Coordinates

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Cartesian Coordinates

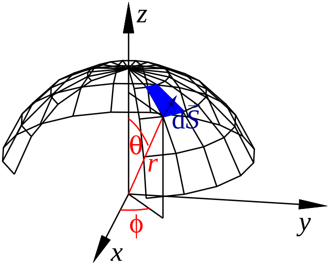
- Cartesian Coordinates, x, y, z: Given are three arbitrary unit vectors $\vec{i}_x$, $\vec{i}_y$, $\vec{i}_z$, which are pairwise perpendicular to each other and oriented in this order by the "right-hand rule"
- each point in space is determined uniquely by its position vector, pointing from the origin of the coordinate system to the point: $\vec{r} = x\vec{i}_x + y\vec{i}_y + z\vec{i}_z$
- dot product: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$
- cross product

$$\begin{aligned}\vec{a} \times \vec{b} =& (a_y b_z - a_z b_y)\vec{i}_x \\ &+ (a_z b_x - a_x b_z)\vec{i}_y \\ &+ (a_x b_y - a_y b_x)\vec{i}_z\end{aligned}$$

- Gradient: $\vec{E} = -\text{grad}\, V$

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\vec{i}_x + \frac{\partial V}{\partial y}\vec{i}_y + \frac{\partial V}{\partial z}\vec{i}_z\right)$$

Spherical Coordinates (Definition)

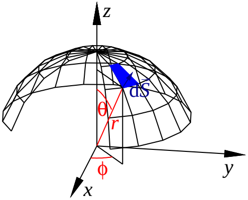


- spherical coordinates, r, θ, ϕ
- determines any point uniquely, not located on the z axis
- Relation to Cartesian Coordinates
  (read them off from the figure!):

$$\vec{r}(r,\theta,\phi) = r(\sin\theta\cos\phi\,\vec{i}_x + \sin\theta\sin\phi\,\vec{i}_y + \cos\theta\,\vec{i}_z)$$

- coordinate ranges: $r > 0$, $\theta \in (0,\pi)$, $\phi \in [0,2\pi)$

Spherical Coordinates (important formulae)



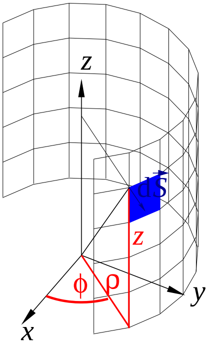
$$\vec{i}_r = \sin\theta\cos\phi\,\vec{i}_x + \sin\theta\sin\phi\,\vec{i}_y + \cos\theta\,\vec{i}_z,$$
$$\vec{i}_\theta = \cos\theta\cos\phi\,\vec{i}_x + \cos\theta\sin\phi\,\vec{i}_y - \sin\theta\,\vec{i}_z,$$
$$\vec{i}_\phi = -\sin\phi\,\vec{i}_x + \cos\phi\,\vec{i}_y.$$

- each vector, not pointing into the z-direction, is then uniquely determined by its components with respect to these unit vectors: $\vec{E} = E_r\,\vec{i}_r + E_\theta\,\vec{i}_\theta + E_\phi\,\vec{i}_\phi$.
- these basis vectors depend on θ and ϕ!
- surface element for sphere of radius r: $\mathrm{d}\vec{S} = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\vec{i}_r$.
- Gradient:

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\vec{i}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\vec{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\vec{i}_\phi\right)$$

- vector products: $\vec{i}_r \times \vec{i}_\theta = \vec{i}_\phi$, $\vec{i}_\theta \times \vec{i}_\phi = \vec{i}_r$, $\vec{i}_\phi \times \vec{i}_r = \vec{i}_\theta$.

Cylinder Coordinates (Definition)

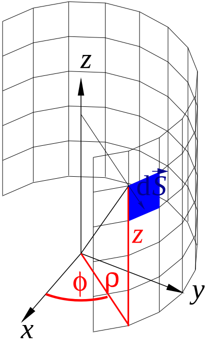


- cylinder coordinates, ρ, ϕ, z
- determines any point uniquely, not located on the z axis
- Relation to Cartesian Coordinates (read them off from the figure!):

$$\vec{r}(\rho, \phi z) = \rho \cos\phi \, \vec{i}_x + \rho \sin\phi \, \vec{i}_y + z \, \vec{i}_z$$

- coordinate ranges: $\rho > 0$, $\phi \in [0, 2\pi)$, $z \in \mathbb{R}$

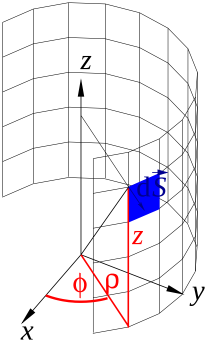
Cylinder Coordinates (important formulae I)



$$\vec{i}_\rho = \cos\phi\, \vec{i}_x + \sin\phi\, \vec{i}_y,$$
$$\vec{i}_\phi = -\sin\phi\, \vec{i}_x + \cos\phi\, \vec{i}_y,$$
$$\vec{i}_z = \vec{i}_z$$

- each vector, not pointing into the z-direction, is then uniquely determined by its components with respect to these unit vectors: $\vec{E} = E_\rho\, \vec{i}_\rho + E_\phi\, \vec{i}_\phi + E_z \vec{i}_z$.
- $\vec{i}_\rho$ and $\vec{i}_\phi$ depend on ϕ!
- surface element for cylinder envelope $\rho =$ const: $\mathrm{d}\vec{S} = \rho\, \mathrm{d}\phi\, \mathrm{d}z\, \vec{i}_\rho$.
- surface element for upper cylinder cap: $\mathrm{d}\vec{S} = \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \vec{i}_z$.

Cylinder Coordinates (important formulae II)



$$\vec{i}_\rho = \cos\phi\,\vec{i}_x + \sin\phi\,\vec{i}_y,$$
$$\vec{i}_\phi = -\sin\phi\,\vec{i}_x + \cos\phi\,\vec{i}_y,$$
$$\vec{i}_z = \vec{i}_z$$

- Gradient:

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho}\vec{i}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{i}_\phi + \frac{\partial V}{\partial z}\vec{i}_z\right)$$

- vector products: $\vec{i}_\rho \times \vec{i}_\phi = \vec{i}_z$, $\vec{i}_\phi \times \vec{i}_z = \vec{i}_\rho$, $\vec{i}_z \times \vec{i}_\rho = \vec{i}_\phi$.