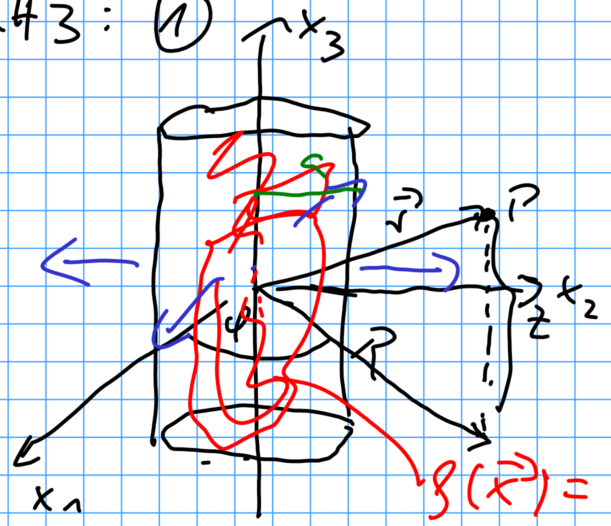
Blatt 3: (1)



$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

$R \in [0,a]$; $\varphi \in [0, 2\pi)$; $z \in \mathbb{R}$

$$\varrho(\vec{x}) = \begin{cases} \varrho & \text{in Zylinder} \\ 0 & \text{außen} \end{cases}$$

(a) Symmetrie: Zylindersymmetrie: Translationssymmetrie bzgl. z-Achse + Drehsymmetrie um z-Achse

$\Rightarrow \quad \Phi(\vec{r}) = V(R)$

(b) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\varrho$; $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\Phi = -\vec{\nabla}V(R) \overset{\text{Anl. A.}}{=} -\vec{e}_R\,\partial_R V(R)$

$= E_R\vec{e}_R \; (+\cancel{E_\varphi \vec{e}_\varphi + E_z \vec{e}_z}^{\,0})$ wegen Symm.

$$\Delta\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi = -\frac{\varrho}{\varepsilon_0}$$

$$\Delta V(R) = \frac{1}{R}\frac{d}{dR}\left(R\frac{dV}{dR}\right) = -\frac{\varrho}{\varepsilon_0}$$

$\underline{R > a} \Rightarrow \varrho = 0$

$$\Rightarrow \frac{d}{dR}\left(R\frac{dV_>}{dR}\right) = 0 \Rightarrow R\frac{dV_>}{dR} = C_1$$

$$\frac{dV_>}{dR} = \frac{C_1}{R} \Rightarrow \boxed{V_>(R) = C_1 \ln\left(\frac{R}{a}\right) + C_2}$$

$\underline{R < a} \Rightarrow \frac{1}{R}\frac{d}{dR}\left(R\frac{dV_<}{dR}\right) = -\frac{\varrho}{\varepsilon_0} = \text{const.}$

$$\Rightarrow \frac{d}{dR}\left(R\frac{dV_<}{dR}\right) = -\frac{\varrho}{\varepsilon_0}R$$

$$R\frac{dV_<}{dR} = -\frac{\varrho}{2\varepsilon_0}R^2 + C_1'$$

$$\frac{dV_<}{dR} = -\frac{\rho}{2\varepsilon_0} R + \frac{c_1'}{R} \Rightarrow V_< = \frac{-\rho}{4\varepsilon_0} R^2 + C_1' \ln\left(\frac{R}{a}\right) + c_2'$$

Randbed. bei $R = a$

$$V_<(a) = V_>(a) \; ; \; E_R = -\frac{dV}{dR} \text{ stetig } E_{R<}(a) = E_{R>}(a)$$

$$V_<(R) = -\frac{\rho}{4\varepsilon_0} R^2 + C_1' \cancel{\ln\left(\frac{R}{a}\right)} + c_2' \quad (0 < R < a)$$

$$\boxed{V_>(R) = C_1 \ln\left(\frac{R}{a}\right) + C_2} \quad (R > a)$$

keine Singularität bei $R = 0 \Rightarrow C_1' = 0$

$$V_<(a) = -\frac{\rho}{4\varepsilon_0} a^2 + C_2' \overset{=V_>(a)}{=} C_2 \Rightarrow C_2' = C_2 + \frac{\rho}{4\varepsilon_0} a^2$$

$$\Rightarrow \boxed{V_<(R) = -\frac{\rho}{4\varepsilon_0}(R^2 - a^2) + C_2}$$

$$E_{R<} = -\vec{\nabla} V_< = -V_<'(R) = \frac{\rho}{2\varepsilon_0} R$$

$$E_{R>} = -V_>'(R) = -\frac{C_1}{R}$$

$$E_{R<}(a) = E_{R>}(a)$$

$$\Rightarrow \frac{\rho}{2\varepsilon_0} a = -\frac{C_1}{a} \Rightarrow C_1 = -\frac{\rho a^2}{2\varepsilon_0}$$

$$\Rightarrow V_< = -\frac{\rho}{4\varepsilon_0}(R^2 - a^2) \cancel{+ C_2}$$

$$V_> = -\frac{\rho a^2}{2\varepsilon_0} \ln\left(\frac{R}{a}\right) \cancel{+ C_2}$$

$$\Rightarrow E_{R<} = -V_<' = \frac{\rho}{2\varepsilon_0} R \; \Big\}$$

$$E_{R>} = -V_>' = \frac{\rho a^2}{2\varepsilon_0 R} \; \Big\}$$

(c) $\vec{\nabla} \cdot \vec{E} = \frac{1}{R} \frac{\partial (R E_R)}{\partial R}$

$$\vec{\nabla}\cdot\vec{E}_< = \frac{1}{R}\frac{d}{dR}\left(\frac{\rho}{2\varepsilon_0}R^2\right) = \frac{1}{\cancel{R}}\left(\frac{\rho}{\varepsilon_0}\cancel{R}\right) = \frac{\rho}{\varepsilon_0} \checkmark$$

$$\vec{\nabla}\cdot\vec{E}_> = \frac{1}{R}\frac{d}{dR}\left(\frac{\rho a^2}{2\varepsilon_0}\right) = 0 \checkmark$$

(d) Gauß in Integralform

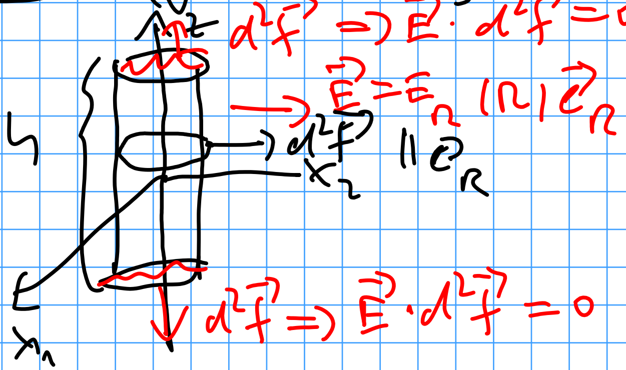
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \int_V d^3x\, \vec{\nabla}\cdot\vec{E} = \int_{\partial V} d^2\vec{f}\cdot\vec{E} = \frac{Q_V}{\varepsilon_0}$$

Symmetrie: $\vec{E} = E_R(R)\,\vec{e}_R$

V: Zylinder mit Radius R; $z \in [0,h]$

$\underline{R<a}$ $\quad Q_V \overset{\rho = \text{const.}}{=} \rho V = \pi R^2 h \rho$

$\underline{R>a}$: $\quad Q_V = \pi a^2 h \rho \checkmark$



Zyl.-Mantel $R = \text{const}$

$$\vec{x} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix};\ \varphi \in [0,2\pi];\ z \in [0,h]$$

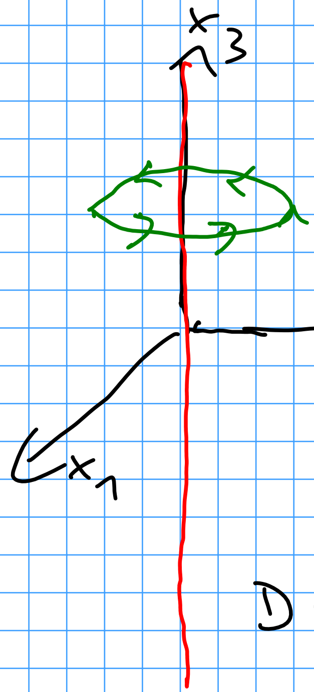
$$d^2\vec{f} = d\varphi\, dz\, \frac{\partial\vec{x}}{\partial\varphi} \times \frac{\partial\vec{x}}{\partial z} = d\varphi\, dz \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= d\varphi\, dz \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ 0 \end{pmatrix} = d\varphi\, dz\, R\,\vec{e}_R$$

$$\frac{Q_V}{\varepsilon_0} = \int_V d^2\vec{f}\cdot\vec{E} = R E_R(R) \int_0^{2\pi} d\varphi \int_0^h dz\, 1 = 2\pi R h E_R$$

$$E_R = \frac{Q_V}{2\pi\varepsilon_0 R h} = \begin{cases} \frac{\rho}{2\varepsilon_0}R & \text{für } R<a \\ \frac{\rho a^2}{2\varepsilon_0 R} & \text{für } R>a \end{cases}$$

Aufgabe 2.



$\vec\nabla \times \vec E = 0 \Rightarrow \vec E = -\vec\nabla V$

für einfach zsh. Gebiete

$\vec V = \frac{\vec e_\varphi}{R}$

$\vec V = \vec e_3 \times \frac{\vec r}{x_1^2 + x_2^2}$

$D \in \mathbb{R}^3 / x_3\text{-Achse} \quad (x_1^2 + x_2^2 = R^2 = 0)$

(a) $x_1^2 + x_2^2 \neq 0$

$$\vec V(\vec x) = \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

$$\vec\nabla \times \vec V = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial_1 \left(\frac{x_1}{x_1^2 + x_2^2} \right) + \partial_2 \frac{x_2}{x_1^2 + x_2^2} \end{pmatrix}$$

$$\Rightarrow \frac{2}{x_1^2 + x_2^2} + (-1) \left(\frac{2x_1^2 + 2x_2^2}{(x_1^2 + x_2^2)^2} \right) = \frac{2}{x_1^2 + x_2^2} - \frac{2}{x_1^2 + x_2^2} = 0$$

$$\vec\nabla \cdot \vec E = \partial_1 E_1 + \partial_2 E_2 = \cdots = 0$$

(b) Zylinderkoord.

$$\vec x = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}, \quad \frac{\partial \vec x}{\partial R} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \vec e_R$$

$$\frac{\partial \vec x}{\partial \varphi} = R \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = R\,\vec e_\varphi \quad ; \quad \vec e_R \cdot \vec e_\varphi = 0$$

$$\frac{\partial \vec x}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec e_z \quad \Rightarrow$$

$$\vec V = \frac{1}{R^2}\begin{pmatrix}-R\sin\varphi\\ R\cos\varphi\\ 0\end{pmatrix} = \frac{1}{R}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = \frac{1}{R}\vec e_\varphi$$

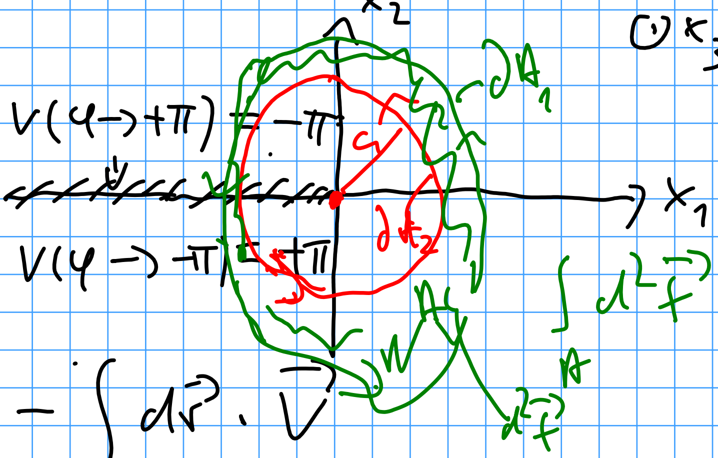
$$\vec V = V_\varphi \vec e_\varphi = \frac{1}{R}\vec e_\varphi \; ; \quad V_R = V_z = 0 , \quad V_\varphi = \frac{1}{R}$$

$$\vec\nabla\cdot\vec V = 0 \; ; \quad \vec\nabla\times\vec V = 0 \quad \text{(Aufg. A)}$$

(c) Potential:

$$V = -\vec\nabla\Phi(\varphi) = -\vec e_\varphi \frac{1}{R} \qquad \Phi'(\varphi) \overset{!}{=} \frac{1}{R}\vec e_\varphi$$

$$\Rightarrow \Phi'(\varphi) = -1 \Rightarrow \Phi(\varphi) = -\varphi$$



$\varphi \in (-\pi, \pi)$

$$\int_A d^2\vec f\cdot\underbrace{\vec\nabla\times\vec V}_{0} = 0 = \int_{\partial A_1} d\vec r\cdot\vec V - \int_{\partial A_2} d\vec r\cdot\vec V = -''- \; - 2\pi$$

$$V(\vec r) = -\int_{C_{\vec r_0\to\vec r}} d\vec r\cdot\vec V$$

Integral über bel. Kreis von Radius a: K_a

$$\vec r(\varphi) = \begin{pmatrix}a\cos\varphi\\ a\sin\varphi\\ 0\end{pmatrix} \Rightarrow d\vec r = d\varphi\frac{\partial\vec r}{\partial\varphi} = d\varphi\, a\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = d\varphi\, a\,\vec e_\varphi$$

$$\oint_{K_a} d\vec r\cdot\vec V = \int_{-\pi}^{\pi} d\varphi\,\frac{1}{\cancel{a}}\vec e_\varphi\cdot\cancel{a}\,\vec e_\varphi = 2\pi = -\Big[\underset{-\pi}{V(\pi)} - \underset{\pi}{V(-\pi)}\Big] = 2\pi$$

$$\int_A d^2\vec f\cdot(\vec\nabla\times\vec V) \overset{\text{Stokes}}{=} \int_{\partial A} d\vec r\cdot\vec V$$

Für bel. geschl. Kurve C, die Fläche begrenzt, die von z-Achse geschnitten wird $\oint_C d\vec r\cdot\vec V = 2\pi \Rightarrow$ topologische Singularität