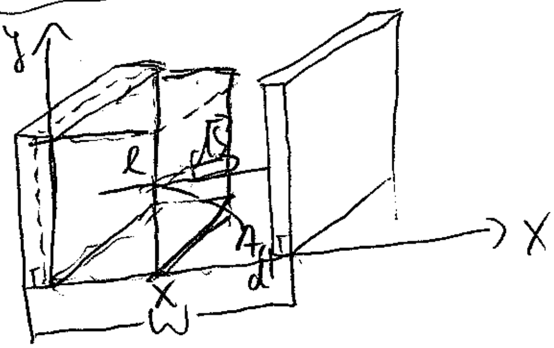
Makeup Exam II – Spring 2008

(1)



$d = W - 2T$

(a) Gaussian Surface: box as indicated.

From symmetry: $\vec{E} = E_x \vec{i}_x$

Only right surface at x gives distributions since in such conductors is no field, and the other areas of the box have $d\vec{S} \perp \vec{E}$. Gauss's law:

$$\Rightarrow \frac{\sigma A}{\varepsilon_0} = \int_A d\vec{S} \cdot \vec{E} = A \vec{i}_x \vec{E} = A E_x$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\varepsilon_0} \vec{i}_x \quad \text{for } 0 < x < d}$$

(b) Voltage between plates

$$V = -\int_{\mathcal{C}} d\vec{r}\, \vec{E}$$

$\ell : \vec{r}(\lambda) = (d - \lambda)\vec{i}_x$ with $\lambda \in (0, d) \Rightarrow \frac{d\vec{r}}{d\lambda} = -\vec{i}_x$

$$V = -\int_0^d \underbrace{d\lambda\,(-\vec{i}_x)}_{d\vec{r}} \cdot \vec{E} = \int_0^d d\lambda\, E_x = E_x d = \frac{\sigma d}{\varepsilon_0}$$

$$Q = \sigma A \Rightarrow V = \frac{Q d}{A \varepsilon_0} = \frac{Q}{C} \Rightarrow \boxed{C = \frac{\varepsilon_0 A}{d}}$$

(c) charge is still the same

$$C_{tot} = \frac{C_1 C_2}{C_1 + C_2} \quad \text{(series)}$$

$$C_1 = \frac{\varepsilon_0 A}{H} \; ; \quad C_2 = \frac{\varepsilon_0 A}{d-(T+H)}$$

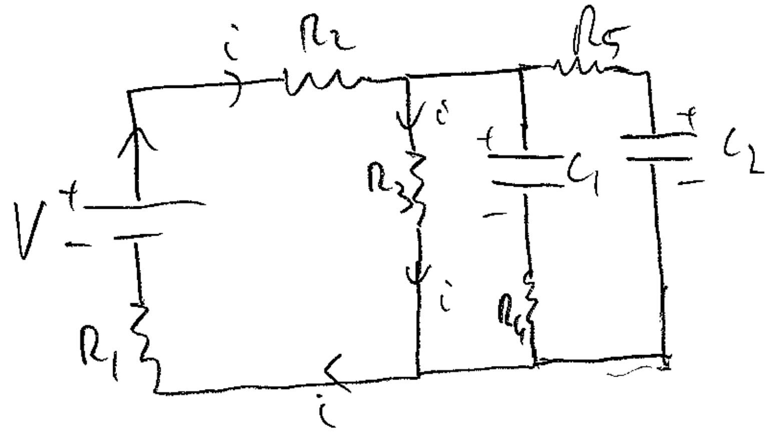
$$C_1 + C_2 = \varepsilon_0 A \left(\frac{1}{H} + \frac{1}{d-(T+H)} \right)$$

$$= \varepsilon_0 A \left(\frac{d-T}{H\,(d-T-H)} \right)$$

$$\Rightarrow C_{tot} = \frac{\varepsilon_0 A}{d-T} \; ; \quad V_{new} = \frac{Q}{C} = \frac{\sigma A (d-T)}{\varepsilon_0 A}$$

$$\boxed{V_{new} = \frac{\sigma (d-T)}{\varepsilon_0}}$$

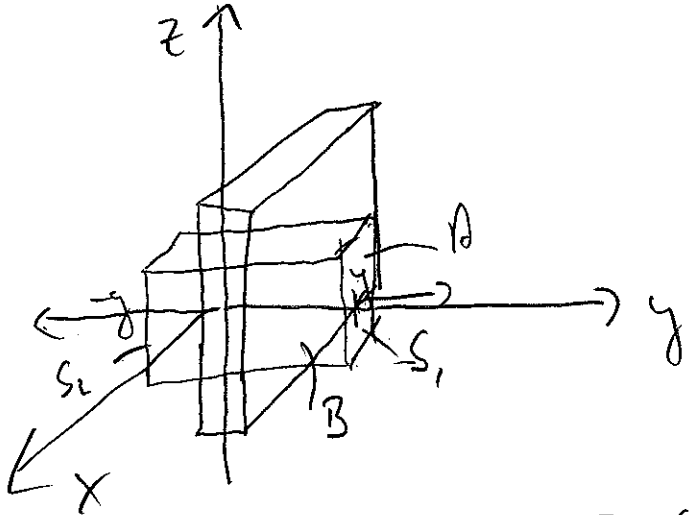
(2)



(a) It's a simple circuit with series resistors

$$\Rightarrow i = \frac{V}{R_1 + R_2 + R_3} \; ; \text{ no currents through } R_4, R_5$$

$$Q_1 = \frac{R_3 C_1}{R_1 + R_2 + R_3} V \; ; \quad Q_2 = \frac{R_3 C_2}{R_1 + R_2 + R_3} V$$



$E_y(-y) = -E_y(y)$ (Symmetry)

$\vec{E} = E_y(y)\,\vec{i}_y$;

Gaussian surface: box symmetric in y

Only contributions: S_1, S_2 (for $y>0$):

$$\oint_B d\vec{S}\,\vec{E} = A\left[E_y(y) - E_y(-y)\right] = 2A\,E_y(y)$$

$$= \frac{Q_{inside}}{\varepsilon_0}$$

if $y > \frac{T}{2} \Rightarrow Q_{inside} = \rho A T \Rightarrow \boxed{E_y(y) = \frac{\rho T}{2\varepsilon_0} \text{ for } y > T/2}$

if $y < \frac{T}{2} \Rightarrow Q_{inside} = \rho A 2y \Rightarrow E_y(y) = \frac{\rho y}{\varepsilon_0}$ for $|y| < \frac{T}{2}$

$$E_y(y) = \begin{cases} \frac{\rho y}{\varepsilon_0} & \text{for } |y| < \frac{T}{2} \\ \text{sign } y \cdot \frac{\rho T}{2\varepsilon_0} & \text{for } |y| > \frac{T}{2} \end{cases}$$

Electric potential

$$\frac{\partial V}{\partial y} = -E_y(y) \Rightarrow V(y) = \begin{cases} -\frac{\rho y^2}{2\varepsilon_0} & \text{for } |y| < \frac{T}{2} \\ -\frac{\rho T|y|}{2\varepsilon_0} + \frac{\rho T^2}{8\varepsilon_0} & \text{for } y > \frac{T}{2} \end{cases}$$

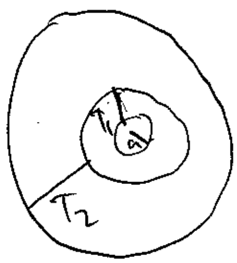
$$V(L+\tfrac{T}{2}) - V(0) = V(L+\tfrac{T}{2})$$

$$= -\frac{\rho T^2}{8\varepsilon_0} - \frac{\rho T(L+\frac{T}{2})}{2\varepsilon_0}$$

$$\boxed{\Delta V = -\frac{\rho T^2}{8\varepsilon_0} - \frac{\rho T L}{2\varepsilon_0}}$$

4a) $\vec{j}(\ = j_r(r)\,\vec{i}_r = \dfrac{i}{4\pi r^2}\,\vec{i}_r \begin{cases} \text{for } a < r < a+T_1+T_2 \\ 0 \text{ elsewhere} \end{cases}$

$$\vec{E} = \rho\,\vec{j}$$



(b) $$\Delta V = \int_a^{a+T_1} dr\left(\frac{\rho_1 i}{4\pi r^2}\right) + \int_{a+T_1}^{a+T_1+T_2} dr\,\frac{\rho_2 i}{4\pi r^2}$$

$$= \frac{i}{4\pi}\left(-\frac{\rho_1}{r}\Big|_a^{a+T_1} + \left(-\frac{\rho_2}{r}\right)_{a+T_1}^{a+T_1+T_2}\right)$$

$$= \frac{i}{4\pi}\left[\rho_1\left(\frac{1}{a} - \frac{1}{a+T_1}\right) + \rho_2\left(\frac{1}{a+T_1} - \frac{1}{a+T_1+T_2}\right)\right]$$

$$\boxed{\Delta V = \frac{i}{4\pi}\left[\frac{\rho_1 T_1}{a(a+T_1)} + \frac{\rho_2 T_2}{(a+T_1)(a+T_1+T_2)}\right]}$$

(5)

(c) $\frac{\sigma}{\varepsilon_0} = \hat{r} \cdot \left[\vec{E}(a+T_1+0^+) - \vec{E}(a+T_1-0^+) \right]$

$$\boxed{\sigma = \frac{\varepsilon_0 (\delta_2 - \delta_1) i}{4\pi (a+T_1)^2}}$$