Physics 208

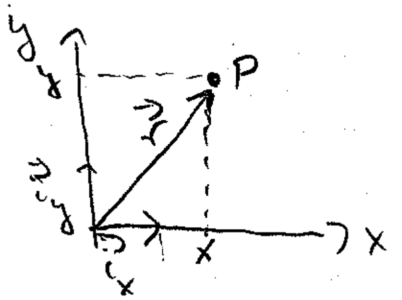
Lecture 1 (Jan 11)

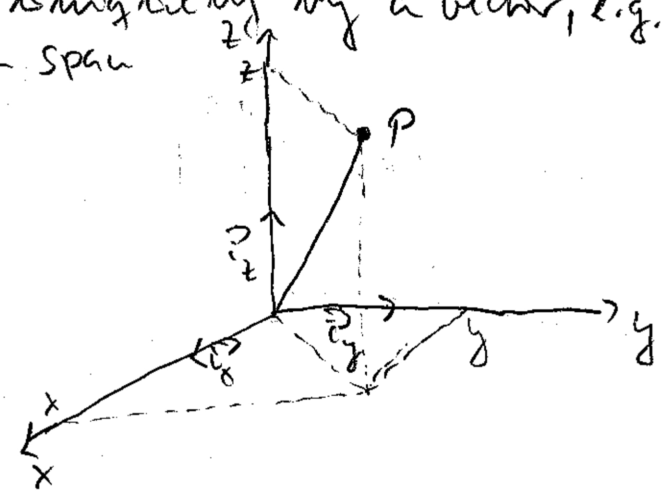
1.1 Reminder about Newtonian Mechanics and vectors

In the previous lecture, Physics 218, the basic laws of Newtonian mechanics were established. They cannot be derived from any "basic principles" but only inferred from experience and finally quantitative laws which have to be checked by experiment.

The basic concepts, introduced without further explanation are space and time. The most fundamental entity is a point particle which is a body "small enough" to be described by a point in space.

It is further assumed that the relations of point like objects in space are described by 'Euclidian' geometry. It turned out that the most adequate description of the motion of a point particle is to introduce an arbitrary fixed point in space and a rectilinear (Cartesian) coordinate system such that the position of the particle is described unambiguously by a vector, e.g., for a particle in a plane or in space





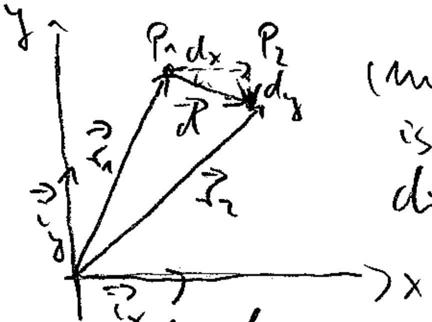
~~Then the~~

The vector can be decomposed in components with respect to this coordinate system (or basis vectors):

$$\vec{r} = x\,\vec{i}_x + y\,\vec{i}_y \quad \text{(plane, 2D)}$$

$$\vec{r} = x\,\vec{i}_x + y\,\vec{i}_y + z\,\vec{i}_z \quad \text{(space, 3D)}$$

Vectors can be added, as defined in the following way: Suppose one has a point particle located at P_1 and is then shifted to another position P_2. For simplicity's sake we work in a plane. Then we have:



(note that $d_y < 0$ because the arrow is pointing "downwards", i.e. opposite direction of $\vec{i}_y$!)

then we obviously have

$$\vec{r}_2 = \vec{r}_1 + \vec{d}$$

Now we have

$$\left.\begin{aligned} \vec{r}_1 &= x_1 \vec{i}_x + y_1 \vec{i}_y \\ \vec{d} &= d_x \vec{i}_x + d_y \vec{i}_y \end{aligned}\right\} \Rightarrow \vec{r}_2 = (x_1 + d_x)\,\vec{i}_x + (y_1 + d_y)\,\vec{i}_y$$

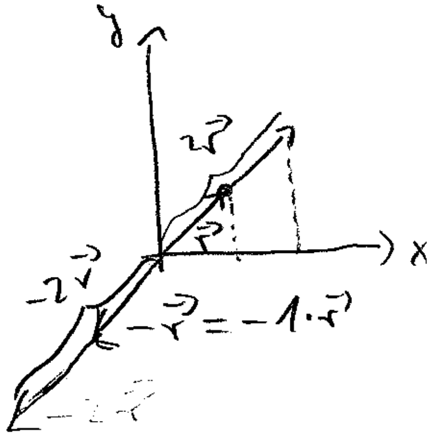
Sometimes one writes the vectors in column form by just writing the components, e.g.,

$$\vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \text{ etc.}$$

Then the formula above reads

$$\vec{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \vec{r}_1 + \vec{d} = \begin{pmatrix} x_1 + d_x \\ y_1 + d_y \end{pmatrix}$$

Vectors can also be multiplied by real numbers. If the number is positive, the resulting vector points in the same direction but the length is multiplied by this number



If the number is negative, the arrow points in the opposite direction. From basic geometrical laws, we find for the components

$$\vec{r} = x\,\vec{i}_x + y\,\vec{i}_y$$

$$\Rightarrow \lambda\vec{r} = \lambda x\,\vec{i}_x + \lambda y\,\vec{i}_y$$

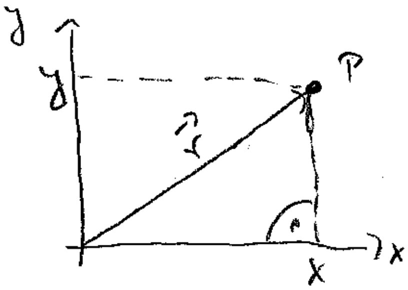
or $\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

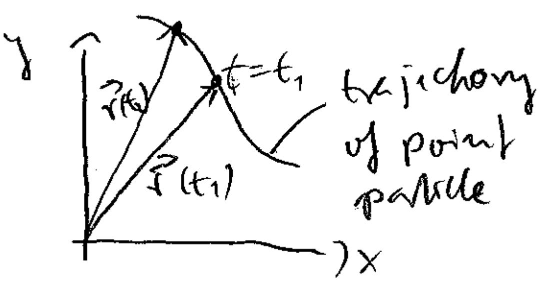
Generally, for the sum of two vectors we find the law

$$\lambda(\vec{r}_1 + \vec{r}_2) = \lambda\vec{r}_1 + \lambda\vec{r}_2$$

The length of a vector is given by its components w.r.t. a Cartesian system due to Pythagoras's theorem

$|\vec{r}| = \sqrt{x^2+y^2}$ (or in 3D $|\vec{r}| = \sqrt{x^2+y^2+z^2}$)





The motion of a particle is described by the vector $\vec{r}$ as a function of time:

$$\vec{r} = \vec{r}(t) = x(t)\,\vec{i}_x + y(t)\,\vec{i}_y + z(t)\,\vec{i}_z,$$

and one can use algebra and analysis (calculus) to describe the motion of this particle.

The velocity is given by the time derivative

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \dot{\vec{r}}(t) = \dot{x}(t)\,\vec{i}_x + \dot{y}(t)\,\vec{i}_y + \dot{z}(t)\,\vec{i}_z$$

($\vec{i}_x, \vec{i}_y, \vec{i}_z$ fixed, i.e., time independent) and the acceleration by

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$$

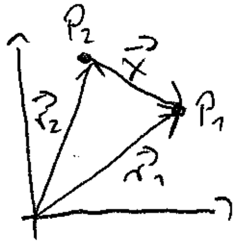
Then Newton's laws read (in an inertial frame of reference)

(1) A particle moves with constant velocity, if no force is changing it.

(2) The force is proportional to the acceleration of the point particle. The proportionality constant is called mass of the particle

$$\vec{F} = m\,\vec{a}$$

Force and mass are not further defined by simpler concepts but are basic notions of mechanics!
An important example of a force is gravity. Two point masses are attracted according to Newton's law of gravitation:



$$|\vec{F}_{12}| = \gamma\,\frac{m_1 m_2}{x^2} \quad \text{with } x = |\vec{x}|$$

or since

$$\vec{r}_1 = \vec{r}_2 + \vec{x} \Rightarrow \vec{x} = \vec{r}_1 - \vec{r}_2$$

$$|\vec{F}_{12}| = \gamma\,\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2}$$

The force is attractive and directed along $\vec{x}$, i.e.

$$\vec{F}_{12} = \gamma\,\frac{m_1 m_2}{x^2}\,\frac{\vec{x}}{x} = \gamma\,\frac{m_1 m_2}{x^2}\,\hat{x}$$

$$= \gamma\,\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2}\,\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_2 - \vec{r}_1|}$$

In the same way one obtains the force, exerted by the point particle 2 on point particle 1:

$$\vec{F}_{21} = \gamma \frac{m_1 m_2}{|\vec{r}_2-\vec{r}_1|^2} \frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|} = -\vec{F}_{12}$$

That's not only true for the gravitational force, but a general law, known as Newton's 3rd law

(3) If particle 1 exerts a force, $\vec{F}_{12}$, on particle 2, then particle 2 exerts a force $\vec{F}_{21}$ on particle 1 which is equal in magnitude but directed in the opposite way:

$$\vec{F}_{21} = -\vec{F}_{12}$$

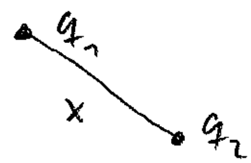
Physics 208

Lecture 2 (Jan/22)

1.2 Electric charge and Coulomb's law

Ancient Greeks, about 2500 years ago discovered that rubbing amber made it attract certain other materials. Since amber is called "electron" such phenomena are called electronic.
To describe such electric phenomena one introduces another basic quantity (which cannot be explained by simpler means today) called electric charge.
In the late 18th century, Coulomb and Priestly found the law of force for electrically charged point particles to be similar to Newton's law of gravitation

$$|\vec{F}_E| = \alpha \frac{|q_1 q_2|}{x^2}$$



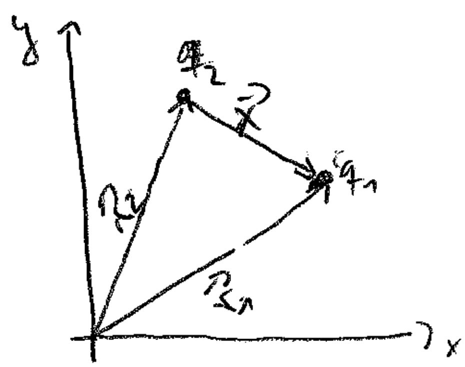
where q_1 and q_2 are the charges of the point particles, and x is the distance between them. α is a constant which determines the unit of charges.
There are many systems of units in use. In this lecture we stick to the officially used "rationalized MKS" system or the SI (Système International de Poids et Mesures).
The basic unit for charges in this system is called Coulomb, abbreviated with C.
In the MKS system the constant $\alpha = \frac{1}{4\pi\varepsilon_0}$, where ε_0 is called "permittivity of free space". Then Coulomb's law reads

$$|\vec{F}_E| = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{x^2}$$

The value is
$$k = \frac{1}{4\pi\varepsilon_0} \simeq 9\cdot 10^9 \frac{N m^2}{C^2}$$
Remember that N is the MKS unit of force
$$1N = 1\,\frac{kg\cdot m}{s^2}$$
m is the unit of length (metre) and kg (kilo grams), the unit of mass and s the unit of time (second).
Charges can appear with both signs, positive and negative, and it is found that charges of

like signs repell
unlike signs attract

each other, and the direction of the force is along the line from one charge to the other, in perfect analogy to Newton's law for the gravitational force.



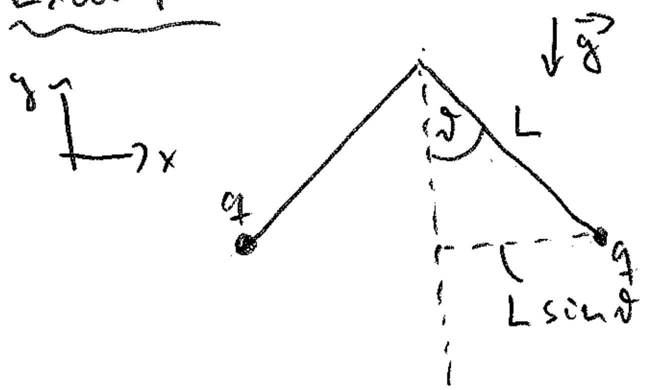
$$\vec{x} = \vec{r}_1 - \vec{r}_2$$

Thus we have
$$\vec{F}_{E12} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{x^2}\,(-\hat{x}) \quad \text{with } \hat{x} = \frac{\vec{x}}{x}$$
no matter which signs q_1 and q_2 might have. Indeed, if q_1 and q_2 have like signs, $q_1 q_2 > 0$, and the force is repulsive while if they have unlike signs the force is attraction.

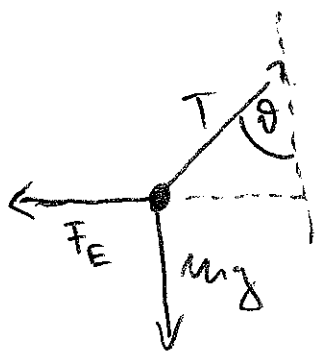
As suggested in the book it can be advantageous to define the symbols q as positive and write $-q$ for negative charges.

Now we can calculate electric forces of point particles and use them in force diagrams as usual.

Example



Force diagram for left charge



$$F_E = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2L\sin\vartheta)^2}$$

Now $\vec{F}_{total} = m\vec{a} \stackrel{!}{=} 0$

x-direction: $-F_E + T\sin\vartheta = 0 \Rightarrow F_E = T\sin\vartheta$

y-direction: $T\cos\vartheta - mg = 0 \Rightarrow T = \frac{mg}{\cos\vartheta}$

Thus we find

$$F_E = \frac{mg}{\cos\vartheta}\sin\vartheta = mg\tan\vartheta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2L\sin\vartheta)^2}$$

Suppose we know q and measure the angle, we can determine

$$\alpha = \frac{1}{4\pi\varepsilon_0} = \frac{mg \tan\vartheta \, (2L\sin\vartheta)^2}{q^2}$$

To get an idea about orders of magnitude, suppose $q = 1C$, $L = 1m$ and $\vartheta = 30^\circ$. Then

$$mg = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q^2}{(2L\sin\vartheta)^2 \tan\vartheta}$$

$$\cong 9 \cdot 10^9 \, \frac{1}{0.5774} \, N \cong 15.6 \cdot 10^9 \, N$$

$$(\cong 3.5 \cdot 10^9 \text{ lbs})$$

That shows that $1C$ is a rather large charge.

Nowadays we know that by rubbing, e.g., rubber with a cat's fur one transfers only some electrons, each of which has a charge of

$$q_{electron} = -e \cong -1.6 \cdot 10^{-19} \, C$$

The mass of an electron is

$$m_{electron} \cong 9 \cdot 10^{-31} \, kg$$

<u>Example</u>

Compare electrical force between 2 electrons which are 10^{-8} m apart to the gravitational force between them

$$F_E = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{x^2} \approx 2.3 \cdot 10^{-12} \, N$$

$$F_G = \gamma \, \frac{m_e^2}{x^2} \approx 540 \cdot 10^{-57} \, N$$

$$\frac{F_E}{F_G} = \frac{\gamma e^2}{4\pi\varepsilon_0 m_e^2} \approx 4.3 \cdot 10^{42}$$

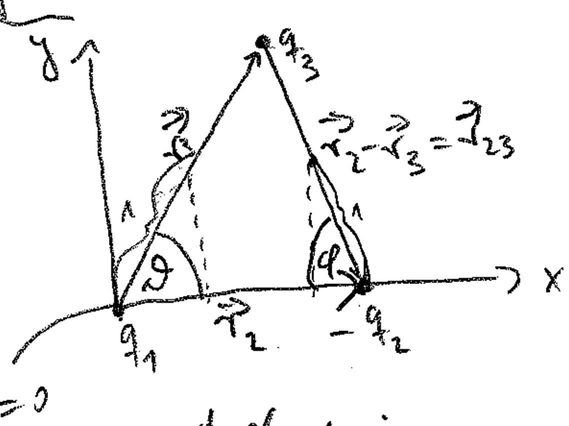
In this sense the electric force is much larger than the gravitational force.

Superposition

If we have only point particles, the Coulomb forces simply add (as vectors!), because the force exerted on one particle by another does not change its inner structure, especially not the distribution of charges. It is important to note that this does not need to hold true for finite objects where the charge distribution can be changed by the presence of other charges.

Example

here all the q's are taken positive. The 2nd charge is thus negative, $-q_2$



Force on the 3rd charge:

$$\vec{F}_{13} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{r_3^2} \hat{r}_3 \quad \text{(like sign} \Rightarrow \text{repulsion!)}$$

$$\vec{F}_{23} = \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{r_{23}^2} \hat{r}_{23} \quad \text{(unlike signs: attraction)}$$

Decomposition into components: the all charges lay in the xy plane. Thus we can work with plane and so all the vectors are including them. vectors:

$$\hat{r}_3 = \cos\vartheta \, \vec{i}_x + \sin\vartheta \, \vec{i}_y$$

$$\hat{r}_{23} = \cos\varphi \, \vec{i}_x - \sin\varphi \, \vec{i}_y$$

$$\vec{F}_{total} = \frac{1}{4\pi\varepsilon_0} \left[\left(\frac{q_1 q_3}{r_3^2} \cos\vartheta + \frac{q_2 q_3}{r_{23}^2} \cos\varphi \right) \vec{i}_x + \left(\frac{q_1 q_3}{r_3^2} \sin\vartheta - \frac{q_2 q_3}{r_{23}^2} \sin\varphi \right) \vec{i}_y \right]$$

Lecture 3-4 (Jan/24 – 26)

2.1 The electric field

Suppose we have any number of charges

$q_1, \dots, q_n$

placed arbitrarily in space at points

$\vec{r}_1, \dots, \vec{r}_n$

Now we ask what force is exerted on a particle q_0 if put at $\vec{r}_0$. From Coulomb's law, we know that, if q_0 is so small that the charges $q_1, \dots, q_n$ keep at the above given places:

$$4\pi\varepsilon_0 \vec{F}_{\text{all charges},0} = \frac{q_0 q_1}{|\vec{r}_0 - \vec{r}_1|^3}(\vec{r}_0 - \vec{r}_1) + \frac{q_0 q_2}{|\vec{r}_0 - \vec{r}_2|^3}(\vec{r}_0 - \vec{r}_2) + \dots + \frac{q_0 q_n}{|\vec{r}_0 - \vec{r}_n|^3}(\vec{r}_0 - \vec{r}_n)$$

$$= \sum_{k=1}^{n} \frac{q_0 q_k}{|\vec{r}_0 - \vec{r}_k|^3}(\vec{r}_0 - \vec{r}_k)$$

$$= q_0 \sum_{k=1}^{n} \frac{q_k}{|\vec{r}_0 - \vec{r}_k|^3}(\vec{r}_0 - \vec{r}_k)$$

Thus we can define, by measuring the force on the charge q_0 in the limit of $q_0 \to 0$ a quantity which depends only on the charges $q_1, \dots, q_n$, what we call the electric field

$$\vec{E}(\vec{r}_0) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\vec{r}_0 - \vec{r}_k|^3} (\vec{r}_0 - \vec{r}_k)$$

The important concept behind this definition is that we can interpret the action of all charges q_k located in $\vec{r}_k$ not only as a force acting at a distance, but as the source of a physical object spread over all space, namely the electric field, $\vec{E}$, which causes the force of all the charges q_k, on another charge q_0, which is small enough not to relocate the other charges, q_k, by

$$\vec{F}_{\text{all other charges, on } q_0} = q_0 \vec{E}(\vec{r}_0)$$

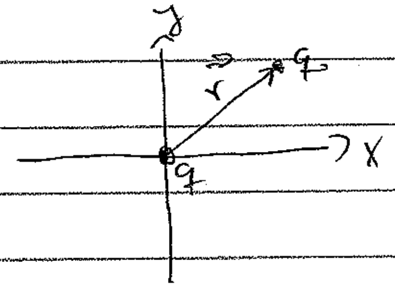
if q_0 is located at $\vec{r}_0$. In the following we call such a charge a test charge (for convenience). This important notion is due to Michael Faraday who investigated experimentally a large number of electromagnetic phenomena.

To calculate an electric field at a point $\vec{r}$, produced by point charges just calculate the Coulomb force on a test charge, q_0, located at $\vec{r}$ and finally divide by q_0 (see the equation above).

Example 1

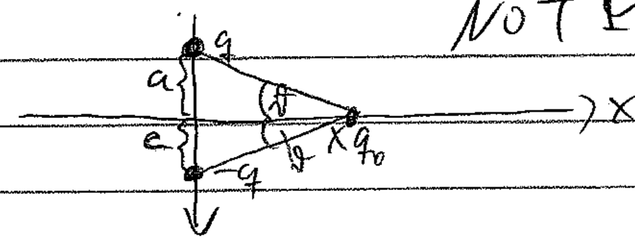
Take one point charge q. Then we can put the origin of the coordinate system at this charge. The electric field on the test charge q_0 located at $\vec{r}$ is then

$$\vec{E}(\vec{r}) = \frac{\vec{F}_{q q_0}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$



Example 2

Dipole:



NOTE: y axis points downwards

What is the electric field along the x axis?

Here we work with components (note also the method in the textbook using angles and trig. functions!)

$$\vec{E}(x\vec{i}_x) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(x^2+a^2)^{3/2}}\left(x\vec{i}_x - (-a)\vec{i}_y\right) + \right.$$

$$\left. + \frac{-q}{(x^2+a^2)^{3/2}}\left(x\vec{i}_x - a\,\vec{i}_y\right)\right]$$

$$\boxed{\vec{E}(x\,\vec{i}_x) = \frac{q}{4\pi\varepsilon_0}\,\frac{2a}{(x^2+a^2)^{3/2}}\,\vec{i}_y}$$

2.2 Motion of a test particle in an electric field

If the charge q is sufficiently small compared to all charges involved in creating the electric field (such that we can consider the locations of these charges as unaffected by the test charge), we can calculate its motion by using Newton's laws

$$m\vec{a} = \vec{F} = q\vec{E}(\vec{x})$$

if we know the initial conditions

$$\vec{x}(0) = \vec{x}_0, \quad \dot{\vec{x}}(0) = \vec{v}_0$$

by solving the set of differential equations

$$m\ddot{\vec{x}} = q\vec{E}(\vec{x})$$

for the components of $\vec{x}$.

Example 1: Motion of a charge in constant el. field

Let $\vec{E} = \text{const}$. Then we may choose the coordinate system such that $\vec{E}$ points in positive x direction

$$\underset{m,q}{\bullet}\longrightarrow \vec{F} = q\vec{E} = qE\,\vec{i}_x$$

Then

$$m\ddot{x} = qE$$
$$m\ddot{y} = 0$$
$$m\ddot{z} = 0$$

The particle initially may be at rest in the coordinate origin:

$$x(0) = y(0) = z(0) = 0$$
$$v_{x0} = v_{y0} = v_{z0} = 0$$

Since $qE = \text{const}$ it is easy to solve the equation for x

$$\ddot{x} = \frac{q}{m}E \Rightarrow \dot{x} = \frac{q}{m}E\int_0^t dt' + v_{x0} = \frac{q}{m}Et$$

Integrating once more

$$x = \int_0^t dt'\,\dot{x}(t') = \frac{q}{m}E\int_0^t dt'\,t' = \frac{qE}{m}\frac{t^2}{2}$$

$$\boxed{x(t) = \frac{qE}{2m}\frac{t^2}{2}}$$

The other 2 components are trivially zero

$$y(t) = z(t) = 0$$

3. The electric potential (week of Jan/29)

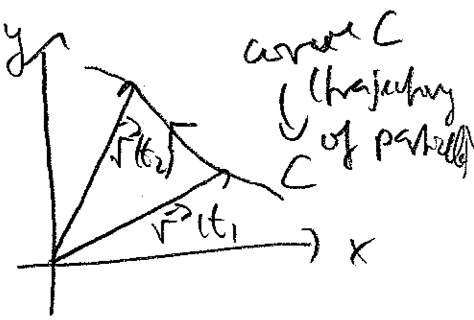
3.1 Review on conservative forces

In mechanics, we have learnt about the concept of energy. Here this idea is reviewed. We start with Newton's 2nd law.

$$m\ddot{\vec{r}} = \vec{F}$$

Now suppose we have solved these equations of motion, i.e., we know the trajectory of the particle

$$C : \vec{r} = \vec{r}(t)$$



Then we multiply the EoM with $\dot{\vec{r}} = \vec{v}$

$$m\dot{\vec{r}}\ddot{\vec{r}} = \dot{\vec{r}}\vec{F}$$

where we used the dot product. In cartesian coordinates it's defined by

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z \in \mathbb{R}$$

which is a real number (a scalar).

Now it is easy to see that we can write

$$\dot{\vec{r}}\ddot{\vec{r}} = \frac{1}{2}\frac{d}{dt}\left(\dot{\vec{r}}^2\right)$$

To prove this, first use the product rule for differentiation.

Thus we can write

$$\frac{d}{dt}\left[\frac{m}{2}\dot{\vec{r}}^2\right] = \dot{\vec{r}}\cdot\vec{F}$$

Now we can integrate this equation from t_1 to t_2

$$\int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{m}{2}\dot{\vec{r}}^2\right) = \frac{m}{2}\dot{\vec{r}}^2\Big|_{t_1}^{t_2} = \frac{m}{2}\left[\dot{\vec{r}}^2(t_2) - \dot{\vec{r}}^2(t_1)\right]$$

Or $$\int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{m}{2}\dot{\vec{r}}^2\right) = \frac{m}{2}\left(\vec{v}_2^{\,2} - \vec{v}_1^{\,2}\right) = E_{kin,2} - E_{kin,1}$$

with $E_{kin} = \frac{m}{2}\vec{v}^2 \quad (\vec{v} = \dot{\vec{r}})$

The RHS is the work done on the particle along its trajectory:

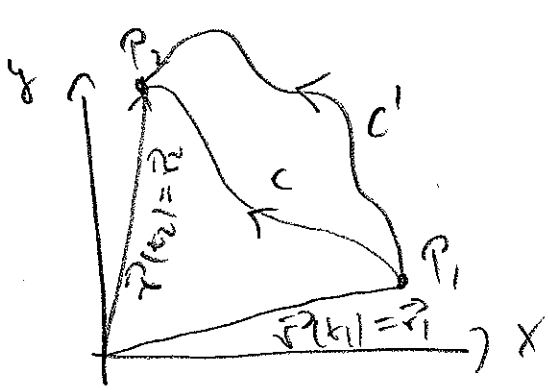
$$W_c = \int_{t_1}^{t_2} dt\, \dot{\vec{r}}\cdot\vec{F} \; ; \quad P = \dot{\vec{r}}\,\vec{F} \text{ (power)}$$

The change in kinetic energy is equal to the work done on the particle moving along its trajectory

<u>Conservative Forces</u>

Matters simplify a lot if $\vec{F}$ depends only on the position of the particle, not its velocity ($\leftrightarrow$ magnetic forces), and if the work done on the particle is independent on the curve connecting $\vec{r}_1$ and $\vec{r}_2$:

$$W_C = W(\vec{r}_1, \vec{r}_2) = \int_{t_1}^{t_2} dt\, \frac{d\vec{r}}{dt}\,\vec{F}[\vec{r}(t)]$$



$$= \int_C d\vec{r}\,\vec{F}(\vec{r})$$

$$= \int_{C'} d\vec{r}\,\vec{F}(\vec{r})$$

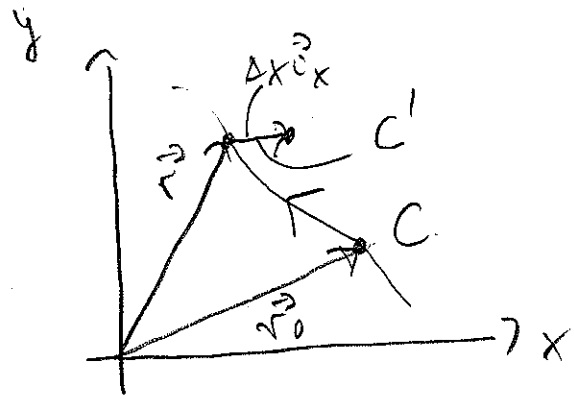
We can reformulate it also in the way that

$$W_C = 0 \quad \text{for all closed curves } C$$

<u>The potential</u>

If $\vec{F}$ is a conservative force we can reconstruct it from the work as a function of its endpoint

$$W(\vec{r}) := \int_{C_{\vec{r}}} d\vec{r}\, \vec{F}(\vec{r})$$



where C is an arbitrary curve connecting some arbitrary fixed point $\vec{r}_0$ with the end point $\vec{r}$.

Now we can ask about the derivatives of W:

$$\frac{\partial W}{\partial x} = \lim_{\Delta x \to 0} \frac{W(\vec{r} + \Delta x \vec{i}_x) - W(\vec{r})}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_{C + C'} d\vec{r}\, \vec{F}(\vec{r}) - \int_C d\vec{r}\, \vec{F}(\vec{r})\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_{C'} d\vec{r}\, \vec{F}(\vec{r})\right]$$

Now we parametrize C' simply with x:

$$C' : \vec{r} = x \vec{i}_x$$

Then

$$\frac{\partial W}{\partial x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} dx\, \vec{i}_x \vec{F}(\vec{r} + x\, \vec{i}_x)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} F_x(\vec{r} + x\, \vec{i}_x)\, dx$$

If now Δx is very small, we can write

$$F_x(\vec{r} + x\, \vec{i}_x) \simeq F_x(\vec{r})$$

and then

$$\frac{\partial W}{\partial x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} dx\, F_x(\vec{r}) = \lim_{\Delta x \to 0} F_x(\vec{r}) \frac{1}{\Delta x} \left[x\right]_0^{\Delta x}$$

$$= F_x(\vec{r}) \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = F_x(\vec{r})$$

In the same way we find

$$F_y(\vec{r}) = \frac{\partial W(\vec{r})}{\partial y}$$

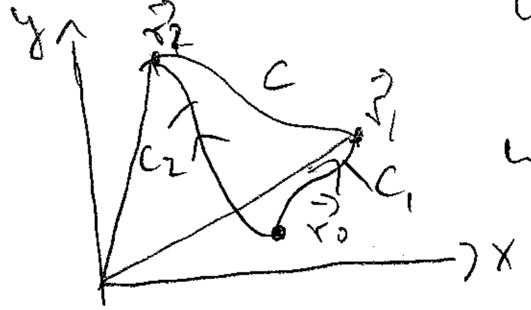
$$F_z(\vec{r}) = \frac{\partial W(\vec{r})}{\partial z}$$

Now we come back to our energy formula

$$E_{kin,2} - E_{kin,1} = \int_C d\vec{r}\, \vec{F}(\vec{r})$$

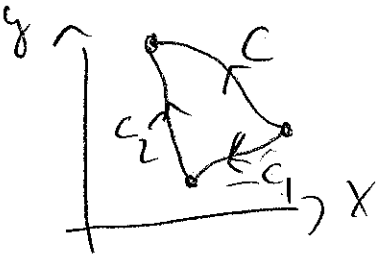
$$W(\vec{r}_1) = \int_{C_1} d\vec{r} \cdot \vec{F}(\vec{r})$$

Indeed:



$$W(\vec{r}_2) = \int_{C_2} d\vec{r}\, \vec{F}(\vec{r})$$

So we have if we just reverse the direction of C_1 to $-C_1$



We have

$$W(\vec{r}_2) - W(\vec{r}_1) = \int_{C_2 - C_1} d\vec{r}\,\vec{F}(\vec{r})$$

On the other hand, since $\vec{F}$ is conservative, the result of the integral is just the work done on the particle along its trajectory:

$$W(\vec{r}_2) - W(\vec{r}_1) = \int_C d\vec{r}\,\vec{F}(\vec{r}) = W_C$$

Thus we have

$$E_{kin,2} - E_{kin,1} = W(\vec{r}_2) - \vec{W}(\vec{r}_1)$$

or

$$E_{kin,2} - W(\vec{r}_2) = E_{kin,1} - W(\vec{r}_1)$$

This holds true for all $\vec{r}_2$ if we keep the initial point $\vec{r}_1$ fixed, i.e., defining

$$E_{pot,2} = -W(\vec{r}_2) := U(\vec{r}_2)$$

we have the law of conservation of energy

$$E_{kin} + E_{pot} = \text{const.}$$ (conservation law: the total energy stays constant)

Or in another way:

$$\frac{m}{2}\vec{v}^2 + U(\vec{r}) = \text{const.}$$

We can easily prove this, by differentiation w.r.t. time

$$\frac{d}{dt}\left[\frac{m}{2}\vec{v}^2\right] = \frac{m}{2}\frac{d(\vec{v}^2)}{dt} = \frac{m}{2}2\vec{v}\cdot\dot{\vec{v}} = m\vec{v}\cdot\vec{a}$$

$$\frac{d}{dt}\left[U[\vec{r}(t)]\right] \overset{\text{chain rule}}{=} \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial t}\dot{z}$$

But now we have seen that

$$F_x = \frac{\partial W}{\partial x} = -\frac{\partial U}{\partial x} \quad \text{etc.}$$

So we have

$$\frac{d}{dt}U[\vec{r}(t)] = -(F_x v_x + F_y v_y + F_z v_z)$$

$$= -\vec{v}\cdot\vec{F} \quad (= -P)$$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{m}{2}\vec{v}^2 + U(\vec{r})\right] = m\vec{v}\cdot\vec{a} - \vec{v}\cdot\vec{F}$$

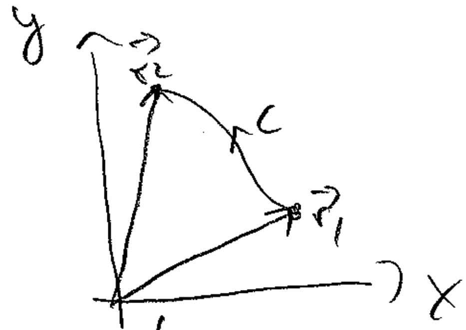
$$= \vec{v}(m\vec{a} - \vec{F}) = 0$$

because of Newton's 2nd law.

On the other hand, if there exists a function with

$$\vec{F}(\vec{r}) = \sum_{k=x}^{z}\left(-\frac{\partial U}{\partial x_k}\vec{i}_k\right)$$

then $\vec{F}$ is conservative:



$$\int_C d\vec{r}\cdot\vec{F}(\vec{r}) = \int_{t_1}^{t_2} dt\, \frac{d\vec{r}}{dt}\vec{F}[\vec{r}(t)]$$

$$= -\int_{t_1}^{t_2} dt \left[\frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y} + \frac{dz}{dt}\frac{\partial u}{\partial z}\right]$$

$$= -\int_{t_1}^{t_2} dt\, \frac{d}{dt} u[\vec{r}(t)]$$

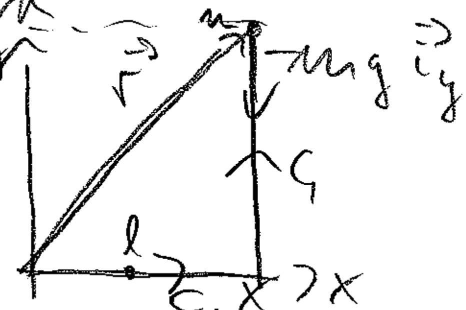
$$= -u[\vec{r}(t)]\Big|_{t_1}^{t_2} = -[u(\vec{r}_2) - u(\vec{r}_1)]$$

independent of the curve C! $\Rightarrow$ $\vec{F}$ is conservative

Examples

(1) Constant force, e.g. gravity near the earth

$$\vec{F} = -mg\,\vec{i}_y$$



Find potential: If it exists we can choose any curve connecting, e.g., the origin with $\vec{r}$. Here we choose $C = C_1 + C_2$ (see figure)

$C_1: \vec{r} = l\,\vec{i}_x$ with $l \in (0, x)$; $C_2: \vec{r} = l'\vec{i}_y + x\vec{i}_x$ (23)

$l' \in (0, y)$

$$U(\vec{r}) = -\int_{C_1} d\vec{r}\,\vec{F}(\vec{r}) - \int_{C_2} d\vec{r}\,\vec{F}(\vec{r})$$

$$= -\int_0^x dl\,\vec{i}_x\,(-\cancel{mg}\,\vec{i}_y)^{\to 0} - \int_0^y dl'\,\vec{i}_y\,(-mg\,\vec{i}_y)$$

$$= mg\,y$$

Energy: $E = \frac{m}{2}\vec{v}^2 + mg\,y.$

check:

$F_x = -\frac{\partial U}{\partial x} = 0$ ✓

$F_y = -\frac{\partial U}{\partial y} = -mg$ ✓

(!) $\vec{F}$ is really conservative

$\Rightarrow E =$ const. of motion

Examples for potentials of forces (Jan 30)

(a) constant force

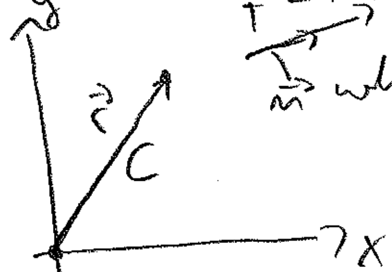
$\vec F = F\vec n$ with $\vec n$, $F = \text{const}$.

How to find the potential of it? Two methods:

(α) take an arbitrary path from a fixed point to $\vec r$

$\vec F = F\vec n$

$\vec n$ where $|\vec n| = 1$



Take an arbitrary path from the origin to $\vec r$ and integrate $\vec F$ along the path. Then check whether really

$$\vec F = -\left(\frac{\partial U}{\partial x}\vec i_x + \frac{\partial U}{\partial y}\vec i_y + \frac{\partial U}{\partial z}\vec i_z\right)$$

The best idea is to use a straight line:

$$C: \vec r(\lambda) = \lambda \vec r \quad ; \ \lambda \in [0,1]$$

$$d\vec r = \frac{d\vec r}{d\lambda}\, d\lambda = \vec r\, d\lambda$$

$$U = -\int_C d\vec r\, \vec F(\vec r) = -F\vec n\vec r \int_0^1 d\lambda = -F\vec n\vec r = -\vec F\vec r = -(F_x x + F_y y + F_z z)$$

check:

$$-\frac{\partial U}{\partial x} = F_x, \ldots$$

☺

(β) Solve partial DEs,

$$F_x = -\frac{\partial U}{\partial x} \quad \text{etc.}$$

Since F_x is constant, that's easy

$$\frac{\partial U}{\partial x} = -F_x \Rightarrow U = -F_x x + U_1(y,z)$$

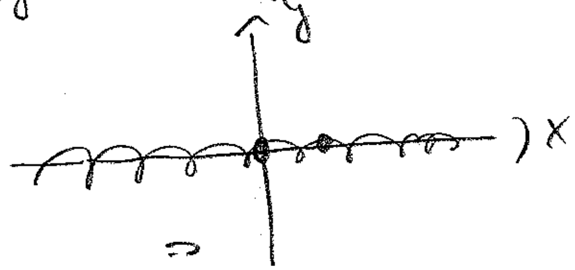
$$\frac{\partial U}{\partial y} = \frac{\partial U_1}{\partial y} = -F_y \Rightarrow U_1 = -F_y y + U_2(z)$$

$$\frac{\partial U}{\partial z} = \frac{\partial U_1}{\partial z} = \frac{\partial U_2}{\partial z} = -F_z \Rightarrow U_2 = -F_z z + \text{const.}$$

$$U(\vec{r}) = -(F_x x + \dots) = -\vec{F}\vec{x} + \text{const}$$

const. is undetermined and physically also unimportant.
In method (α) it's hidden in the arbitrariness of the choice of the initial point.

(γ) Force of a spring



$$\vec{F} = -Dx\,\vec{e}_x$$

Here the 2nd method is easiest: We can assume $U = U(x)$ because there's only an x-directed force

So we want

$$\frac{dU}{dx} = +Dx \Rightarrow U(x) = +\frac{D}{2}x^2 + \text{const.}$$

Constant $\vec{E}$-field

That we have already solved just a page ago

$$\vec{F} = q\vec{E} = \text{const}$$

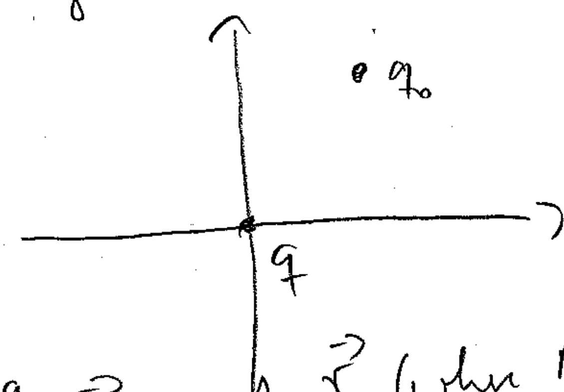
$$U = -q\vec{E}\vec{r}$$

$\Rightarrow$

Coulomb force

(α) Cartesian coordinate system

The statement in the book that the potential cannot be determined in Cartesian coordinates is incorrect!
Here's the proof why



$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^3}\vec{r} := A\frac{\vec{r}}{r^3} \quad \left(\text{where } A = \frac{1}{4\pi\varepsilon_0}qq_0 = \text{const}\right)$$

With the method (β) it's easy!

$$\frac{\partial U}{\partial x} = -F_x = -A\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

Thus we integrate by x

$$U = -A\int dx\,\frac{x}{(x^2+y^2+z^2)^{3/2}} + U_1(y,z)$$

Substitution: $u = x^2 \Rightarrow x\,dx = \frac{1}{2}du$

$$U = -\frac{A}{2}\int\frac{du}{(u+y^2+z^2)^{3/2}} + U_1(y,z)$$

$$u = -\frac{A}{2} \int du \, (u + y^2 + z^2)^{-3/2} + u_1(y, z)$$

$$= -\frac{A}{2} \left(-\frac{3}{2}+1\right)(u + y^2 + z^2)^{-3/2+1} \Big|_{u = x^2} + u_1(y, z)$$

$$= A\,(x^2 + y^2 + z^2)^{-1/2} = \frac{A}{r} + u_1(y, z)$$

$$U = \frac{1}{4\pi\varepsilon_0} \frac{q\, q_0}{r} + u_1(y, z)$$

$$\frac{\partial U}{\partial y} = -\frac{1}{4\pi\varepsilon_0} \frac{q\, q_0}{r^2} \frac{\partial r}{\partial y} + \frac{\partial u_1(y, z)}{\partial y}$$

$$= -\frac{1}{4\pi\varepsilon_0} \frac{q\, q_0}{r^3}\, y + \frac{\partial u_1(y, z)}{\partial y}$$

$$= -F_y + \frac{\partial u_1}{\partial y}$$

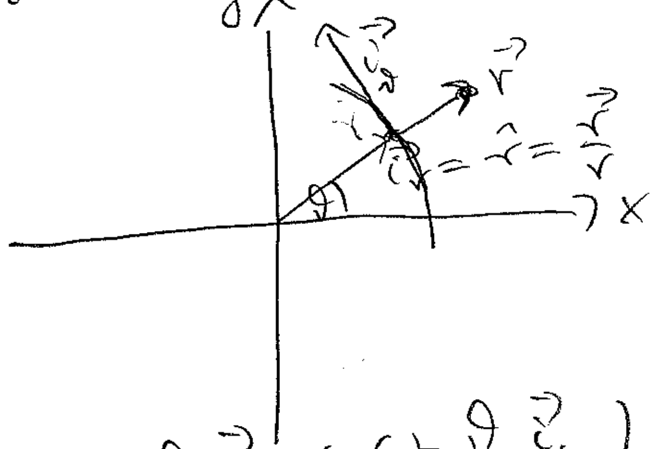
$\Rightarrow$ I can choose $\frac{\partial u_1}{\partial y} = 0 \Rightarrow u_1 = u_2(z)$

In the same way: $\frac{\partial U}{\partial z} = -F_z + \frac{du_2}{dz} \Rightarrow \frac{du_2}{dz} = 0$

$$\Rightarrow \boxed{U = \frac{1}{4\pi\varepsilon_0} \frac{q\, q_0}{r}}$$

(b) Line integral method à la text book

Use polar coordinates. I cannot choose the origin as starting point for my integral because $\vec{F}$ is singular at $\vec{r}=0$:



$$\vec{r} = r(\cos\vartheta\,\vec{i}_x + \sin\vartheta\,\vec{i}_y)$$

Unit vectors in direction of the coordinate lines are given by derivatives of $\vec{r}$ w.r.t. to the independent parameters r, ϑ:

$$\vec{i}_r = \frac{\partial\vec{r}}{\partial r}\,\frac{1}{\left|\frac{\partial\vec{r}}{\partial r}\right|}$$

$$\frac{\partial\vec{r}}{\partial r} = \cos\vartheta\,\vec{i}_x + \sin\vartheta\,\vec{i}_y$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\cos^2\vartheta + \sin^2\vartheta} = 1$$

$$\vec{i}_r = \cos\vartheta\,\vec{i}_x + \sin\vartheta\,\vec{i}_y$$

$$\vec{i}_\vartheta = \frac{\partial\vec{r}}{\partial\vartheta}\,\frac{1}{\left|\frac{\partial\vec{r}}{\partial\vartheta}\right|}$$

$$\frac{\partial \vec{r}}{\partial \vartheta} = r\,(-\sin\vartheta\,\vec{i}_x + \cos\vartheta\,\vec{i}_y)$$

$$\left|\frac{\partial \vec{r}}{\partial \vartheta}\right| = r$$

$$\Rightarrow \vec{i}_\vartheta = (-\sin\vartheta\,\vec{i}_x + \cos\vartheta\,\vec{i}_y)$$

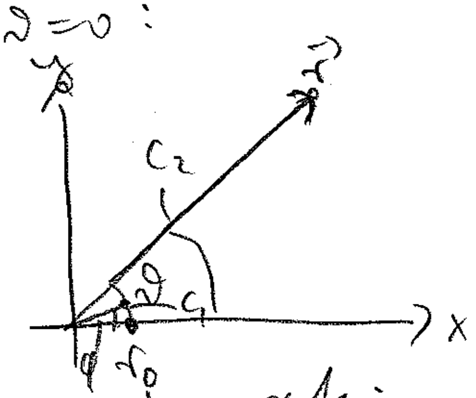
note that

$\vec{i}_r \vec{i}_\vartheta = 0$ as it should be since the radius of a circle is always $\perp$ to the tangent at the point it touches the circle!

The force in the new coordinates is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\,\frac{q\,q_0}{r^2}\,\vec{i}_r$$

and now the path is easily chosen. As a starting point take $r = r_0$, $\vartheta = 0$:



The path consists of two parts:

$$C_1: \vec{r}(\varphi) = r_0(\cos\varphi\,\vec{i}_x + \sin\varphi\,\vec{i}_y) \text{ with } \varphi \in [0, \vartheta]$$

$$d\vec{r} = \frac{d\vec{r}}{d\varphi}\,d\varphi = r_0\,\vec{i}_\varphi$$

$$\vec{r}(\varphi) = r_0\,\vec{i}_r(\varphi)$$

$$C_2: \vec{r}(s) = s\,\vec{i}_r(\vartheta)\;;\; d\vec{r} = \frac{d\vec{r}}{ds}\,ds = \vec{i}_r(\vartheta)\,ds$$

$$U = -\int_{C_1+C_2} d\vec{r}\,\vec{F}(\vec{r})$$

$$= \underbrace{-\int_0^{\vartheta} d\varphi\, r_0\, \vec{i}_\varphi(\varphi) \frac{A}{r_0^2}\, \vec{i}_r(\varphi)}_{0 \text{ because } \vec{i}_\vartheta(\varphi)\,\vec{i}_r(\varphi) = 0} - \int_{r_0}^{r} d\rho\, \vec{i}_r(\vartheta) \frac{A}{\rho^2}\, \vec{i}_r(\vartheta)$$

$$= -\int_{r_0}^{r} d\rho\, \frac{A}{\rho^2} = \frac{A}{\rho}\Big|_{r_0}^{r} = A\left(\frac{1}{r} - \frac{1}{r_0}\right).$$

This is the same result as with the other method with a constant $-\frac{A}{r_0}$ which is irrelevant. Usually one chooses such a potential to vanish for $r \to \infty$ i.e. one lets $r_0 \to \infty$

then

$$U_{r_0 \to \infty} = \frac{A}{r} = \frac{1}{4\pi\varepsilon_0} \frac{q\, q_0}{r}.$$

Potentials (Ctd.) (Feb 12)

Note that we must be careful when we like to calculate the force from the potential in general coordinates which are not cartesian.

In cartesian coordinates we get the Force by the partial derivatives

$$\vec{F} = -\vec{i}_x \frac{\partial U}{\partial x} - \vec{i}_y \frac{\partial U}{\partial y} - \vec{i}_z \frac{\partial U}{\partial z} \quad (1)$$

We can write this in a coordinate-independent form

$$dU = -d\vec{r}\,\vec{F} \quad (2)$$

(1) and (2) tell us the same since $\vec{i}_x^{\,2} = 1$, $\vec{i}_x \cdot \vec{i}_y = 0$ etc.

Now suppose we have given U in terms of polar coordinates (in the plane). Then we can get the components of $\vec{F}$ for a problem in the following way:

$$dU = \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \vartheta}$$

$$= -d\vec{r} \cdot \vec{F}$$

$$= -\vec{F}\left[\frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \vartheta} d\vartheta\right]$$

Now we define

$$g_r = \left|\frac{\partial \vec{r}}{\partial r}\right| \quad \text{and} \quad g_\vartheta = \left|\frac{\partial \vec{r}}{\partial \vartheta}\right|$$

So we have by definition

$$\frac{\partial \vec{r}}{\partial r} = g_r \vec{i}_r \, ; \quad \frac{\partial \vec{r}}{\partial \vartheta} = g_\vartheta \vec{i}_\vartheta$$

In the lecture before, we have seen that

$$g_r = 1 \text{ and } g_\vartheta = r$$

Thus we have

$$\therefore \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \vartheta} d\vartheta = -\vec{F}\left(g_r \vec{i}_r dr + g_\vartheta \vec{i}_\vartheta d\vartheta\right)$$

$$= -\vec{F}\left(\vec{i}_r dr + \vec{i}_\vartheta r d\vartheta\right)$$

Now since

$$\vec{i}_r^{\,2} = \vec{i}_\vartheta^{\,2} = 1 \text{ and } \vec{i}_r \cdot \vec{i}_\vartheta = 0,$$

we have

$$\vec{F} = F_r \vec{i}_r + F_\vartheta \vec{i}_\vartheta$$

$$\Rightarrow \vec{F} \vec{i}_r = F_r \text{ and } \vec{F} \vec{i}_\vartheta = F_\vartheta$$

So we find

$$\frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \vartheta} d\vartheta = -F_r dr - r F_\vartheta d\vartheta$$

Now we can set $d\vartheta = 0$ and $dr \neq 0$

$$\Rightarrow \frac{\partial u}{\partial r} dr = -F_r dr \Rightarrow \boxed{F_r = -\frac{\partial u}{\partial r}}$$

or we can set $dr = 0$ and $d\vartheta \neq 0$ $\Rightarrow$

$$\Rightarrow \frac{\partial u}{\partial \vartheta} d\vartheta = -r F_\vartheta d\vartheta$$

$$r F_\vartheta = -\frac{\partial u}{\partial \vartheta} \Rightarrow \boxed{F_\vartheta = -\frac{1}{r} \frac{\partial u}{\partial \vartheta}}$$

Now we can show directly in polar coordinates

$$U(r) = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r}$$

is the correct potential for a Coulomb force exerted by a charge q at the origin on a charge q_0

$$\vec{F} = -\vec{i}_r \frac{\partial U}{\partial r} = -\vec{i}_r \cdot \frac{q q_0}{4\pi\varepsilon_0}\left(-\frac{1}{r^2}\right)$$

$$= \frac{q q_0}{4\pi\varepsilon_0 r^2} \vec{i}_r \quad ☺$$

Superposition principle for potentials

From chapter 1 we know that the forces on a charge exerted by various point charges add up

$$\vec{F}_{all,0} = \vec{F}_{10} + \vec{F}_{20} + \dots + \vec{F}_{n0}$$

$$= \sum_{k=1}^{n} \vec{F}_{k0}$$

Suppose we know all the potentials of the various $\vec{F}$'s:

$$\vec{F}_{k0} = -\left(\frac{\partial U_k}{\partial x}\vec{i}_x + \frac{\partial U_k}{\partial y}\vec{i}_y + \frac{\partial U_k}{\partial z}\vec{i}_z\right)$$

Then, since the derivative of a sum is the sum of the derivatives we can write

$$F_{k0,x} = -\frac{\partial U_k}{\partial x}$$

$$\Rightarrow F_{tot0,x} = \sum_{k=1}^{n} F_{k0,x} = -\sum_{k=1}^{n} \frac{\partial U_k}{\partial x} = -\frac{\partial}{\partial x}\sum_{k=1}^{n} U_k$$

and analogously for y and z

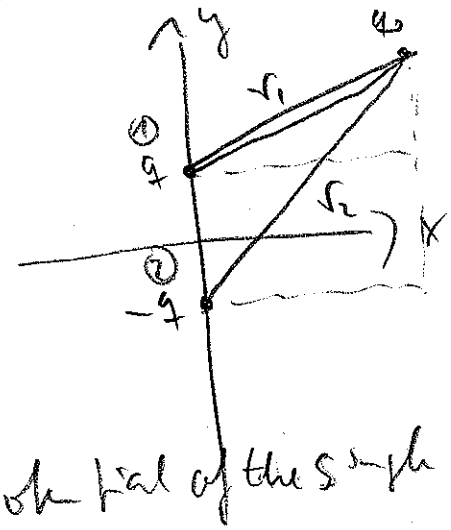
So

$$U_{tot} = \sum_{k=1}^{n} U_k$$

The potential for the total force is thus given by the sum of the potentials for the single charges.

<u>Example</u>: Dipole again

This time in the usual coordinate system :-)



We know the potential of the single charges namely by

$$U_1 = \frac{q q_0}{4\pi\varepsilon_0 r_1} = \frac{q q_0}{4\pi\varepsilon_0 \sqrt{x^2 + (y-a)^2}}$$

$$U_2 = \frac{(-q) q_0}{4\pi\varepsilon_0 r_2} = \frac{q q_0}{4\pi\varepsilon_0 \sqrt{x^2 + (y+a)^2}}$$

So the potential for the total force is

$$U = U_1 + U_2 = \frac{q q_0}{4\pi\varepsilon_0} \left[\frac{1}{\sqrt{x^2 + (y-a)^2}} - \frac{1}{\sqrt{x^2 + (y+a)^2}} \right]$$

$$F_x = -\frac{\partial U}{\partial x} = \frac{q q_0}{4\pi\varepsilon_0} \left[\frac{x}{\sqrt{x^2 + (y-a)^2}^{\,3}} - \frac{x}{\sqrt{x^2 + (y+a)^2}^{\,3}} \right]$$

$$F_y = -\frac{\partial u}{\partial y} = \frac{q q_0}{4\pi\varepsilon_0}\left(\frac{y-a}{[x^2+(y-a)^2]^{3/2}} - \frac{y+a}{[x^2+(y+a)^2]^{3/2}}\right)$$

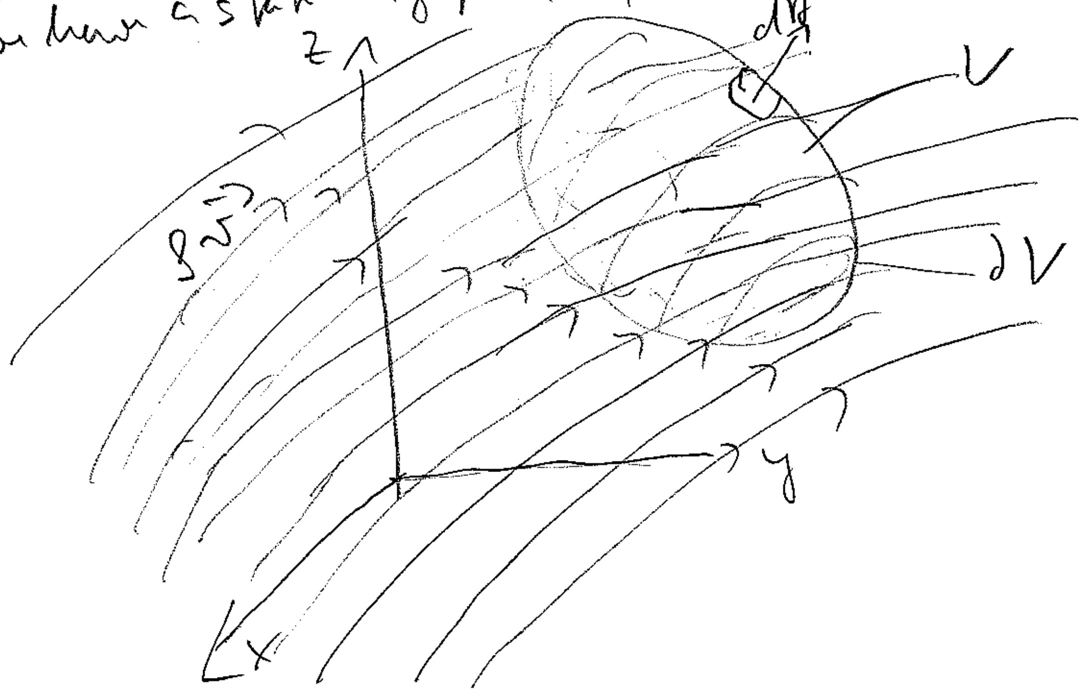
For $y=0$ we obtain the same result as in chapt. II.

Gauss's Law (Lect. Feb 15–) (Own version)

I chose my own way of representing the material since the way in the book is quite complicated.

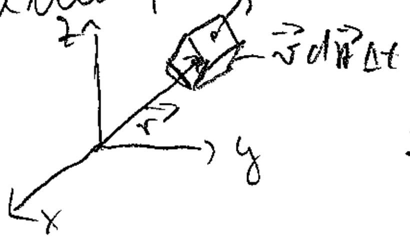
4.1 Statement of the theorem

We ask the question whether we can find the charge distribution in space by measuring the electric field. This is indeed the case. To state the answer, known as Gauss's law, we have to define a new quantity, "electric flux". It is easier to picture the concept by thinking a moment about the flow of particles, described by the density of particles, $n(\vec r)$, streaming through a volume V. At each point in space we also need to know the velocity of the fluid particles at this point. We assume that we have a stationary flow, i.e., $\vec v$ is independent of time



Current: $n\vec v$

Now we ask, how many particles pass through the surface element $d\vec A$.



$d\vec A$ has the magnitude of the surface element and points outwards the volume and is always $\perp$ to the surface element.

Now in a little time element dt, all the particles contained in the volume element $\vec{v}\cdot d\vec{A}\,dt$ stream through the surface element $d\vec{A}$. That means

$$dN_{d\vec{A}} = n(\vec{r})\,dV = n(\vec{r})\,\vec{v}(\vec{r})\cdot d\vec{A}\,dt$$

Thus per unit time we have

$$\frac{dN_{d\vec{A}}}{dt} = n(\vec{r})\,\vec{v}(\vec{r})\,d\vec{A}$$

particles streaming through the surface element $d\vec{A}$.
To get all particles streaming through the whole surface, we just have to sum over all surface elements:

$$\frac{dN_{\partial V}}{dt} = \sum_{\text{all } d\vec{A}\text{'s}} n(\vec{r})\,\vec{v}(\vec{r})\cdot d\vec{A}$$

That becomes accurate making the $d\vec{A}$'s infinitesimally small leading to a surface integral:

$$\frac{dN_{\partial V}}{dt} = \int_{\partial V} d\vec{A}\; n(\vec{r})\,\vec{v}(\vec{r}).$$

If there is no source of fluid inside the volume, we must have

$$\int_{\partial V} d\vec{A}\; n(\vec{r})\,\vec{v}(\vec{r}) = 0 \quad \text{(no sources)}$$

Suppose there is some source inside the volume providing at each point in space S particles per unit volume and per unit time. Then we have

$$\int_{\partial V} d\vec{A}\; n(\vec{r})\,\vec{v}(\vec{r}) = \int_V dV\, s(\vec{r}) = \frac{dN_{\text{produced in } V}}{dt}$$

Now, since we have learnt that the sources of el. fields are the el. charges of particles, we might think that the total charge contained in a volume is given by a surface integral over its boundary. One should of course not think that the electric field is the flow of some kind of particles. It's just a mathematical analogy. And indeed, as Carl Friedrich Gauß has shown, we have always

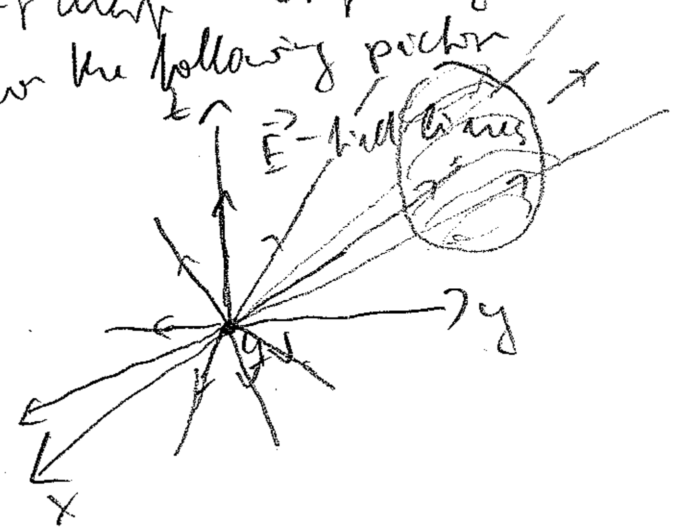
$$\int_{\partial V} d\vec{A} \cdot \vec{E} = \frac{Q_V}{\varepsilon_0} \quad \text{(Gauss' Law)}$$

where Q_V is the charge contained in the volume V. Note that ε_0 is just a conversion factor introduced by the choice of units for the charges.

4.2 Proof for a single point charge

(a) Point charge not contained in the volume

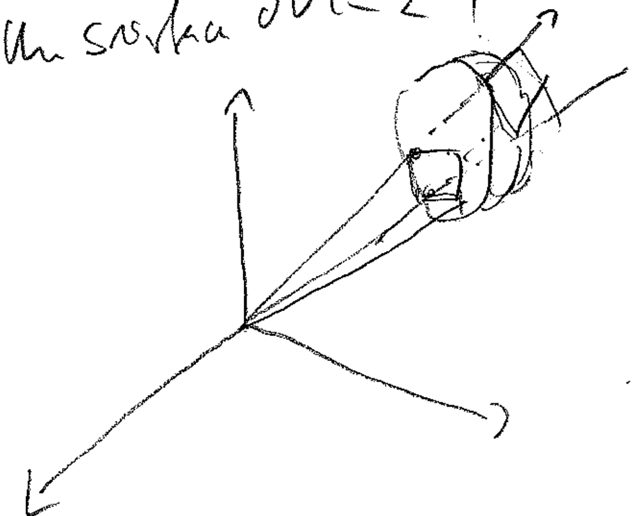
Put the point charge in origin of coordinate system. Then we have the following picture



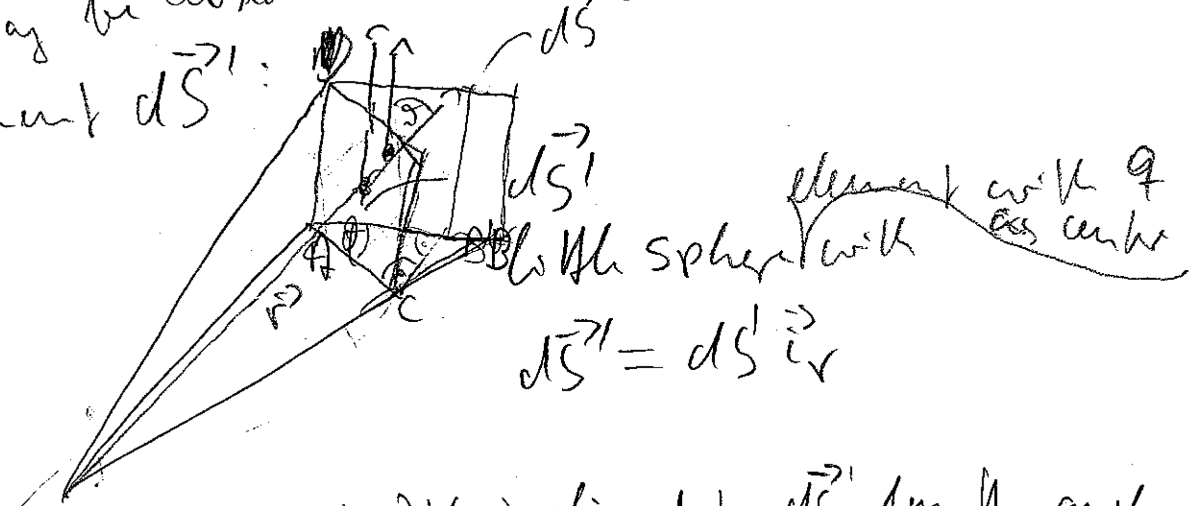
We like to show that then

$$\int_V d\vec{A}\cdot\vec{E} = 0$$

For that purpose we draw a cone through the origin, intersecting the surface ∂V in 2 points



The cone may be closed at the end by a little spherical surface element $d\vec{S}'$:



$$d\vec{S}' = dS'\,\vec{i}_r$$

The surface element of ∂V is inclined to $d\vec{S}'$ by the angle φ, and since ΔABC is rectangular, because the radius through the sphere is always $\perp$ to the sphere. Thus

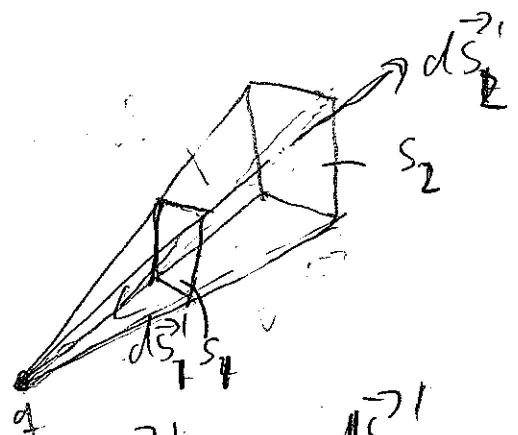
$$|d\vec{S}'| = |AD|\cdot|AC| = |AD|\cdot|AB|\cos\varphi = |d\vec{S}|\cos\varphi$$

Now $d\vec{S}\cos\varphi = \vec{i}_r\, d\vec{S} = |d\vec{S}'|$

So we have, because of

$$d\vec{S} \cdot \vec{E}(\vec{r}) = d\vec{S} \cdot \vec{i}_r E_r = |d\vec{S}'| E_r = d\vec{S}' \vec{E}$$

So we have to show the whole thing only for a conelike element like this



note that $d\vec{S}$ is always pointing out of the volume so $d\vec{S}_2$ is oppositely directed (towards q) than $d\vec{S}_1$

Now: $\dfrac{d\vec{S}_1'}{r_1^2} = -\dfrac{d\vec{S}_2'}{r_2^2}$

from the fact that the surface elements of a sphere scale with its radius squared. Thus we have

$$d\vec{S}_1 \cdot \vec{E}(\vec{r}_1) = d\vec{S}_1' \cdot \vec{E}(\vec{r}_1) = \vec{i}_r \, d\vec{S}_1' \frac{q}{4\pi\varepsilon_0} \frac{1}{r_1^2}$$

$$d\vec{S}_2 \cdot \vec{E}(\vec{r}_2) = d\vec{S}_2' \vec{E}(\vec{r}_2) = +\vec{i}_r \cdot d\vec{S}_2' \frac{q}{4\pi\varepsilon_0} \frac{1}{r_2^2}$$

$$= -\vec{i}_r \frac{d\vec{S}_1'}{r_1^2} \frac{q}{4\pi\varepsilon_0} = -d\vec{S}_1 \vec{E}(\vec{r}_1)$$

$$\Rightarrow d\vec{S}_1 \vec{E}(\vec{r}_1) + d\vec{S}_2 \vec{E}(\vec{r}_2) = 0$$

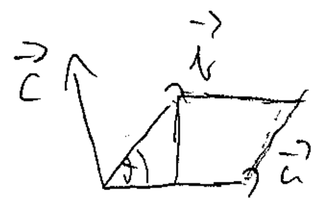
Now we sum over all surface elements of ∂V leading to the conclusion that for each surface element there exists another one which cancels the contribution from it. Thus

$$\int_{\partial V} d\vec{S}\, \vec{E} = 0 \quad (\text{if } q \text{ is not contained in } V).$$

(b) Proof for charge inside a spherical volume

First we choose a spherical volume. Then we can calculate the electric flux without much trouble. We shall see how to do such calculations in practice.

First we remember the vector product. Two vectors $\vec{a}$ and $\vec{b}$, in 3-dim space give a vector $\vec{c} = \vec{a} \times \vec{b}$ which has the magnitude of the surface of the parallelogram defined by $\vec{a}$ and $\vec{b}$ and the direction perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$, following the right-hand rule: putting your thumb in direction of $\vec{a}$ and your index finger in direction of $\vec{b}$, your middle finger points in the direction of $\vec{c}$:

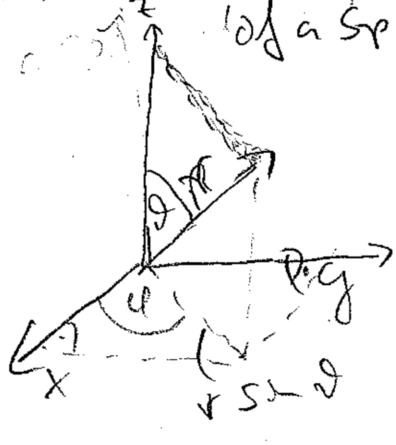


Cartesian coordinates

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\vartheta$$

Now our sphere surface around the origin can be parameterized with the angles ϑ, φ of a spherical coordinate system



$$\vec{r} = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix} \qquad \begin{matrix} 0 \le \varphi \le 2\pi \\ 0 \le \vartheta \le \pi \end{matrix}$$

$\partial V: \vec{r}(\vartheta,\varphi) = R\begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}$ (R: radius of the sphere).

The surface element we calculate by these coordinates:

$$d\vec{S} = \frac{\partial\vec{r}}{\partial\vartheta}\times\frac{\partial\vec{r}}{\partial\varphi}\, d\vartheta\, d\varphi$$

$$d\vec{S} = R\begin{pmatrix}\cos\vartheta\cos\varphi\\ \cos\vartheta\sin\varphi\\ -\sin\vartheta\end{pmatrix}\times R\begin{pmatrix}-\sin\vartheta\sin\varphi\\ \sin\vartheta\cos\varphi\\ 0\end{pmatrix}$$

$$= R^2\begin{pmatrix}\sin^2\vartheta\cos\varphi\\ \sin^2\vartheta\sin\varphi\\ \sin\vartheta\cos\vartheta\end{pmatrix} = R^2\sin\vartheta\begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}$$

$$= R^2\sin\vartheta\,\vec{i}_r$$

This choice of variables and order of cross product gives the correct direction of $d\vec{S}$ radially out of the sphere:

Surface of sphere

$$S = \int_0^\pi d\vartheta\int_0^{2\pi} d\varphi\,|d\vec{S}| = \int_0^\pi d\vartheta\int_0^{2\pi} d\varphi\, R^2\sin\vartheta$$

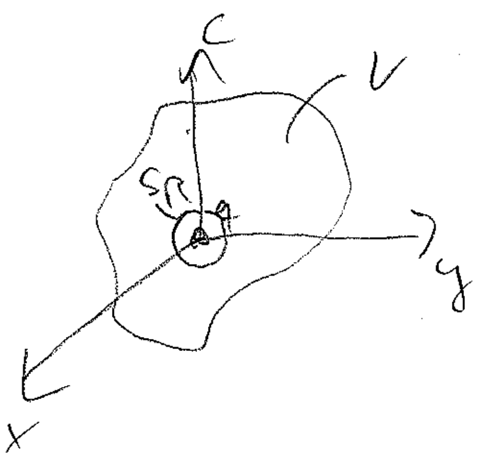
$$= R^2 2\pi\int_0^\pi d\vartheta\,\sin\vartheta = 2\pi R^2(-\cos\vartheta)\Big|_0^\pi = 4\pi R^2 \text{ ☺}$$

$$\Phi_q = \int_0^\pi d\vartheta\int_0^{2\pi} d\varphi\, d\vec{S}\cdot\vec{i}_r\,\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}$$

$$= \int_0^\pi d\vartheta\int_0^{2\pi} d\varphi\, R^2\sin\vartheta\,\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\int_0^\pi d\vartheta\int_0^{2\pi} d\varphi\,\sin\vartheta = \frac{q}{\varepsilon_0}. \text{ ☺}$$

(c) One point charge in arbitrary volume



We just take out a sphere from the volume, containing the charge completely included inside it. Then the volume $V-S_R$ does not contain the charge, and we have shown that then

$$\int_{\partial(V-S_R)} d\vec{S}\cdot\vec{E} = 0$$

Now for the sphere we would have to take the $d\vec{S}$ inwards, because we want to point it out of the volume $V-S_R$. Thus writing it in the usual way with $d\vec{S}$ outwards, we have an additional − sign:

$$\int_{\partial V} d\vec{S}\cdot\vec{E} - \int_{\partial S_R} d\vec{S}\,\vec{E} = 0$$

$$\text{or} \int_{\partial V} d\vec{S}\cdot\vec{E} = \int_{\partial S_R} d\vec{S}\cdot\vec{E} = \frac{q}{\varepsilon_0}$$

This proves Gauss's law for a point charge.

4.3 Proof for arbitrary charge distribution

(a) several point charges

That's easy since

$$\vec{E}_{total} = \sum_{r=1}^{n} \vec{E}_r$$

with $\vec{E}_r = \frac{q_r}{4\pi\varepsilon_0} \frac{\vec{r}-\vec{r}_r}{|\vec{r}-\vec{r}_r|^3}$

the field of charge, q_r.

Thus:

$$\Phi^{(E)}_{total} = \int_{\partial V} d\vec{S} \sum_{r=1}^{n} \vec{E}_r$$

$$= \sum_{r=1}^{n} \int_{\partial V} d\vec{S} \cdot \vec{E}_r = \sum_{r=1}^{n} \Phi^{(E)}_r$$

Now $\Phi^{(E)}_r = \begin{cases} \frac{q_r}{\varepsilon_0} & \text{if } q_r \text{ is in } V \\ 0 & \text{if } q_r \text{ is outside } V \end{cases}$

$$\Rightarrow \Phi^{(E)}_{total} = \int_{\partial V} d\vec{S}\,\vec{E}_{total} = \sum_{\substack{\text{all } r \\ \text{with } q_r \text{ in } V}} \frac{q_r}{\varepsilon_0}$$

☺

(b) arbitrary volume charge

$$\vec{E} = \int_{\text{all space}} dV' \, \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\Rightarrow \Phi^{(E)} = \int_{\partial V} d\vec{S} \cdot \int_{\text{all space}} \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\Phi^{(2)} = \int_{\text{all space}} dV' \, \rho(\vec{r}') \oint_{\partial V} \frac{d\vec{S}}{4\pi\varepsilon_0} \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

but the surface integral is the electric flux of a point charge with $q=1$, i.e., this gives

$$\Phi^{(2)} = \int_{\text{allspace}} dV' \frac{\rho(\vec{r}')}{\varepsilon_0} \chi_V(\vec{r}') = \int_V dV' \frac{\rho(\vec{r}')}{\varepsilon_0} = \frac{Q_V}{\varepsilon}$$

where

$$\chi(\vec{r}') = \begin{cases} 1 \text{ for } \vec{r}' \in V \\ 0 \text{ for } \vec{r}' \notin V \end{cases}$$

and Q_V is the charge inside of V. ☺

Mechanics Review (lecture 2/12 in prep. for exam)

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}} = q\vec{E}(\vec{r})$$

Motion of a particle with mass m in an electrostatic field. We know that $\vec{E}$ is conservative, and thus also the force

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$$

is conservative, i.e., there exists a potential, u:

$$\vec{F} = -\left(\frac{\partial u}{\partial x}\vec{i}_x + \frac{\partial u}{\partial y}\vec{i}_y + \frac{\partial u}{\partial z}\vec{i}_z\right)$$

(written in Cartesian coordinates).

Since the electric potential, V, is defined by

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\vec{i}_x + \frac{\partial V}{\partial y}\vec{i}_y + \frac{\partial V}{\partial z}\vec{i}_z\right),$$

we have the relation

$$u = qV$$

Energy-conservation law

With a conservative force we have

$$m\ddot{\vec{r}} = -\left(\frac{\partial u}{\partial x}\vec{i}_x + \frac{\partial u}{\partial y}\vec{i}_y + \frac{\partial u}{\partial z}\vec{i}_z\right)$$

Suppose we have solved the equation of motion. Then we have

$$m\dot{\vec{r}}\ddot{\vec{r}} = \frac{d}{dt}\left(\frac{m}{2}\vec{v}^2\right) = -\left(\dot{x}\frac{\partial u}{\partial x} + \dot{y}\frac{\partial u}{\partial y} + \dot{z}\frac{\partial u}{\partial z}\right)$$

$$= -\frac{d}{dt}u[\vec{r}(t)]$$

or $$\frac{d}{dt}\left(\frac{m}{2}\vec{v}^2 + u[\vec{r}(t)]\right) = 0.$$

This means that

$$E = \frac{m}{2}\vec{v}^2 + U[\vec{r}(t)]$$

is constant during the whole motion of the particle.

For two points on the trajectory, t_1, t_2, we thus have

$$\frac{m}{2}\vec{v}_1^{\,2} + U(\vec{r}_1) = \frac{m}{2}\vec{v}_2^{\,2} + U(\vec{r}_2),$$

where $\vec{r}_1 = \vec{r}(t_1)$ etc.

We also know that due to the existence of the potential, its value depends only on the initial and final point, not on the specific path which connects them:

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{C(\vec{r}_1 \to \vec{r}_2)} d\vec{r}\cdot\vec{F}(\vec{r}).$$

Especially with the true trajectory of the particle we see that it is identical with the work done on the particle by the force, $\vec{F}$ (resp. to this sign):

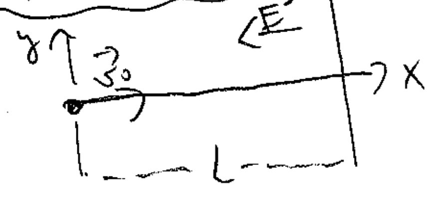
$$W = \int_{C_{traject}(\vec{r}_1 \to \vec{r}_2)} d\vec{r}\,\vec{F}(\vec{r}) = \int_{t_1}^{t_2} dt\,\vec{v}(t)\,\vec{F}[\vec{r}(t)]$$

$$\Rightarrow \boxed{W = -[U(\vec{r}_2) - U(\vec{r}_1)]}$$ or for electrostatics $$\boxed{W = -q\,[V(\vec{r}_2) - V(\vec{r}_1)]}$$

Examples from book

Chapter II Problem 6 (with energy law)

$$\vec{E} = -\underbrace{E}_{\text{const.}}\,\vec{e}_x \Rightarrow V = Ex$$



Initial state: $x = 0$, $v_x = v_0$

$\Rightarrow \mathcal{E} = \frac{m}{2} v_0^2$; $\mathcal{E}$ = energy, because E is used for el. field :-).

Final state: $x = L$, $v_x = 0$

$$\mathcal{E} = q E L$$

$$\Rightarrow \frac{m}{2} v_0^2 = q E L \Rightarrow E = \sqrt{\frac{m v_0^2}{2 q L}}$$

II: Ex 2, 5; Ex 7 → solutions to exercises

II: Ex 6 with energy law

$$\vec{E} = -E \vec{i}_x \Rightarrow V = Ex$$

$$\mathcal{E} = 0 = \frac{m}{2} v^2 - e E a$$

$$\Rightarrow v = \sqrt{\frac{2 e E a}{m}} = 3.25 \cdot 10^7 \frac{m}{s}$$

III. Probs 2, 6; Ex 1b, 5, 6 → see solutions to exercises