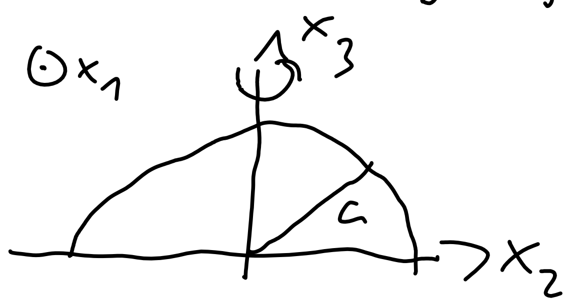
Aufg. 2

Volumen einer Halbkugel



kartesische: $x_1^2+x_2^2+x_3^2=a^2$

(a) $x_3 = +\sqrt{a^2-x_1^2-x_2^2}$; $x_1^2+x_2^2 \le a^2$

(b)
$$V = \int_{-a}^{a} dx_1 \int_{-\sqrt{a^2-x_1^2}}^{\sqrt{a^2-x_1^2}} dx_2 \sqrt{\underbrace{a^2-x_1^2}_{b^2}-x_2^2}$$

$$= \int_{-a}^{a} dx_1 \, \frac{\pi}{2}(a^2-x_1^2) = \pi \int_0 dx_1 (a^2-x_1^2)$$

$$= \pi \left[a^2 x_1 - \frac{1}{3}x_1^3\right]_0^a = \frac{2}{3}\pi a^3$$

(c) Kreis: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ 0 \end{pmatrix}$; $R \in [0,a]$, $\varphi \in [0, 2\pi]$

$$d^2\vec{f} = dR\, d\varphi\, \partial_R \vec{x} \times \partial_\varphi \vec{x}$$

$$= dR\, d\varphi\, R \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$= dR\, d\varphi\, R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow d^2 f = |d^2\vec{f}| = dR\, d\varphi\, R$$

$$V = \int_{K_a} d^2 f \sqrt{a^2 - R^2} = \int_0^a dR \int_0^{2\pi} d\varphi\, R \sqrt{a^2 - R^2}$$

$$= \underline{2\pi} \int_0^a \underline{dR}\ \underline{R}\underbrace{\sqrt{a^2 - R^2}}_{u}$$

substt.: $u = a^2 - R^2 \Rightarrow du = -2R\,dR$

$$V = -\pi \int_{a^2}^{0} du \sqrt{u} = \pi \int_0^{a^2} du \sqrt{u} = \frac{2}{3} \pi u^{3/2} \Big|_0^{a^2}$$

$$= \frac{2}{3} \pi a^3$$

(d) $\vec{r} = r \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$; $d^3 r = dr\, d\vartheta\, d\varphi\, r^2 \sin\vartheta$

$(r = |\vec{r}|)$

$$V = \int_0^a dr \int_0^{\pi/2} d\vartheta \int_0^{2\pi} d\varphi\, 1\, r^2 \sin\vartheta$$

$$= 2\pi \int_0^a dr \int_0^{\pi/2} d\vartheta\, r^2 \sin\vartheta$$

$$= 2\pi \int_0^a dr\, r^2 \left[-\cos\vartheta\right]_{\vartheta=0}^{\pi/2}$$

$$= 2\pi \int_0^a dr\, r^2 = 2\pi \left[\frac{1}{3} r^3\right]_{r=0}^{a} = \underline{\underline{\frac{2}{3}\pi a^3}}$$