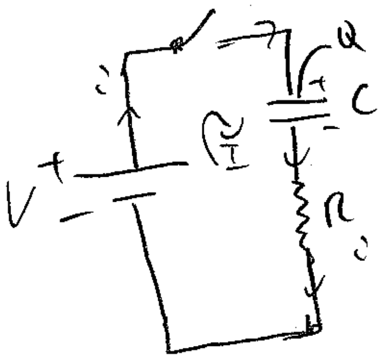
RC-Circuits

(a) DC series circuit



$Q(0) = 0$ at $t=0$ switch closed

(1) $-V + \frac{Q}{C} + Ri = 0$

$i = +\frac{dQ}{dt} = \dot{Q}$ (+, because current goes towards the positive plate)

$$\dot{Q} + \frac{1}{RC} Q = \frac{V}{R}$$

$\Rightarrow$

general solution of homog. eq.

$$\dot{Q} = -\frac{1}{RC} Q \Rightarrow \frac{dQ}{Q} = -\frac{dt}{RC}$$

$$\Rightarrow \ln\left(\frac{Q}{A}\right) = -\frac{t}{RC} \Rightarrow Q(t) = A \exp\left(-\frac{t}{RC}\right)$$

Particular solution of inh. eq.

$Q = A_1 =$ const (ansatz of type of rhs of equation)

$$\Rightarrow \frac{A_1}{RC} = \frac{V}{R} \Rightarrow A_1 = CV$$

General solution of full eq.

$$Q(t) = CV + A \exp\left(-\frac{t}{RC}\right)$$

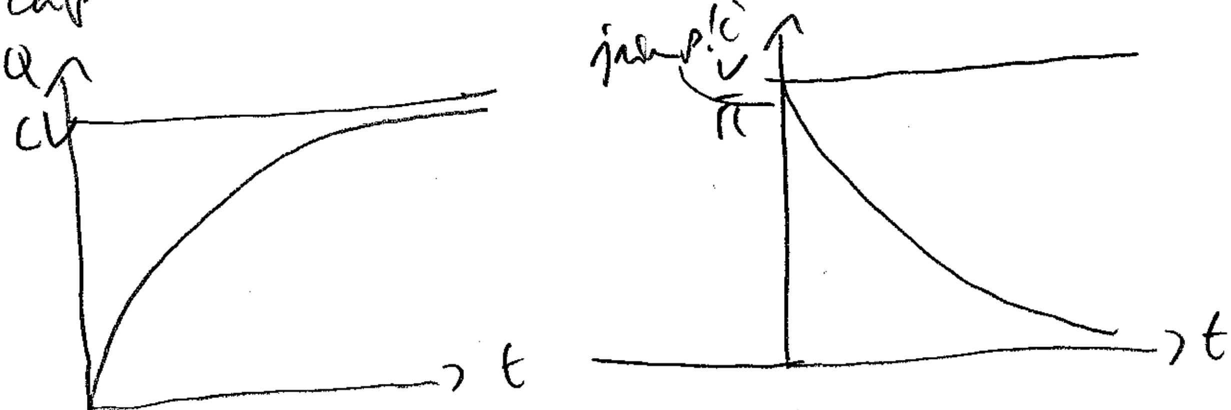
(2)

$Q(0) = 0 \Rightarrow CV + A = 0 \Rightarrow A = -CV$

$$\boxed{Q(t) = CV\left[1 - \exp\left(-\frac{t}{RC}\right)\right]}$$

$$\boxed{i(t) = +\frac{V}{R}\exp\left(-\frac{t}{RC}\right)}$$

For first moment circuit behaves as if there were no capacitor.

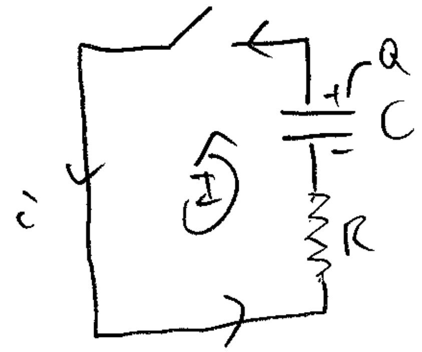


Long-time limit: goes to steady state, i.e., no current and $Q = CV$, i.e., voltage of the capacitor becomes the voltage of Battery.

Limit $R \to 0$

Since $t > 0$ here, the exponentials simply vanish, and we have immediately the steady-state limit.
The sudden jumps in Q (or the current in the case of finite resistance) is due to the negligence of the circuit's self inductance.

Discharge of a capacitor



$Q(0) = Q_0 > 0$

$\mathrm{I}: \; Ri - \frac{Q}{C} = 0$

$i = -\dot{Q}$

↑ current runs away from + plate!

$$\Rightarrow \dot{Q} + \frac{Q}{RC} = 0$$

homogeneous eq. from above

$$\Rightarrow Q(t) = A \exp\left(-\frac{t}{RC}\right)$$

Initial condition: $Q(0) = Q_0 = A$

$$Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right)$$

$$i(t) = -\dot{Q}(t) = \frac{Q_0}{RC} \exp\left(-\frac{t}{RC}\right)$$

Energy stored in capacitor dissipates as heat in resistor

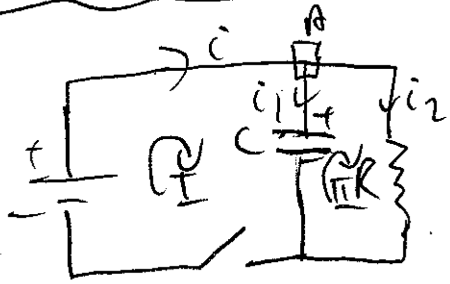
$$P = Ri^2 = \frac{Q_0^2}{RC^2} \exp\left(-\frac{2t}{RC}\right)$$

$$E = \int_0^\infty dt\, P(t) = \frac{Q_0^2}{RC^2} \int_0^\infty \exp\left(-\frac{2t}{RC}\right)$$

$$= -\frac{Q_0^2}{2C} \exp\left(-\frac{2t}{RC}\right)\Big|_0^\infty = \frac{Q_0^2}{2C}$$

$\boxed{E = \frac{Q_0^2}{2C}}$ has been stored in the electric field before we closed the switch. This we have already found out earlier in the course in a different way.

(b) DC parallel circuit



Initial conditions

$Q(0) = 0$
all currents 0
at $t = 0$

A: $i = i_1 + i_2$

(I) $-V + \frac{Q}{C} = 0$

(II) $-\frac{Q}{C} + R i_2 = 0$

$$i_1 = \dot{Q}$$

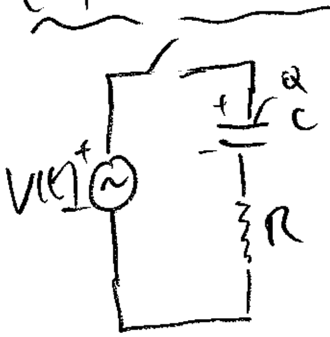
(I) + (II) $\Rightarrow -V + R i_2 = 0 \Rightarrow \boxed{i_2 = \frac{V}{R}}$

(I) $\Rightarrow \frac{Q}{C} = V = \text{const} \Rightarrow \boxed{Q = CV}$

$$i_1 = \dot{Q} = 0 \Rightarrow i = i_2 = \frac{V}{R}$$

The currents and the charges jump to their steady-state values. Reason: Negligence of R in C branch and of L in both loops.

(c) AC series



Same eq. as in (a) but with

$$V(t) = V_0 \cos(\omega t)$$

$$\Rightarrow \dot{Q} + \frac{1}{RC} Q = \frac{V_0}{R} \cos(\omega t)$$

Need only to find new particular solution of

Inhom. eq.

"Ansatz of type of right-hand side of equation"

$$Q = A_1 \cos(\omega t) + B_1 \sin(\omega t)$$

$$\dot{Q} = -A_1 \omega \sin(\omega t) + B_1 \omega \cos(\omega t)$$

$$-A_1 \omega \sin \omega t + B_1 \omega \cos(\omega t) + \frac{1}{RC}\left[A_1 \cos(\omega t) + B_1 \sin(\omega t)\right] =$$

$$= \frac{V_0}{R} \cos(\omega t)$$

$$\Rightarrow \frac{A_1}{RC} + B_1 \omega = \frac{V_0}{R}$$

$$-A_1 \omega + \frac{B_1}{RC} = 0 \Rightarrow B_1 = A_1 RC\omega$$

$$\Rightarrow \frac{A_1}{RC} + A_1 RC \omega^2 = \frac{V_0}{R}$$

$$A_1 \left(\frac{1 + R^2 C^2 \omega^2}{\cancel{R}C}\right) = \frac{V_0}{\cancel{R}}$$

$$\Rightarrow A_1 = \frac{C V_0}{1 + (RC\omega)^2}$$

$$B_1 = \frac{RC^2 \omega V_0}{1 + (RC\omega)^2}$$

Solution

$$Q(t) = A \exp\left(-\frac{t}{RC}\right) + A_1 \cos(\omega t) + B_1$$

Initial condition

$$Q(0) = A + A_1 = 0 \Rightarrow A = -A_1$$

$$Q(t) = \frac{C V_0}{1+(RC\omega)^2}\left[\cos(\omega t) - \exp\left(-\frac{t}{RC}\right)\right]$$

$$+ \frac{RC^2\omega V_0}{1+(RC\omega)^2} \sin(\omega t)$$

$$i(t) = \dot{Q}(t) = C\frac{V_0}{1+(RC\omega)^2}\left[-\omega \sin(\omega t) + \frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\right]$$

$$+ \frac{RC^2\omega^2 V_0}{1+(RC\omega)^2} \cos(\omega t)$$

Long term limit ("stationary state") $(t \gg RC)$

$$i_\infty(t) = -A_1 \omega \sin(\omega t) + B_1 \omega \cos(\omega t)$$

Amplitude and phase shift

$$i_\infty(t) = A_{i\infty} \cos(\omega t + \varphi_0)$$

$$= A_{i\infty}\left[\cos(\omega t)\cos\varphi_0 - \sin(\omega t)\sin\varphi_0\right]$$

$$\Rightarrow \begin{aligned} A_{i\infty} \cos\varphi_0 &= B_1 \omega \\ A_{i\infty} \sin\varphi_0 &= +A_1 \omega \end{aligned}$$

$$A_{i0}^2 = (A_1^2 + B_1^2)\omega^2$$

$$= \frac{\omega^2 V_0^2}{[1+(RC\omega)^2]^2}\left(C^2 + R^2 C^4 \omega^2\right)$$

$$= \frac{\omega^2 V_0^2 C^2}{[1+(RC\omega)^2]^{\cancel{2}}}\left[1 + \cancel{(RC\omega)^2}\right]$$

$$\Rightarrow \boxed{A_{i0} = \frac{\omega C}{\sqrt{1+(RC\omega)^2}} V_0}$$

$$\varphi_0 = \mathrm{sign}(\omega A_1)\arccos\left(\frac{\omega B_1}{A_{i0}}\right) = +\arccos\left(\frac{\cancel{V_0} R\,\omega^{\cancel{2}} C^{\cancel{2}}}{\cancel{\omega C}\sqrt{1+(RC\omega)^2}}\right)$$

$$\boxed{\varphi_0 = \arccos\left(\frac{RC\omega}{\sqrt{1+(RC\omega)^2}}\right)}$$

For $\omega \to 0$ we get the static limit and

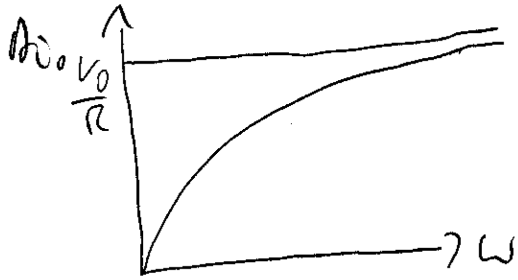
$A_{i0}(\omega = 0) = 0$ (no current through resistor)

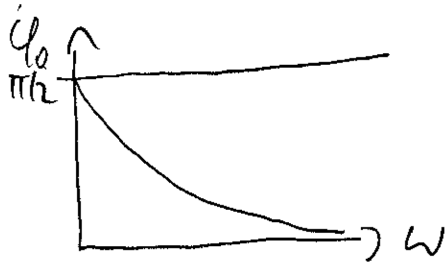
$\varphi_0 \to \frac{\pi}{2}$ (no physical meaning since current $= 0$)

For high-frequency limit $\omega \to \infty$

$A_{i0}(\omega \to \infty) = \frac{V_0}{R}$ (capacitance becomes negligible)

$\varphi_0(\omega \to \infty) = 0$ (no phase shift)





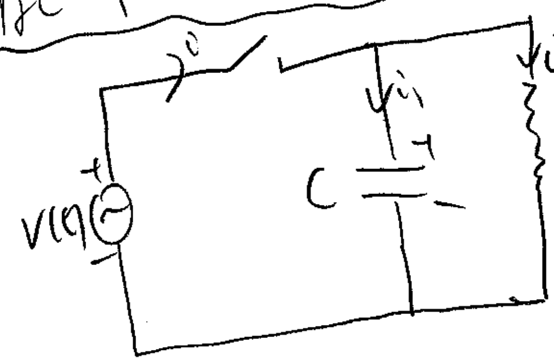
For $R \to 0$

$$A_{i\omega}|_{R\to 0} = \omega C V_0$$

$$\varphi_0 = \frac{\pi}{2}$$

φ_0 is always positive $\Rightarrow$ current advances voltage in stationary limit.

(d) RC-parallel circuit



$Q(0) = 0$
$t = 0 \Rightarrow$ all currents 0
$V(t) = V_0 \cos(\omega t)$

From (a)

(I) $Q = V_0 C \cos(\omega t)$

(I) + (II) $i_2 = \frac{V_0}{R} \cos(\omega t)$

$$i_1 = \dot{Q} = -V_0 \omega C \sin(\omega t)$$

$$i = i_1 + i_2 = V_0 \left[\frac{1}{R} \cos(\omega t) - \omega C \sin(\omega t)\right]$$

Jumps instantaneously to stationary state (same reasons as discussed in case (b)).

Current through resistor

$$A_{oi_2} = \frac{V_0}{R} \; ; \; \varphi_{oi_2} = 0$$

Current through capacitor

$$i_1 = -V_0 \omega C \sin(\omega t) = V_0 \omega C \cos(\omega t + \tfrac{\pi}{2})$$

$$\Rightarrow A_{oi_1} = V_0 \omega C$$

$$\varphi_{oi_1} = \frac{\pi}{2}$$

Total current

$$i(t) = V_0 \left[\frac{1}{R} \cos(\omega t) - \omega C \sin(\omega t) \right]$$

$$= A_{oi} \cos(\omega t + \varphi_{oi})$$

$$\Rightarrow A_{oi}^2 = \left(\frac{1}{R^2} + \omega^2 C^2 \right) V_0^2 = \frac{1 + (R\omega C)^2}{R^2} V_0^2$$

$$\Rightarrow \boxed{A_{oi} = \frac{\sqrt{1 + (RC\omega)^2}}{R} V_0}$$

$$\boxed{\varphi_{oi} = + \arccos\left(\frac{1}{R A_{oi}} \right) = \arccos\left(\frac{1}{\sqrt{1 + (RC\omega)^2}} \right)}$$

$$\omega \to 0 \Rightarrow A_{oi} \to \frac{V_0}{R}$$

$$\Rightarrow \varphi_{oi} \to 0$$

$$\omega \to \infty \Rightarrow A_{oi} \underset{\omega \to \infty}{\approx} V_0 C \omega$$

$$\Rightarrow \varphi_0 \to \frac{\pi}{2}$$

$$A_{0i} \underset{\omega\to\infty}{\cong} V_0 C \omega$$