GRADING

Physics 208 Final Exam

May 07, 2007

Instructor: Dr. Hendrik van Hees, Cyclotron Institute, 845-1411

Print your name **neatly:**

Last Name: VAN HEES

First Name: HENDRIK

Sign your name: ....

Stud. ID (UIN): ☐☐☐ ☐☐ ☐☐☐☐

IMPORTANT

- There are 8 problems totaling 200 points and a formula sheet. Check your exam to make sure you have all the pages. Work each problem on the page the problem is on. You may use the back. If you need extra pages, I have plenty up front. Write your name on each page!
- **Indicate what you are doing!** We cannot give full credit for merely writing down the answer. **Neatness counts!** I will give generous partial credit **if** I can tell that you are on the right track. **Especially indicate the directions of all used vectors, currents, integration paths, etc.**
- Each problem is self explanatory. If you must ask a question, then come to the front, being as discrete as possible so as not to disturb others.

1	2	3	4	5	6	7	8	total

Useful formulae

Maxwell's equations (neglecting the displacement current)

$$\text{Gauss's Law for electric fields:} \quad \oint_{\partial V} \mathrm{d}\vec{S} \cdot \vec{E} = \frac{1}{\epsilon_0} \int_V \mathrm{d}V \rho = \frac{Q_{\text{inside}}}{\epsilon_0}$$

$$\text{Gauss's Law for magnetic fields:} \quad \oint_{\partial V} \mathrm{d}\vec{S} \cdot \vec{B} = 0$$

$$\text{Ampère's Circuital Law:} \quad \oint_{\partial S} \mathrm{d}\vec{r} \cdot \vec{B} = \mu_0 \int_S \mathrm{d}\vec{S} \cdot \vec{j} = \mu_0 \, i_{\text{enclosed}}$$

$$\text{Faraday's Law:} \quad \oint_{\partial S} \mathrm{d}\vec{r} \cdot \vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathrm{d}\vec{S} \cdot \vec{B} = -\frac{\mathrm{d}\Phi_{\vec{B}}}{\mathrm{d}t}$$

Remember the relative directions of the surface-normal vectors, $\mathrm{d}\vec{S}$: For Gauss's Laws always **out of the volume**; for Ampère's and Faraday's Laws always **corresponding to the direction of the boundary, ∂S, according to the right-hand rule!**

Coulomb force from a point charge, q_1, at position $\vec{r}_1$ on a point charge, q_2, at position $\vec{r}_2$

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3}$$

Electric potential of a point charge, q, at position $\vec{r}\,'$

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}\,'|}$$

Lorentz force for a point charge in an electromagnetic field

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Force on a current-conducting wire in a magnetic field

$$\vec{F} = i\vec{l} \times \vec{B}$$

Biot-Savart Law for the $\vec{B}$ field from a current-conducting wire

$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{4\pi} \int \mathrm{d}\vec{r}\,' \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

The differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + c_1 x = c_2,$$

where $c_1 \neq 0$ and c_2 are constants, has the **general** solution

$$x(t) = A\exp(-c_1 t) + \frac{c_2}{c_1}, \quad A = \text{const.}$$

Here x might be a charge, current, or any other quantity!

Potentially useful Integrals

$$\int \mathrm{d}x \frac{1}{[(x+a)^2 + b]^{3/2}} = \frac{x+a}{b\sqrt{(x+a)^2 + b}}, \quad \int \mathrm{d}x \frac{x}{[(x+a)^2 + b]^{3/2}} = -\frac{a(x+a) + b}{b\sqrt{(x+a)^2 + b}}$$

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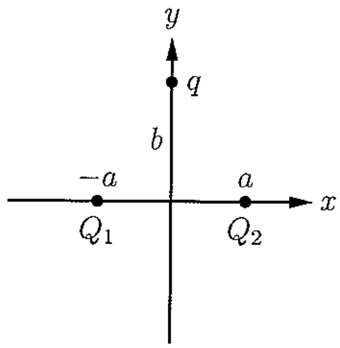
1. (25 points) Two point charges, Q_1 and Q_2, are located at $x = -a$ and $x = +a$ on the x axis of a Cartesian coordinate system (x, y):



(a) Calculate the electric potential, $V(x, y)$, everywhere!

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{\sqrt{(x+a)^2+y^2}} + \frac{Q_2}{\sqrt{(x-a)^2+y^2}}\right] \qquad (5)$$

(b) Determine the electric field, $\vec{E}(x, y)$, from the electric potential!

$$\vec{E} = -\text{grad}\, V = -\left(\frac{\partial V}{\partial x}\vec{i}_x + \frac{\partial V}{\partial y}\vec{i}_y\right)$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1(x+a)}{[(x+a)^2+y^2]^{3/2}} + \frac{Q_2(x-a)}{[(x-a)^2+y^2]^{3/2}}\right] \qquad (15)$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1 y}{[(x+a)^2+y^2]^{3/2}} + \frac{Q_2 y}{[(x-a)^2+y^2]^{3/2}}\right]$$

(c) What is the force on a test charge, q, located on the y axis at $y = b$?

$$\vec{F} = q\vec{E}\Big|_{x=0,\, y=b}$$

$$F_x = \frac{q}{4\pi\varepsilon_0}\left[\frac{Q_1 a}{(a^2+b^2)^{3/2}} - \frac{Q_2 a}{(a^2+b^2)^{3/2}}\right] \qquad (5)$$

$$F_y = \frac{q}{4\pi\varepsilon_0}\left[\frac{Q_1 b}{(a^2+b^2)^{3/2}} + \frac{Q_2 b}{(a^2+b^2)^{3/2}}\right]$$

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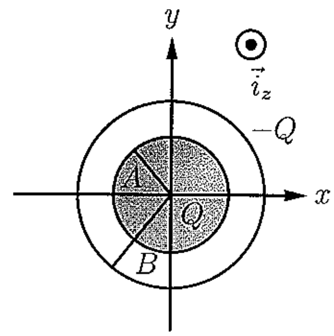
2. (25 points) A charge, Q, is uniformly distributed in a sphere of radius, A. At radius $B > A$ is a very thin spherical shell carrying a uniformly spread surface charge of total magnitude $-Q$.



(a) Determine the electric field, $\vec{E}$, everywhere.

Gauss's Law with spheres as surfaces

$\vec{E} = E_r(r)\,\vec{i}_r$ (3) (3) for integral

$$\int_{S_r} d\vec{S}\cdot\vec{E} = 4\pi r^2 E_r(r) = \frac{Q_{inside}}{\varepsilon_0}$$ (15)

$$\Rightarrow E_r(r) = \frac{Q}{4\pi\varepsilon_0}\begin{cases} \frac{r}{A^3} & \text{for } 0 \le r < A \quad (3) \\ \frac{1}{r^2} & \text{for } A \le r \le B \quad (3) \\ 0 & \text{for } r > B \quad (3) \end{cases}$$

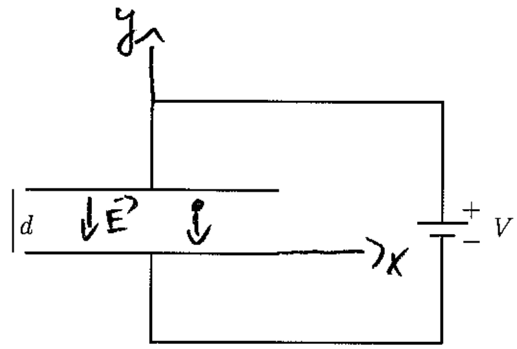
(b) Calculate the potential difference between the center of the sphere and a point outside the shell (at a distance $R > B$ from the center).

$$V(R) = -\int_0^R dr\, E_r(r) \quad (2)$$

$$= -\left[\int_0^A dr \frac{Q}{4\pi\varepsilon_0}\frac{r}{A^3} + \int_A^B dr \frac{Q}{4\pi\varepsilon_0 r^2}\right] \quad (4)$$ (10)

$$= -\frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{2A} - \frac{1}{B}\right] \quad (4)$$

Name: ....



3. (25 points) Two parallel conducting plates (distance, d) with area, A, are hooked to a battery of voltage, V, for a long time.

(a) At $t = 0$, a particle of charge $q > 0$ and mass m is at rest very close to the upper plate. What is its velocity when it reaches the lower plate? Neglect gravity.

$$\frac{m}{2} v_{end}^2 = qV \Rightarrow v_{end} = \sqrt{\frac{2qV}{m}} \qquad (5)$$

(b) What is the electric field between the plates? You can neglect edge effects!

$$\vec{E} = -\frac{V}{d}\vec{i}_y = const \qquad (5)$$

(c) At which time $t > 0$ does the particle, described in part (a), hit the lower plate?

$$m\frac{d^2y}{dt^2} \overset{(2)}{=} qE_y = -\frac{qV}{d} \overset{(3)}{\Rightarrow} \frac{d^2y}{dt^2} = -\frac{qV}{md} = -a_y$$

$$y(t) = -\frac{a_y}{2}t^2 + d \; (5) \quad \left(\text{since } \frac{dy}{dt} = 0 \text{ for } t = 0\right) \qquad (15)$$

$$y(t_{end}) = 0 \Rightarrow t_{end} = \sqrt{\frac{2d}{a_y}} = \sqrt{\frac{2m}{qV}}\, d \quad (5)$$

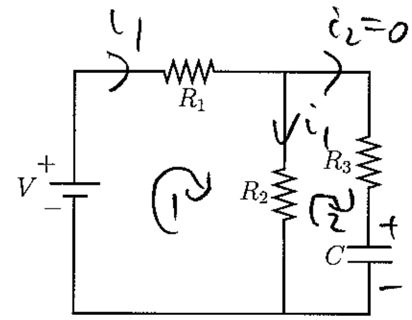
Alternative way: use result from a

$$v_{end} = a_y t_{end} \Rightarrow t_{end} = \frac{v_{end}}{a_y} = \sqrt{\frac{2qV}{m}}\,\frac{md}{qV}$$

$$\Rightarrow t_{end} = \sqrt{\frac{2m}{qV}}\, d$$

Name: ....

4. (25 points) The circuit is hooked up to the battery as shown for a very long time.



(a) What are the currents through each resistor?

$$\oint_{C_1} d\vec{r}\,\vec{E} = -V + (R_1 + R_2)\, i_1 = 0$$

$$\Rightarrow i_1 = \frac{V}{R_1+R_2}$$

No current through R_3 since steady state (10)

(b) What is the voltage across the capacitor? What is the charge at its positively charged plate?

$$\oint_{C_2} d\vec{r}\,\vec{E} = \frac{Q}{C} - R_2 i_1 \Rightarrow V_c = \frac{Q}{C} = R_2\, i_1$$

$$V_C = \frac{R_2}{R_1+R_2} V \qquad (10)$$

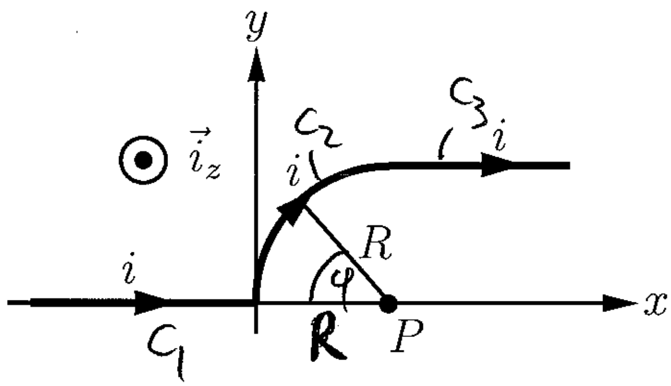
(c) How much power is used by this circuit?

$$P = (R_1+R_2)\, i_1^2 = \frac{V^2}{R_1+R_2} \qquad (5)$$

Hint: Label all currents and the signs of charges at the capacitor in the circuit diagram!

Name: ....

5. (25 points) A thin wire in the xy plane of a Cartesian coordinate system is shaped as shown in the figure: an infinitely long piece lies along the x axis from $x \to -\infty$ to $x = 0$, then a piece is shaped as a quarter of a circle of radius, R. Finally, another infinitely long piece is placed at $y = R$ parallel to the x axis from $x = R$ to $x \to \infty$.



Calculate the magnetic field, $\vec{B}$, at the point P, located at $x = R$, $y = z = 0$, which is the center of the quarter circle!

$$\vec B = \frac{\mu_0 i}{4\pi} \int_C d\vec r' \times \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \qquad (2)$$

Along C_1: $d\vec r' \times (\vec r - \vec r') = 0 \Rightarrow \vec B_1 = 0$ (3)

C_2: $\vec r' = \vec r + R[-\cos\varphi\, \vec i_x + \sin\varphi\, \vec i_y]$ $\qquad \varphi \in (0, \frac{\pi}{2})$

$d\vec r' = d\varphi\, R\, [\sin\varphi\, \vec i_x + \cos\varphi\, \vec i_x]$

$d\vec r \times (\vec r - \vec r') = d\varphi\, R^2 [\sin\varphi\, \vec i_x + \cos\varphi\, \vec i_y] \times [\cos\varphi\, \vec i_x - \sin\varphi\, \vec i_y]$

$= -d\varphi\, R^2\, \vec i_z$ (10)

$$\Rightarrow \vec B_2 = -\frac{\mu_0 i}{4\pi} \vec i_z \int_0^{\pi/2} d\varphi \frac{R^2}{R^3} = -\frac{\mu_0 i}{8R} \vec i_z$$

C_3: $\vec r' = x' \vec i_x + R \vec i_y$; $x' > R$

$d\vec r' = dx'\, \vec i_x$ (10)

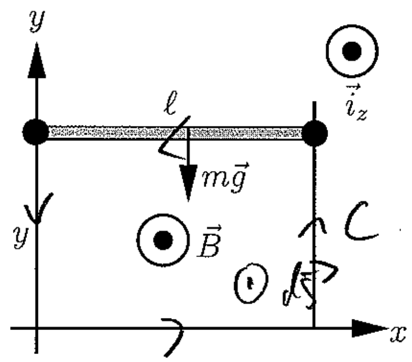
$d\vec r' \times (\vec r - \vec r') = -dx'\, R\, \vec i_z$

$$\Rightarrow \vec B_3 = \frac{\mu_0 i}{4\pi} \int_R^\infty \frac{-R \vec i_z}{[(x' - R)^2 + R^2]^{3/2}} = -\frac{\mu_0 i}{4\pi R} \vec i_z$$

$$\boxed{\vec B = \vec B_2 + \vec B_3 = -\frac{\mu_0 i}{4\pi R}\left[\frac{\pi}{2} + 1\right] \vec i_z}$$

Name: ....

6. (25 points) A rod with resistance, R, length, ℓ and mass, m, can fall in the gravitational field of the earth along ideally conducting wires with negligible friction as shown in the figure. There is a homogeneous magnetic field $\vec{B}$ pointing perpendicularly out of the plane.



(a) Derive the current induced in the rod in the given position at y. The momentary velocity of the rod can be assumed to be $v_y = \mathrm{d}y/\mathrm{d}t$.

$$\oint_C d\vec{r}\,\vec{E} = -\frac{d\Phi_B}{dt} \ (2) \Rightarrow iR = -\ell B \frac{dy}{dt} \ (5) \Rightarrow \qquad (10)$$

$$i = -\frac{\ell B}{R}\frac{dy}{dt} \ (3)$$

(b) Determine the total force on the rod and write down the equation of motion for the rod. The self-inductance of the circuit can be neglected. **You do not need to solve the differential equation!**

$$\vec{F}_M = -i\ell\,\vec{i}_x \times B\vec{i}_z \ (2) = i\ell B\,\vec{i}_y \ (3), \quad \vec{F}_G = -mg\,\vec{i}_y \ (2)$$

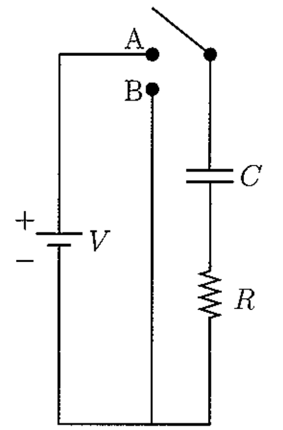
$$m\frac{d^2y}{dt^2} = i\ell B - mg = -\frac{\ell^2B^2}{R}\frac{dy}{dt} - mg \ (3) \qquad (10)$$

(c) Which **constant** terminal velocity does the rod reach after a long time?

$$\frac{d^2y}{dt^2} \overset{(2)}{=} 0 \Rightarrow \frac{dy}{dt} = -\frac{mgR}{\ell^2B^2} \ (3) \qquad (5)$$

Name: ....

7. (25 points) (a) In the circuit shown in the figure, the switch has been in position A for a long time. What is the charge at the upper plate of the capacitor?



Steady state $\Rightarrow$ no current (3)

$$V_c = V \Rightarrow Q_0 = CV$$

(b) At $t = 0$ the switch is set into position B. Calculate the charge, $Q(t)$, at the upper plate of the capacitor as a function of time!

$$\oint d\vec{l}\,\vec{E} = 0 \quad \text{(neglect self-inductance)}$$

$$-\frac{Q}{C} + Ri = 0 \; ; \; i = -\frac{dQ}{dt} \; (2) \Rightarrow \frac{dQ}{dt} = -\frac{1}{RC} Q \; (2) \qquad (10)$$

$$Q(t) = A \exp\left(-\frac{t}{RC}\right) \; ; \; Q(t=0) = Q_0 \Rightarrow A = Q_0 = CV \; (2)$$

(c) What is the current, $i(t)$, through the resistor, as a function of time (for $t > 0$)?

$$i(t) = -\frac{dQ(t)}{dt} = \frac{Q_0}{RC} \exp\left(-\frac{t}{RC}\right) \qquad (5)$$

(d) Calculate the power, $P(t)$, dissipated into heat in the resistor as a function of time.

$$P(t) = R i^2 = \frac{V^2}{R} \exp\left(-\frac{2t}{RC}\right) \qquad (3)$$

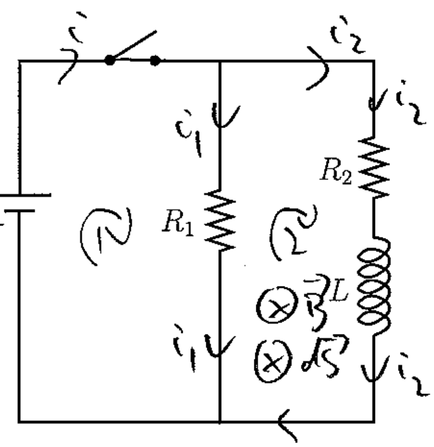
(e) What is the total heat energy produced in the resistor after a very long time ($t \to \infty$)? Comment briefly on the energy-conservation law in this situation!

$$E = \int_0^\infty dt\, P(t) = -\frac{V^2}{R}\frac{RC}{2} \exp\left(-\frac{2t}{RC}\right)\Bigg|_{t=0}^{t\to\infty} = \frac{C}{2} V^2 \; (3) \qquad (4)$$

That was the energy stored in the (electric field of the) capacitor $\Rightarrow$ Energy conservation ok. (1)

Name: ....

8.(25 points) (a) Suppose the switch has been open for a very long time. At $t = 0$ it is closed. Calculate the currents through the resistors as a function of time! All self-inductances except that of the coil, L, can be neglected.



$$\oint_{C_1} d\vec{r}\,\vec{E} = -V + R_1 i_1 = 0 \Rightarrow i_1 = \frac{V}{R_1} = \text{const} \quad 5$$

i_1 jumps at $t=0$ (because self-ind. is ignored)

$$\oint_{C_2} d\vec{r}\,\vec{E} = -R_1 i_1 + R_2 i_2 = -L\frac{di_2}{dt} \quad 5$$

$$L\frac{di_2}{dt} + R_2 i_2 = R_1 i_1 = V \quad 2$$

$$\frac{di_2}{dt} + \frac{R_2}{L} i_2 = \frac{V}{L}$$

(22)

$$i_2(t) = A\exp\left(-\frac{R_2}{L}t\right) + \frac{V}{R_2} \quad 5$$

$$i_2(t=0) = 0 \Rightarrow A = -\frac{V}{R_2}$$

$$i_2(t) = \frac{V}{R_2}\left[1 - \exp\left(-\frac{R_2}{L}t\right)\right] \quad 5$$

(b) Show that after a long time ($t \to \infty$) the currents reach the values to be expected from a steady-state situation.

$$i_2(t) \to \frac{V}{R_2} \text{ for } t \to \infty \quad 3$$

(3)

which is the steady-state value