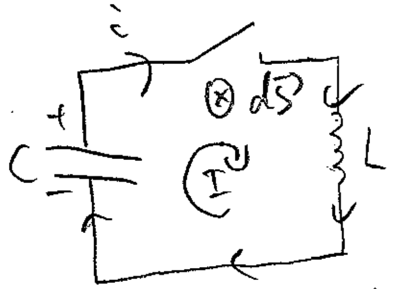
The CL circuit

(a) The simple oscillator



Suppose the capacitor is charged with a certain charge Q_0 at $t=0$, and there is no current flowing.

Now we close the switch at $t=0$.

To analyse this problem we assume a current running as indicated and use loop (I) in Faraday's law. The surface vector $d\vec{S}$ must point into the paper (RHR!). The magnetic field induced by the current also points in. Thus we have

$$\Phi_{\vec{B}} = +Li$$

$$\Rightarrow \oint_{\partial S} d\vec{r}\cdot\vec{E} = -\frac{Q}{C} = -\frac{d}{dt}(Li) = -L\frac{di}{dt}$$

Since i takes charges from the + plate of the capacitor we have:

$$i = -\frac{dQ}{dt} \Rightarrow \frac{di}{dt} = -\frac{d^2Q}{dt^2}$$

$$\Rightarrow \frac{d^2Q}{dt^2} = -\frac{1}{LC}Q := -\omega_0^2 Q$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(Thomson formula)}$$

The general solution for the 2nd-order ode is

$$Q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

The constants A and B are determined by the initial conditions:

$$Q(0) = Q_0$$
$$i(0) = 0$$

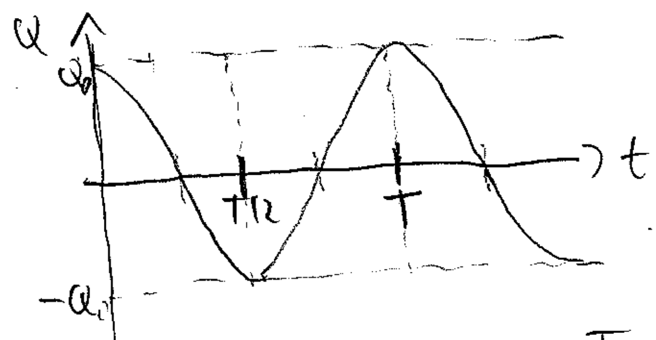
$$\Rightarrow \quad Q(0) = A = Q_0 \; ; \; i(0) = -\left.\frac{dQ}{dt}\right|_{t=0} = -B\omega_0 = 0$$

$$\Rightarrow B = 0$$

$$\Rightarrow Q(t) = Q_0\cos(\omega_0 t),$$

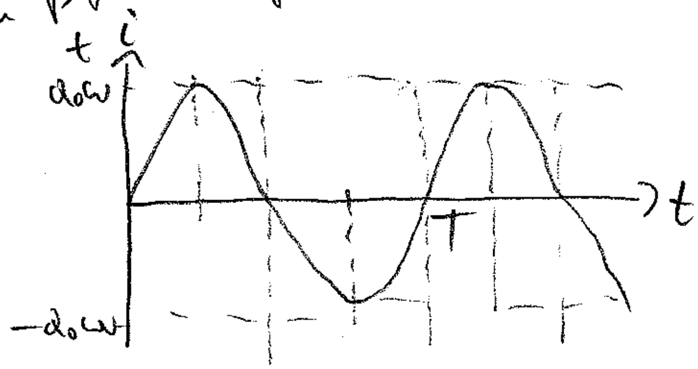
i.e. the charge oscillates back and forth. The current is

$$i(t) = -\frac{dQ}{dt} = Q_0\omega_0\sin(\omega_0 t)$$



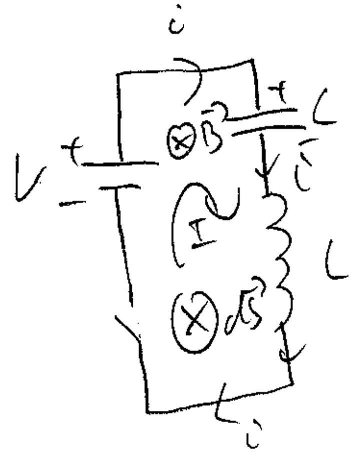
T : is the period of the oscillations

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC}$$



The current has of course the same period.

(b) Battery in series



In the same way as before we get

$$-V + \frac{Q}{C} = -L\frac{di}{dt}$$

but now $i = +\frac{dQ}{dt}$

$$\Rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = V$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{V}{L}$$

We only need a particular solution for the other equation, since we solved the homogeneous one already above. "Imitate of the right-hand side"

$$Q = A_1 = \text{const} \Rightarrow \frac{A_1}{LC} = \frac{V}{L} \Rightarrow A_1 = CV$$

The general solution thus is (with $\omega = \frac{1}{\sqrt{LC}}$ again)

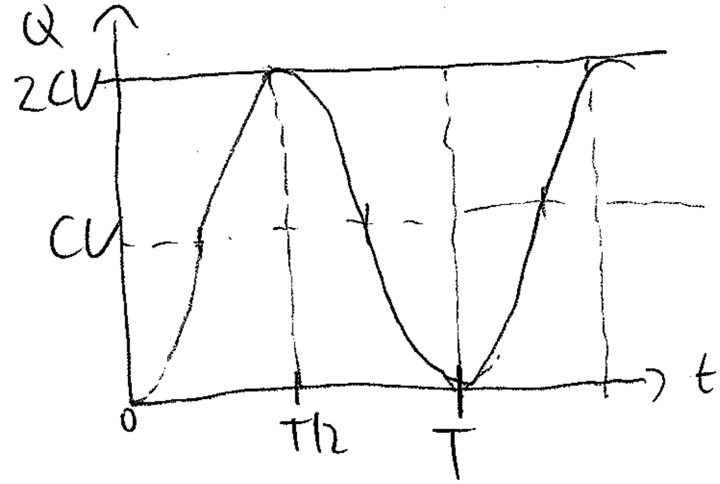
$$Q(t) = CV + A\cos(\omega t) + B\sin(\omega t)$$

Initial conditions: $t=0$: $Q(0)=0$, $i(0)=0$

$$\Rightarrow \begin{matrix} CV + A = 0 \\ B\omega = 0 \end{matrix} \Rightarrow A = -CV,\ B = 0$$

$$Q(t) = CV\,[1-\cos(\omega_0 t)]$$

So we have a peculiar behavior like this

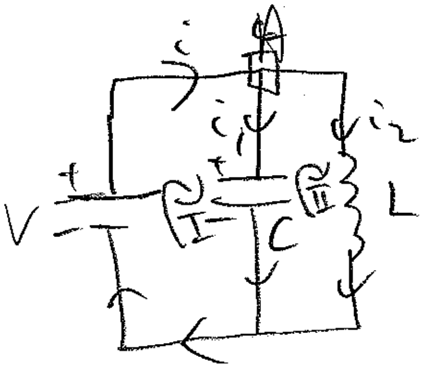


The current is

$$i(t) = \frac{dQ}{dt} = CV\omega \sin(\omega_0 t)$$

as above.
It's of course unphysical since in reality we always have a finite resistance, and the oscillations are damped away such that you end with a stable longer time limit with the capacitor "fully charged" to $Q_{t\to\infty} = CV$.

(c) Battery in parallel



initial condition

$$Q = i = i_1 = i_2 = 0$$

$$\text{I}: -V + \frac{Q}{C} = 0 \quad \text{(no L in this loop)}$$

$$\text{II}: -\frac{Q}{C} = -L\frac{di_2}{dt}$$

$$A: \ i = i_1 + i_2$$

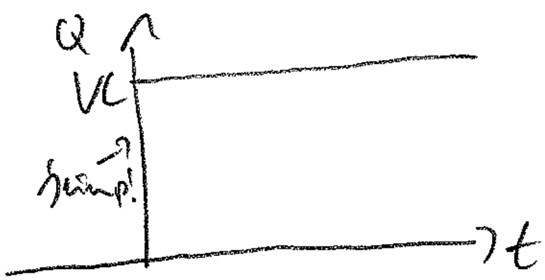
$i_1 = \frac{dQ}{dt} = 0$ due to (I)! $\Rightarrow$ $Q = \text{const.} = VC$

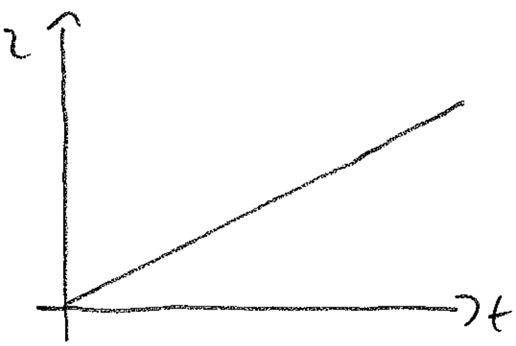
$\Rightarrow V = L\frac{di_2}{dt} \Rightarrow i_2 = \frac{V}{L}t + A$

$i_2(0) = 0 \Rightarrow i_2 = \frac{V}{L}t$

$i = i_2$

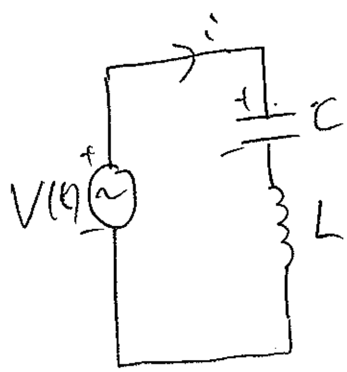
Thus we found





The jump comes from the negligence of the self-inductance of loop ①. The rise of $i_2 \to \infty$ for $t \to \infty$ can also never happen, because of finite resistance. We shall discuss the more realistic case later in this lecture.

(d) AC Voltage (series)



$V(t) = V_0 \cos(\omega t)$

We only need to plug V(t) in the equation instead of the constant V:

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{V_0}{L}\cos(\omega t),$$

and we only need a new particular solution for the inhomogeneous equation. Here it's clear that

$$Q(t) = A_2 \cos(\omega t)$$

should work:

$$\left[-A_2\omega^2 + A_2\omega_0^2\right]\cos(\omega t) = \frac{V_0}{L}\cos(\omega t)$$

$$\Rightarrow \quad \Rightarrow \quad A_2 = \frac{V_0}{L}\,\frac{1}{\omega_0^2-\omega^2}$$

This works only for $\omega \neq \omega_0$. We'll see later what to do in this case

The general solution of the full eq. is thus

$$Q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{V_0}{L}\frac{1}{\omega_0^2-\omega^2}\cos(\omega t)$$

Initial conditions

$$Q(0) = A + \frac{V_0}{L}\frac{1}{\omega_0^2-\omega^2} \overset{!}{=} 0 \Rightarrow A = -\frac{V_0}{L}\frac{1}{\omega_0^2-\omega^2}$$

$$\dot{Q}(0) = \omega_0 B = 0 \Rightarrow B = 0$$

$$Q(t) = \frac{V_0}{L}\frac{1}{\omega_0^2-\omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right]$$

The current is

$$i(t) = \frac{dQ}{dt} = -\frac{V_0}{L}\frac{1}{\omega_0^2-\omega^2}\left[\omega\sin(\omega t) - \omega_0\sin(\omega_0 t)\right]$$

The resonance case ($\omega = \omega_0$)

For the same initial conditions we can take the limit $\omega \to \omega_0$

$$Q(t) = \lim_{\omega\to\omega_0}\frac{V_0}{L}\frac{1}{\omega_0^2-\omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right]$$

de L'Hospital's formula for "$\frac{0}{0}$" limit

$$= \lim_{\omega\to\omega_0}\frac{V_0}{L}\frac{1}{+2\omega}\left[+t\sin(\omega t)\right]$$

$$Q(t) = \frac{V_0}{2\omega_0 L}t\sin(\omega_0 t)$$

$$i(t) = \dot{Q}(t) = \frac{V_0}{2\omega_0 L}\left[\omega_0 t\cos(\omega_0 t) + \sin(\omega_0 t)\right]$$

Check whether this solves the equation:

$$\ddot{Q}(t) = \frac{V_0}{2\omega_0 L}\left[2\omega_0 \cos(\omega_0 t) - \omega_0^2 t \sin(\omega_0 t)\right]$$

$$\ddot{Q} + \omega_0^2 Q = \frac{V_0}{L}\cos(\omega_0 t)$$ ☺

Both, current and charge on the capacitor oscillate with an infinitely growing amplitude ("resonance catastrophe"). In reality this never happens, because of a finite resistance

(e) LC (parallel)

$$\frac{di_2}{dt} = \frac{V_0}{L}\cos(\omega t) \Rightarrow i_2 = \frac{V_0}{\omega L}\sin(\omega t) + A$$

$$i_2(0) = 0 \Rightarrow A = 0 \Rightarrow i_2 = \frac{V_0}{\omega L}\sin(\omega t)$$

$$Q = CV_0 \cos(\omega t)$$

$$i_1 = \frac{dQ}{dt} = -V_0 \omega C \sin(\omega t)$$

$$i = i_1 + i_2 = V_0\left(\frac{1}{\omega L} - \omega C\right)\sin(\omega t)$$

$$i = \frac{V_0}{\omega L}\left(1 - \omega^2 LC\right)\sin(\omega t)$$

$$i = \frac{V_0}{\omega}\left(1 - \frac{\omega^2}{\omega_0^2}\right)\sin(\omega t)$$

For $\omega \to \omega_0$ we get

$$\delta = 0$$

$$i_2 = -i_1 = \frac{V_0}{\omega_0 L} \sin(\omega_0 t)$$

$$Q = CV_0 \cos(\omega_0 t)$$