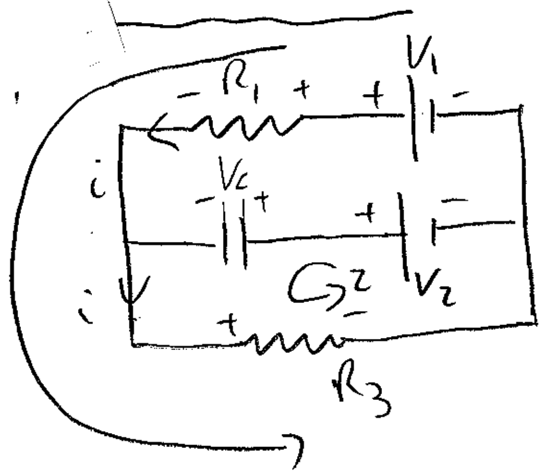
ice Exam III



1: $V_1 - iR_1 - iR_3 = 0$

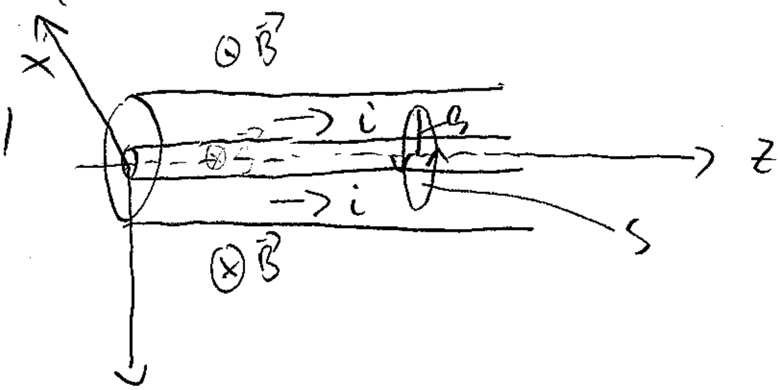
2: $V_2 - V_C - iR_3 = 0$

No current through capacitor in the steady state!

$$① \Rightarrow i = \frac{V_1}{R_1+R_3} \quad ; \quad V_C = V_2 - iR_3 = V_2 - \frac{R_3}{R_1+R_3}V_1$$

$$Q = CV_c = C\left(V_2 - \frac{R_3}{R_1+R_3}V_1\right)$$

(2)



In our usual cylinder coordinates, for $\vec B$ we expect the form

$$\vec B = B_\varphi(\rho)\,\vec i_\varphi$$

To find B_φ we use Ampère's law

$$\oint_{\partial S} \vec B\, d\vec r = \int_S d\vec S\, \vec j\, \mu_0$$

For S we use a circular disc ∥ to the xy plane with radius ρ: Then we have, with φ as the parameter of ∂S

$$\vec{r} = \rho\,\vec{i}_\rho = \rho\,(\cos\varphi\,\vec{i}_x + \sin\varphi\,\vec{i}_y)$$

$$d\vec{r} = d\varphi\,\rho\,\vec{i}_\varphi = d\varphi\,\rho\,(-\sin\varphi\,\vec{i}_x + \cos\varphi\,\vec{i}_y)$$

$$\oint_{\partial S} d\vec{r}\,\vec{B} = \int_0^{2\pi} d\varphi\,\rho\,\vec{i}_\varphi\,B_\varphi(\rho)\,\vec{i}_\varphi = 2\pi\rho\,B_\varphi(\rho)$$

The current density vector is given by

$$\vec{j} = \begin{cases} 0 & \text{for } 0<\rho<a \\ \dfrac{i\,\vec{i}_z}{\pi(b^2-a^2)} & \text{for } a<\rho<b \\ 0 & \text{for } \rho>b \end{cases}$$

Thus

$$\int_S d\vec{S}\,\vec{j}\,\mu_0 = \begin{cases} 0 & \text{for } 0<\rho<a \\ \dfrac{i\pi(\rho^2-a^2)}{\pi(b^2-a^2)} = i\,\dfrac{\rho^2-a^2}{b^2-a^2} & \text{for } a<\rho<b \\ i & \text{for } \rho>b \end{cases}$$

$$\Rightarrow B_\varphi(\rho) = \begin{cases} 0 & \text{for } 0<\rho<a \\ \dfrac{\mu_0 i}{2\pi\rho}\,\dfrac{\rho^2-a^2}{b^2-a^2} & \text{for } a<\rho<b \\ \dfrac{\mu_0 i}{2\pi\rho} & \text{for } \rho>b \end{cases}$$

(a) For a particle with $\vec{v} = v\,\vec{i}_z$ along $\rho=0$ we have

$$\vec{F}_{mag} = q\,\vec{v}\times\vec{B} = 0$$

because at $\rho=0$: $\vec{B}=0$

(b) For the particle outside the cylinder we have

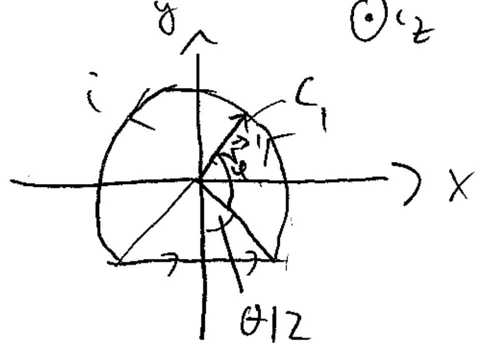
$$\vec{v} = v_1\,\vec{i}_z + v_2\,\vec{i}_\rho \; ; \; \vec{r} = \rho\,\vec{i}_\rho + z\,\vec{i}_z \text{ with } \rho>b$$

$$\vec{F}_{mag} = q\,\vec{v}\times\vec{B} = q\,(v_1\,\vec{i}_z + v_2\,\vec{i}_\rho)\times B_\varphi\,\vec{i}_\varphi$$

$$= -q\,v_1\,B_\varphi\,\vec{i}_\rho + q\,v_2\,B_\varphi\,\vec{i}_z$$

$$\vec{F}_{mag} = \frac{4\mu_0 i}{2\pi s}\left(-v_1 \vec{i}_s + v_2 \vec{i}_z\right)$$

(3)



Biot-Savart law

(a) along circle

$$\vec{B}_1 = \frac{\mu_0 i}{4\pi}\int_{C_1} d\vec{r}' \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \quad ; \vec{r}=0$$

$C_1 : \vec{r}' = R\,\vec{i}_s(\varphi)$ with φ (counted as on the figure!)

$$\varphi \in (0, 2\pi-\theta)$$

$$d\vec{r}' = R\,\vec{i}_\varphi(\varphi)$$

$$\vec{B}_1 = -\frac{\mu_0 i}{4\pi}\int_0^{2\pi-\theta} d\varphi\, R\,\vec{i}_\varphi \times \frac{R\vec{i}_s}{R^3}$$

$$= -\frac{\mu_0 i}{4\pi R}(-\vec{i}_z)\int_0^{2\pi-\theta} d\varphi = +\frac{\mu_0 i}{4\pi R}(2\pi-\theta)\,\vec{i}_z$$

(b) along straight line

$\vec{r}' = -R\cos\left(\frac{\theta}{2}\right)\vec{i}_y + x'\vec{i}_x$ with x from $-R\sin\left(\frac{\theta}{2}\right) \to R\sin\left(\frac{\theta}{2}\right)$

$$d\vec{r}' = dx'\,\vec{i}_x$$

$$\vec{B}_2 = \frac{\mu_0 i}{4\pi}\int_{-R\sin(\frac{\theta}{2})}^{R\sin(\theta/2)} dx'\,\vec{i}_x \times \frac{R\cos\left(\frac{\theta}{2}\right)\vec{i}_y}{\left[x'^2 + R^2\cos^2\left(\frac{\theta}{2}\right)\right]^{3/2}}$$

$$\vec{B}_2 = \frac{\mu_0 i}{4\pi} R \cos\left(\frac{\theta}{2}\right) \vec{i}_z \, 2\int_0^{R\sin(\theta/2)} dx' \frac{1}{(x'^2 + R^2\cos^2\frac{\theta}{2})^{3/2}}$$

$$= \frac{\mu_0 i}{2\pi} \frac{R^2 \sin\left(\frac{\theta}{2}\right)}{R^3 \cos\left(\frac{\theta}{2}\right)} \vec{i}_z$$

$$\vec{B}_2 = \frac{\mu_0 i \tan\left(\frac{\theta}{2}\right)}{2\pi R} \vec{i}_z$$

$$\vec{B} = \left[\frac{\mu_0 i}{4\pi R}(2\pi - \theta) + \frac{\mu_0 i}{2\pi R}\tan\left(\frac{\theta}{2}\right)\right]\vec{i}_z$$

(4) $\vec{B} = -\vec{i}_z(\alpha + \beta x)$

(a) $$\Phi_{\vec{B}} = -\int_0^H dy \int_0^D dx \, (\alpha + \beta x)$$

$$= -H\left(\alpha D + \frac{\beta}{2}D^2\right)$$

(b) $$\oint_{\partial S} d\vec{r}\,\vec{E} = +V_c = -\dot{\Phi}_B - L\frac{di}{dt} = +\frac{Q}{C}$$

$$-\dot{\Phi}_B = H \quad C_1 D$$

$$\frac{Q}{C} = -L\ddot{Q} + HC_1 D$$

(c) For $L=0 \Rightarrow Q = CHC_1D$

There is a constant EMF induced and thus the charge is constant in time

For the general case ($L \neq 0$) we have to add the solution of the homogeneous equation:

$$\ddot{Q}_H = -\frac{Q_H}{LC} \Rightarrow Q_H = a\cos(\omega t + \varphi_0)$$

$$\text{with } \omega = \frac{1}{\sqrt{LC}}$$

If for $t=0$; $Q=0$ (initial condition) we have

$\varphi_0 = 0$; $a = -CH q_1 D$

$$Q(t) = CHq_1D\left[1-\cos(\omega t)\right]$$