3/2}} &= -\frac{1}{\sqrt{\rho^2 + z^2}}, \\ \int \mathrm{d}z \frac{1}{(\rho^2 + z^2)^{3/2}} &= \frac{z}{\rho^2 \sqrt{\rho^2 + z^2}}. \end{aligned} \tag{2}$$

Solution

(a) According to Gauss's Law the flux through the cylinder is the enclosed charge $\times 1/\epsilon_0$:

$$\Phi_E = \frac{q}{\epsilon_0}.$$

(b) To evaluate the flux we have to parametrize the three surfaces of the cylinder, i.e., the top and bottom caps and the envelope. This is most easily done in our standard cylinder coordinates ρ, ϕ, and z. In these coordinates the electric field reads

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0(\rho^2 + z^2)^{3/2}} \left(\rho \vec{i}_\rho + z \vec{i}_z \right). \tag{3}$$

For the upper cap the parameters are $\rho \in (0, R)$ and $\phi \in (0, 2\pi)$, and $z = h/2 = \text{const}$. The surface-normal vector is $\mathrm{d}\vec{A} = \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \vec{i}_z$. Plugging all this into the flux integral, gives

$$\Phi_{\text{upper}} = \int_0^{2\pi} \mathrm{d}\phi \int_0^R \mathrm{d}\rho \, \frac{\rho h}{2} \frac{q}{4\pi\epsilon_0(\rho^2 + h^2/4)^{3/2}}, \tag{4}$$

and using the first of the integrals, given in Eq. (2), leads to

$$\Phi_{\text{upper}} = \frac{q}{4\epsilon_0} \left[2 - \frac{h}{\sqrt{R^2 + h^2/4}} \right]. \tag{5}$$

The lower cap gives the same result due to symmetry:

$$\Phi_{\text{lower}} = \Phi_{\text{upper}}. \tag{6}$$

For the envelope of the cylinder the right parametrization is given by $z \in (-h/2, h/2)$, $\phi \in (0, 2\pi)$, while $\rho = R = \text{const}$, and the surface-normal vector is given by $\mathrm{d}\vec{A} = R\,\mathrm{d}\phi\,\mathrm{d}z\,\vec{i}_\rho$. With help of the second integral, given in Eq. (2), we find

$$\Phi_{\text{env}} = \int_0^{2\pi} \mathrm{d}\phi \int_{-h/2}^{h/2} \mathrm{d}z \frac{q}{4\pi\epsilon_0} \frac{R^2}{(R^2+z^2)^{3/2}} = \frac{q}{2\epsilon_0} \frac{h}{\sqrt{R^2+h^2/4}}. \tag{7}$$

For the total flux, we have to add our results (5-7):

$$\Phi_{\text{tot}} = \Phi_{\text{upper}} + \Phi_{\text{lower}} + \Phi_{\text{env}} = 2\Phi_{\text{upper}} + \Phi_{\text{env}} = \frac{q}{\epsilon_0}, \tag{8}$$

which shows that Gauss's Law is correct.