S-MATRIX APPROACH TO THE THERMODYNAMICS OF HADRONS

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S-MATRIX APPROACH

R. Dashen, S. K. Ma and H. J. Bernstein,
Phys. Rev. 187 (1969) 345.

HADRON RESONANCE GAS MODEL

- Confinement

physical quantities → hadronic states representation

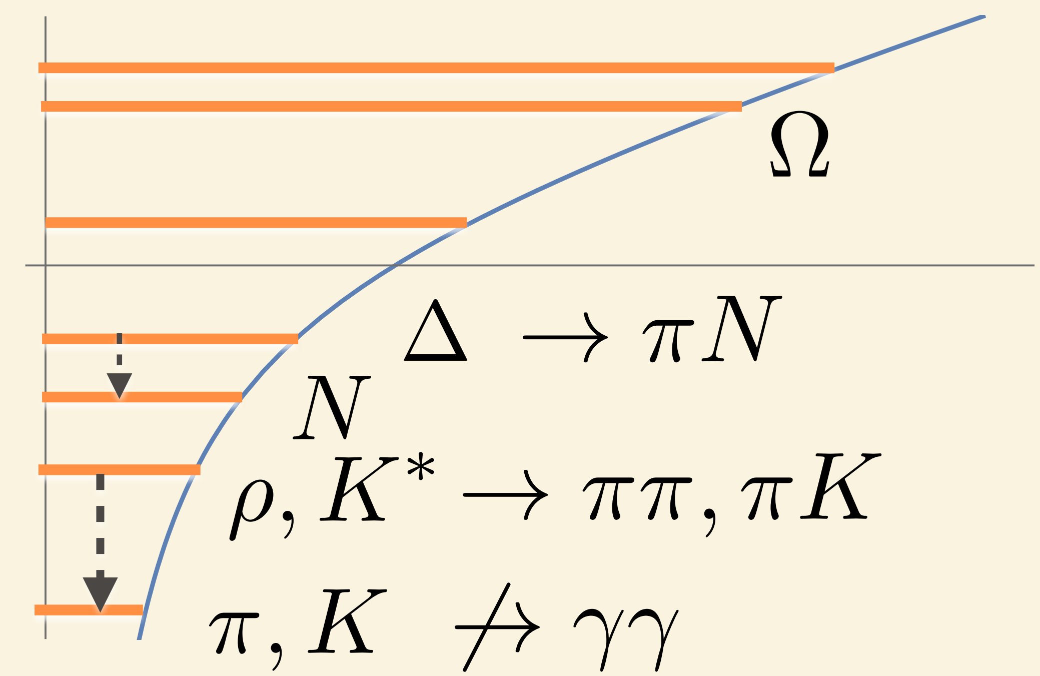
$$Z = \sum_{\alpha = B, M} \langle \alpha | e^{-\beta H} | \alpha \rangle$$

HADRON RESON MODEL

- Confinement

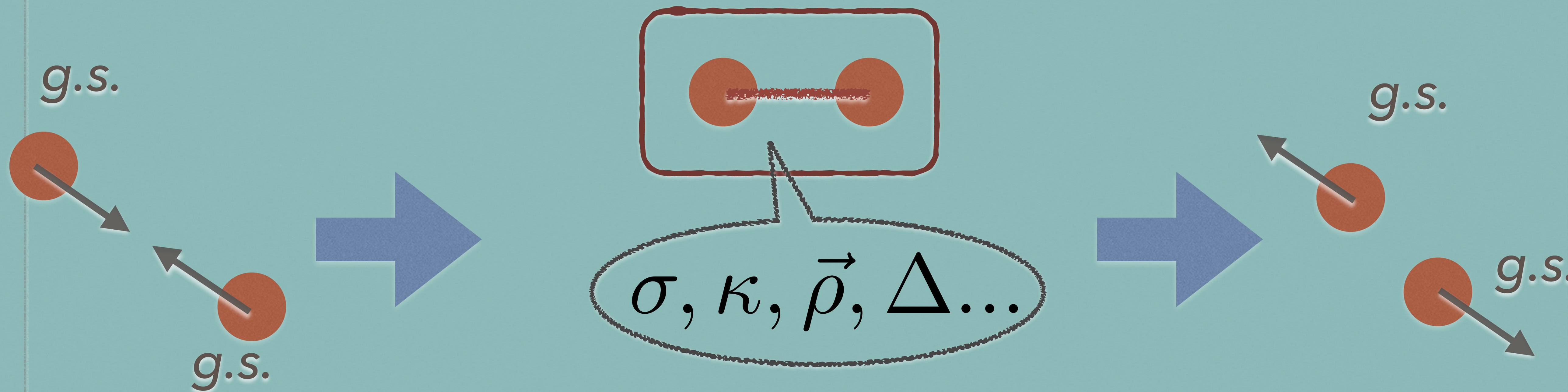
physical quantities

QCD spectrum



$$Z = \sum_{\alpha = B, M} \langle \alpha | e^{-\beta H} | \alpha \rangle$$

S-MATRIX APPROACH

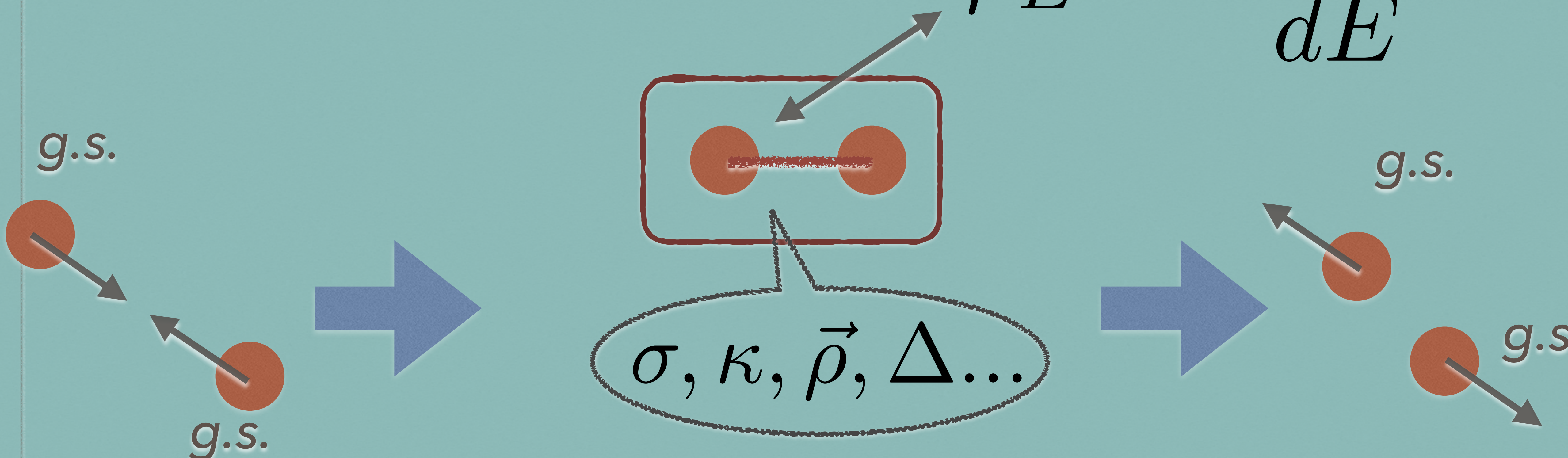


+ *repulsive forces*

consistent treatment of both
attractive and repulsive forces

S-MATRIX APPROACH

$$\rho_E \sim 2\frac{d\delta}{dE}$$



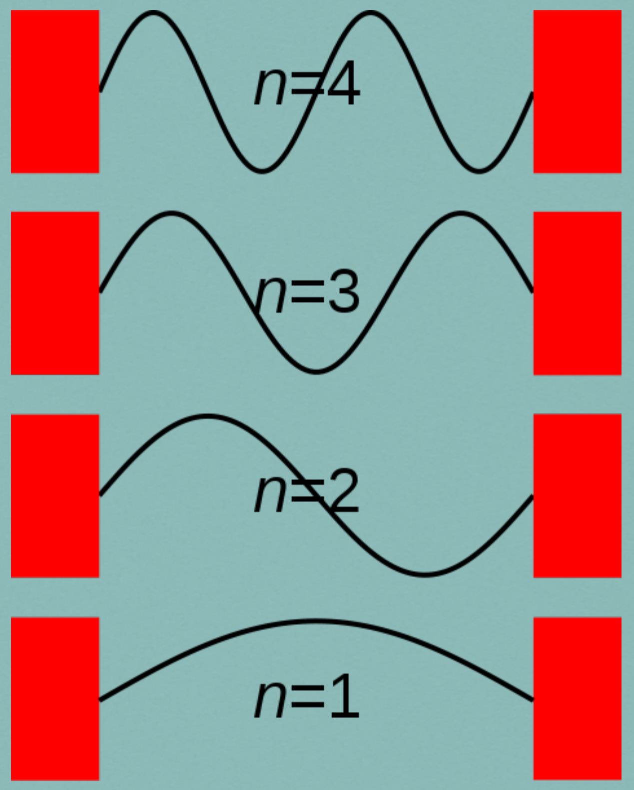
+ repulsive forces

consistent treatment of both
attractive and repulsive forces

PHASE SHIFT AND DENSITY OF STATES

particle in a box

$$\psi \sim \sin(k^{(0)}x) \qquad k^{(0)} = \frac{n\pi}{L}$$



PHASE SHIFT AND DENSITY OF STATES

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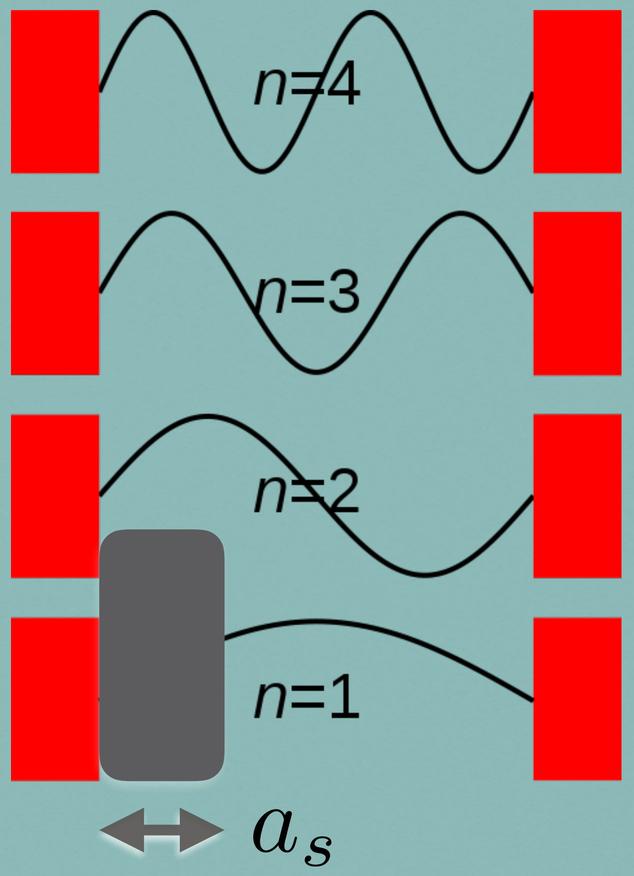


in the presence of a scattering potential

$$\psi \sim \sin(kx + \delta(k))$$

density of states

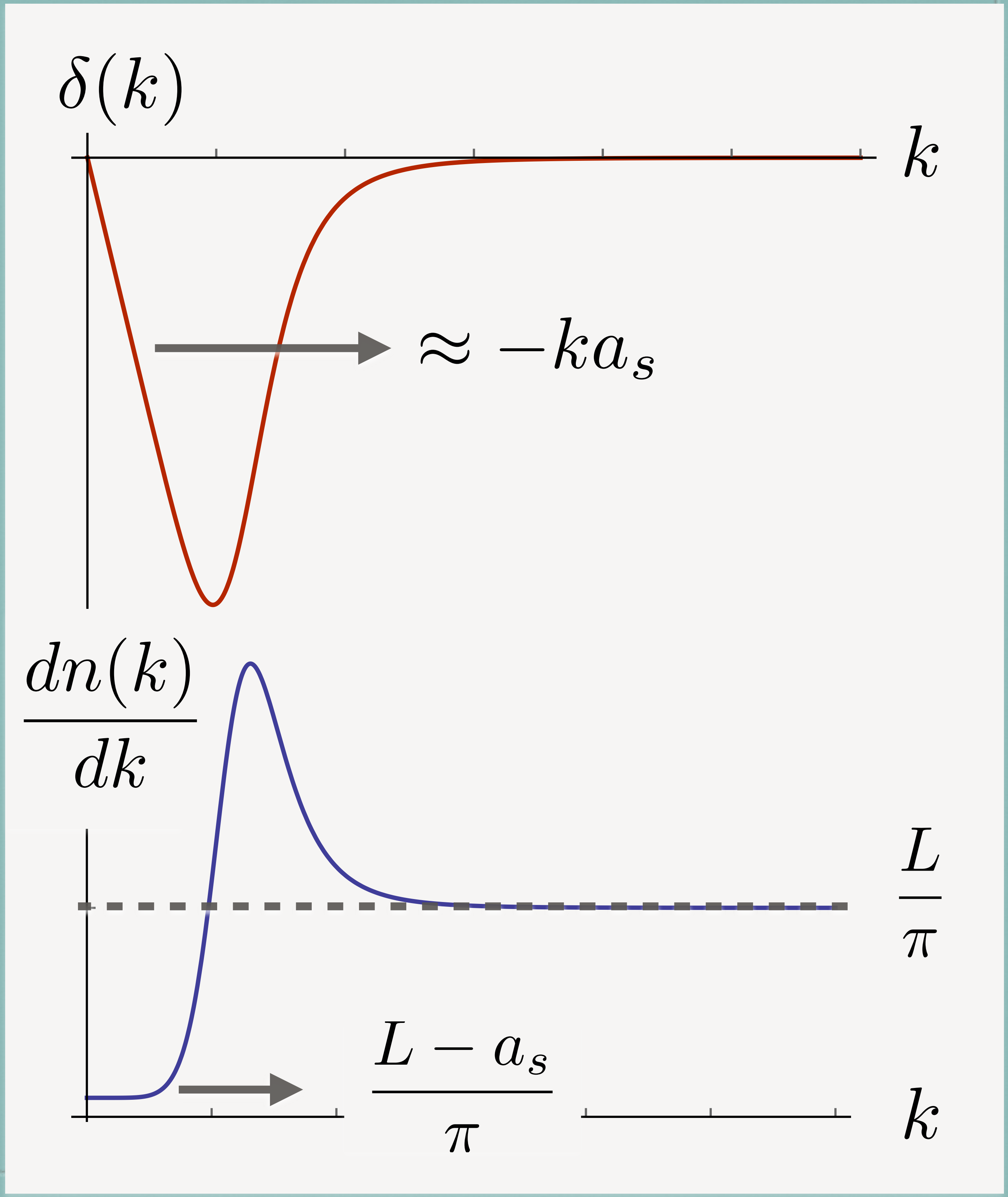
$$kL + \delta(k) = n\pi \quad \Rightarrow \quad \frac{dn(k)}{dk} = \frac{L}{\pi} + \frac{1}{\pi}\frac{d\delta}{dk}$$

PHASE SHIFT AND DENSITY OF STATES

$$\frac{dn(k)}{dk} = \frac{L}{\pi} + \frac{1}{\pi}\frac{d\delta}{dk}$$

change in d.o.s. due to int.

Effect of repulsive interaction: pushing states from low k to high k



phase shift and d.o.s. (schematics)

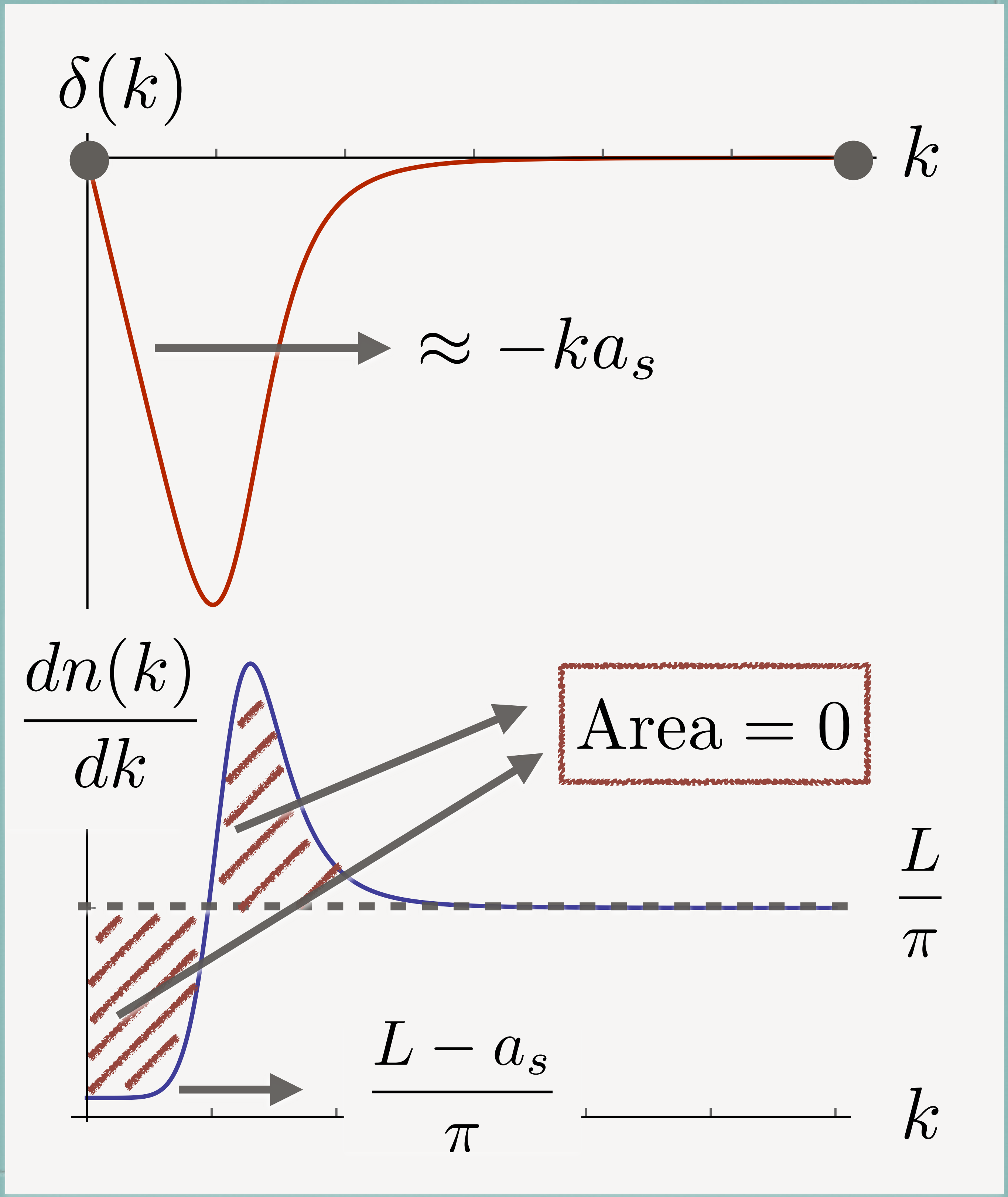
PHASE SHIFT AND DENSITY OF STATES

$$\frac{dn(k)}{dk} = \frac{L}{\pi} + \frac{1}{\pi}\frac{d\delta}{dk}$$

sum rule
(Levinson's theorem)

$$\int_0^\infty dk\,\frac{1}{\pi}\delta' = \frac{\delta(\infty)-\delta(0)}{\pi}$$

n_{int}



phase shift and d.o.s. (schematics)

S-MATRIX FORMULATION OF THERMODYNAMICS

$$\Delta \ln Z = \int dE\, e^{-\beta E} \frac{1}{4\pi i} \operatorname{tr} \left\{ S_E^{-1} \overleftrightarrow{\frac{\partial}{\partial E}} S_E \right\}_c$$

R. Dashen, S. K. Ma and H. J. Bernstein,
Phys. Rev. 187 (1969) 345.

A SIMPLE TRICK

$$S_E = e^{2i\delta_E}$$

$$\frac{1}{4\pi i}\,\mathrm{tr}\left\{S_E^{-1}\frac{\overleftrightarrow{\partial}}{\partial E}S_E\right\}_c$$

$$= \frac{1}{2\pi} \times 2\,\frac{\partial}{\partial E}\,\frac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\left\{\ln S_E\right\}$$

$$\Delta \ln Z = \int dE\, e^{-\beta E} \times \frac{1}{\pi}\frac{\partial}{\partial E}\mathrm{tr}\left(\delta_E\right).$$

E. Beth and G. Uhlenbeck,
Physica (Amsterdam) 4, 915 (1937).

A SIMPLE TRICK

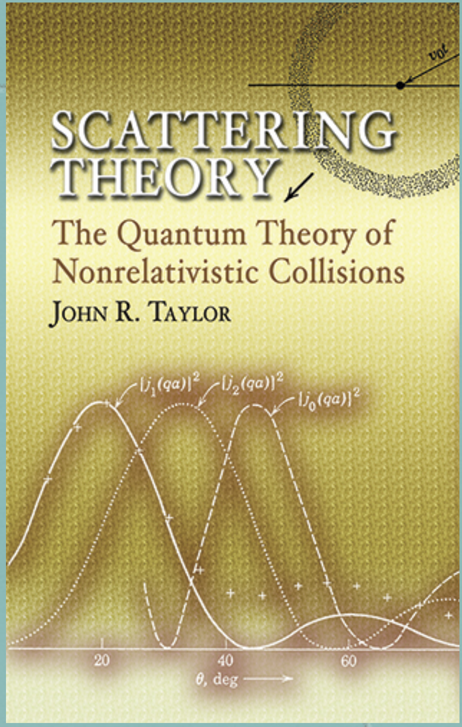
$$\frac{1}{4\pi i}\,\mathrm{tr}\left\{S_E^{-1}\frac{\overleftrightarrow{\partial}}{\partial E}S_E\right\}_c \qquad S_E = e^{2i\delta_E}$$

$$= \frac{1}{2\pi}\times 2\,\frac{\partial}{\partial E}\,\frac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\,\{\ln S_E\}$$

$\mathcal{Q}(E)$ ⟶ *Generalised phase shift*

$B = 2\frac{\partial}{\partial E}\mathcal{Q}(E)$ ⟶ *Generalised spectral function*

EXERCISE: QM SCATTERING OPERATOR



show that

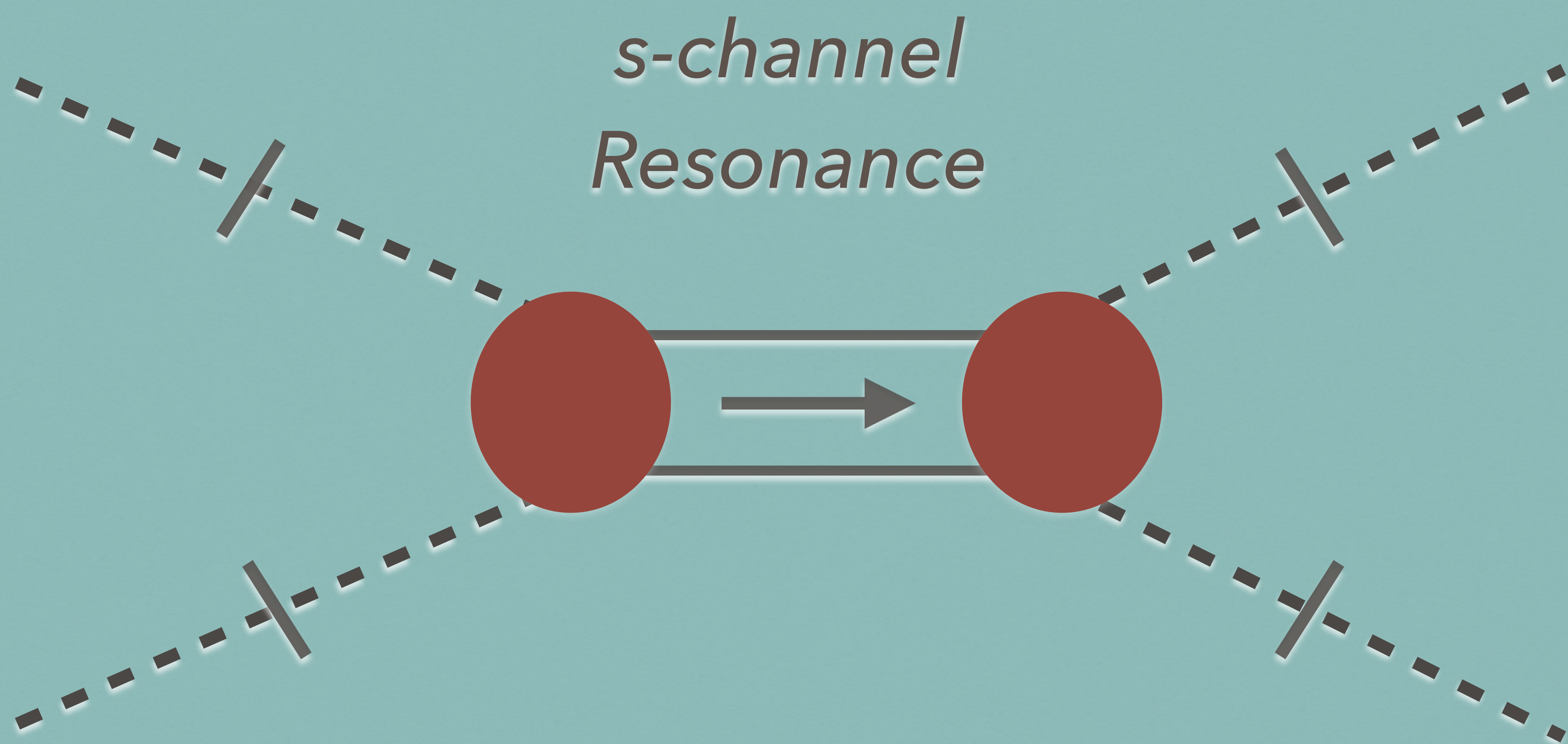
$$S_E = G_0^* \, G^{*\,-1} \, G \, G_0^{-1}$$
$$= 1 - 2\pi i \times \delta(E - H_0) \times T_E$$

$$G = \frac{1}{E - H + i\epsilon}$$

Verify $$\Delta \ln Z = \int dE \, e^{-\beta E} \frac{1}{4\pi i} \operatorname{tr} \left\{ S_E^{-1} \overleftrightarrow{\frac{\partial}{\partial E}} S_E \right\}_c$$

Alternative way to obtain the Beth-Uhlenbeck result!

ILLUSTRATION: S-MATRIX FOR RELATIVISTIC RESONANCES



$$i\mathcal{M}_E \approx (-ig)\frac{i}{E^2 - m_{\rm res}^2 + iE\,\gamma_E}(-ig)$$

$$\mathcal{Q}(E) = \frac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\,\{\ln S_E\}$$
$$= \frac{1}{2}\,\mathrm{Im}\;\ln\left[1 + \int d\phi_2\, i\mathcal{M}_E\right]$$

$$\int d\phi_2\, i\mathcal{M}_E = \frac{-i\,2\,E\,\gamma_E}{E^2 - m_{\rm res}^2 + iE\,\gamma_E}$$

$$= 2\,i\,\sin\delta_E\, e^{i\delta_E}$$

then…

$$\mathcal{Q}(M) = \frac{1}{2}\,\mathrm{Im}\,\left[\ln\left(1 + 2\,i\sin\delta_E e^{i\delta_E}\right)\right]$$

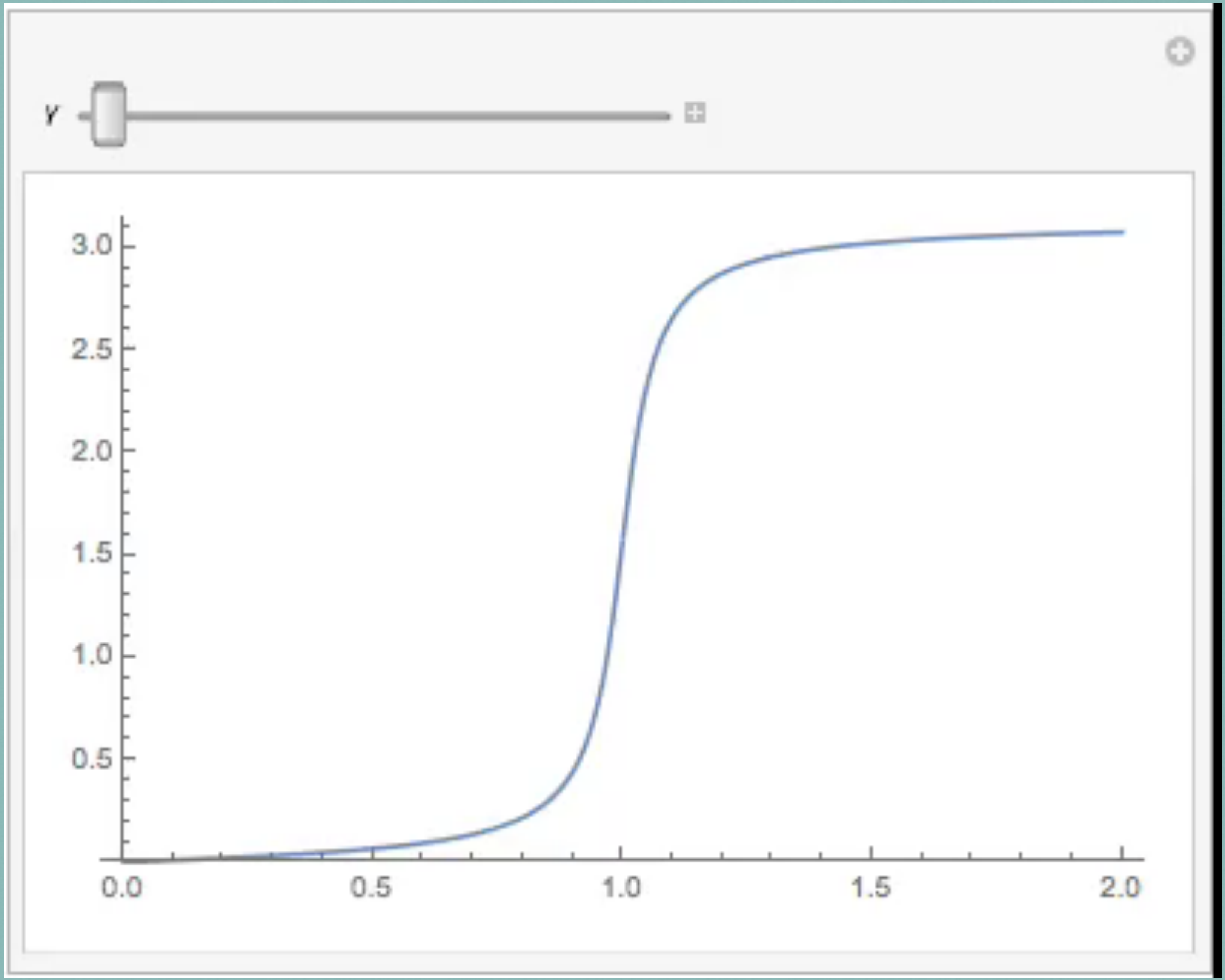
$$= \frac{1}{2}\,\mathrm{Im}\,\ln e^{2i\delta_E}$$

$$= \delta_E$$

with

$$\delta_E = \tan^{-1}\frac{-E\gamma_E}{E^2 - m_{\rm res}^2}$$

$$\delta_E = \tan^{-1} \frac{-E\gamma_E}{E^2 - m_{\mathrm{res}}^2}$$



HRG approx.

$$\delta_E = \pi \times \theta(E - m_{\mathrm{res}})$$

FORMULATION

given the exact phase shift $\delta_j(M)$

from theory
or
from experiment

thermodynamics

$$B_j = 2\frac{d}{dM}\delta_j$$

eff. spectral function

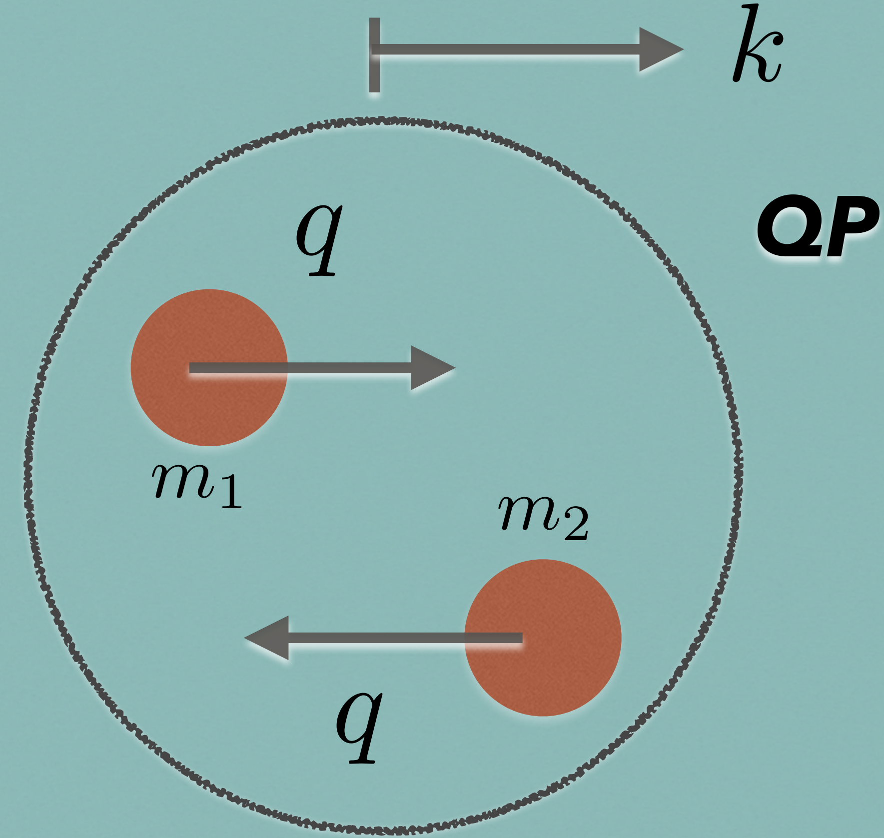
$$P = P^{(0)} + \Delta P^{\mathrm{B.U.}}$$

free gas + interaction

FORMULATION

dynamical **statistical (thermal weight)**

$$\Delta P^{\mathrm{B.U.}} = (2j+1) \int \frac{dM}{2\pi} B_j(M) \int \frac{d^3k}{(2\pi)^3} T \ln\left(1 + e^{-\beta E(k,q,m_i)}\right)$$



$$E = \sqrt{k^2 + M^2}$$

$$q = \frac{M}{2}\sqrt{(1 - \frac{(m_1+m_2)^2}{M^2})\,(1 - \frac{(m_1-m_2)^2}{M^2})}$$

$$B_j = 2\frac{d}{dM}\delta_j$$

$$M(q) = \sqrt{q^2 + m_1^2} + \sqrt{q^2 + m_2^2}$$

WHAT'S IN A NAME? THAT WHICH WE CALL A RESONANCES?

- A resonance is MORE than a MASS and a **WIDTH**

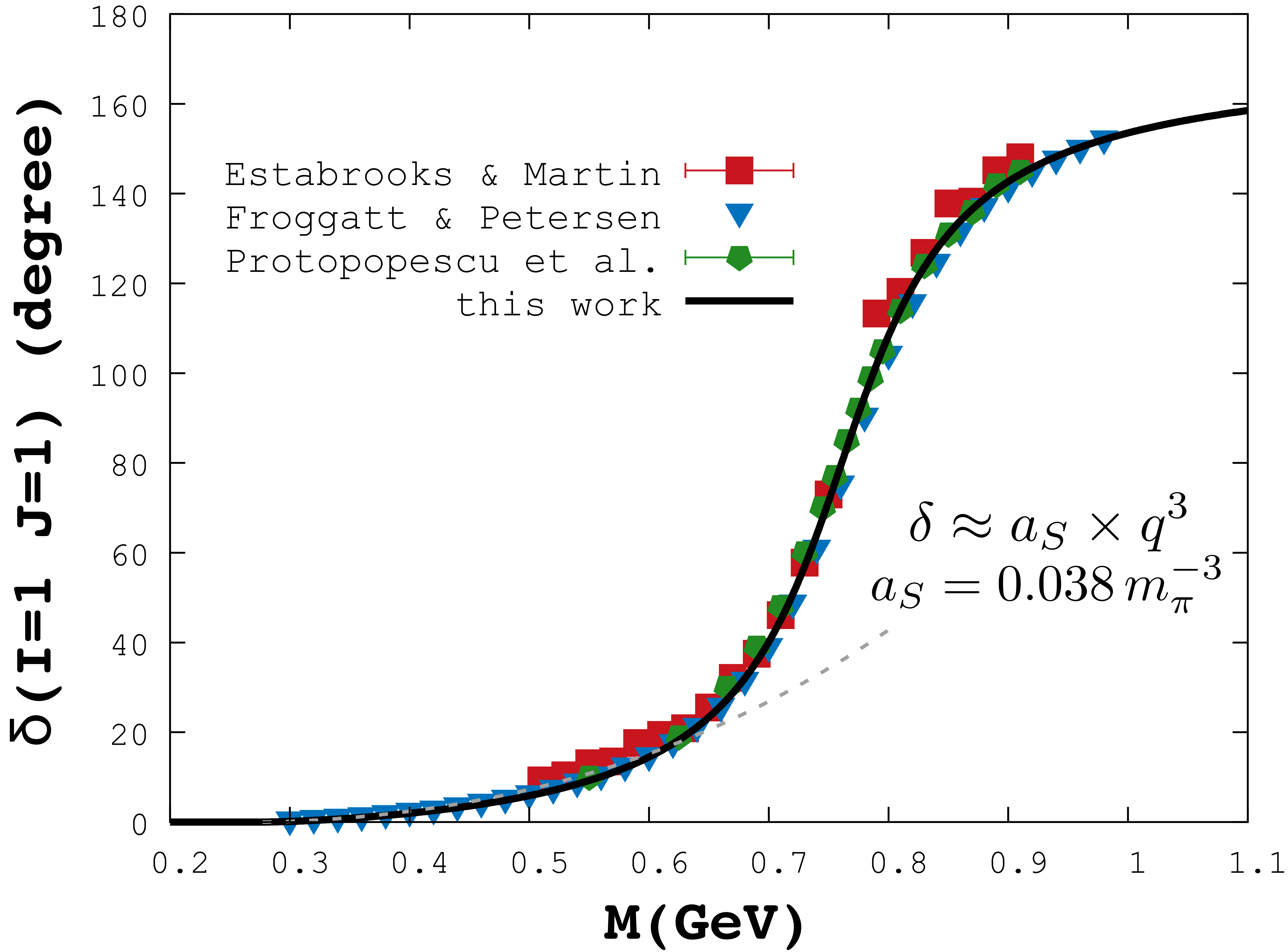
ρ(770) [h]

$I^G(J^{PC}) = 1^+(1^{--})$

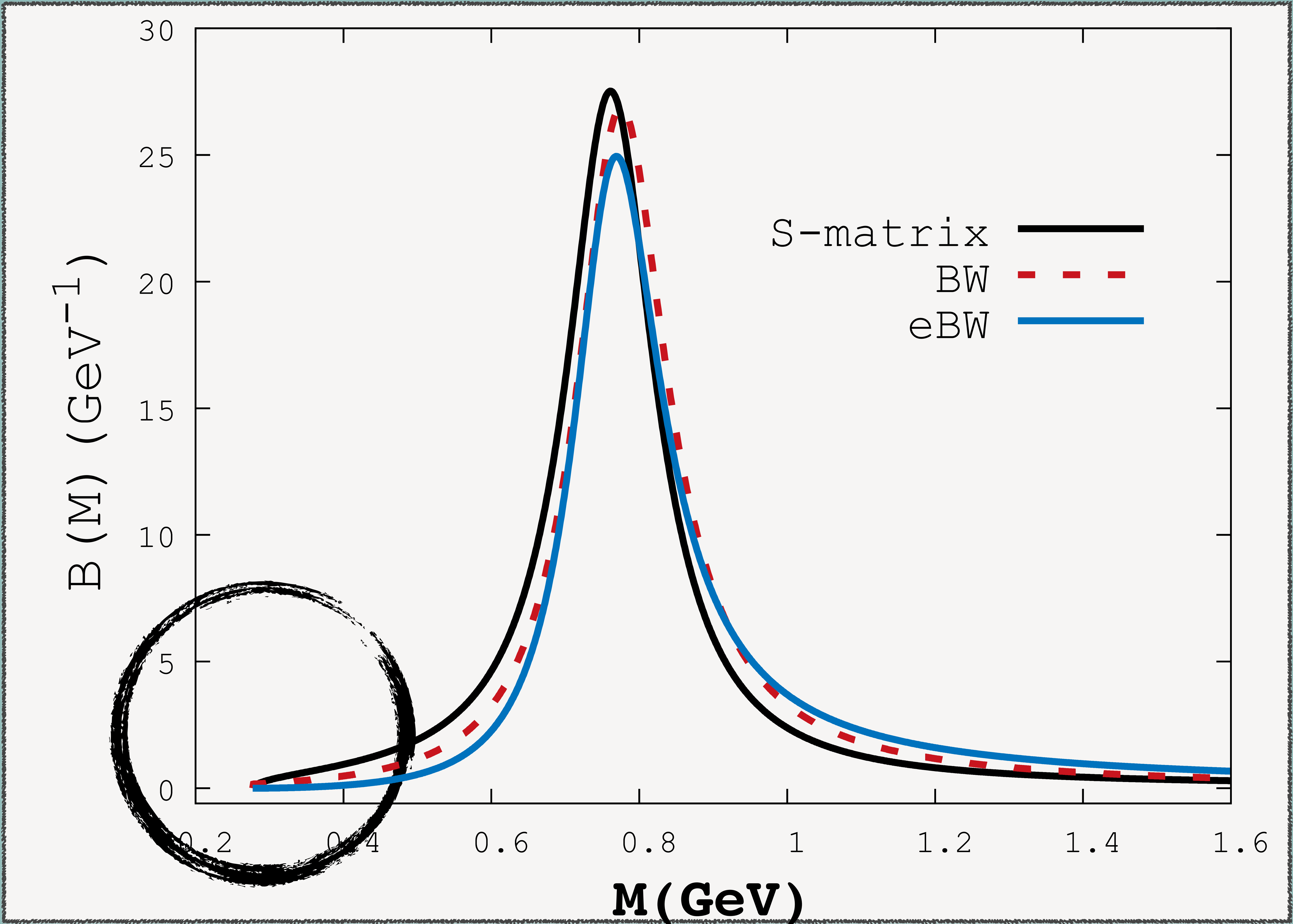
Mass $m = 775.26 \pm 0.25$ MeV

Full width $\Gamma = 149.1 \pm 0.8$ MeV

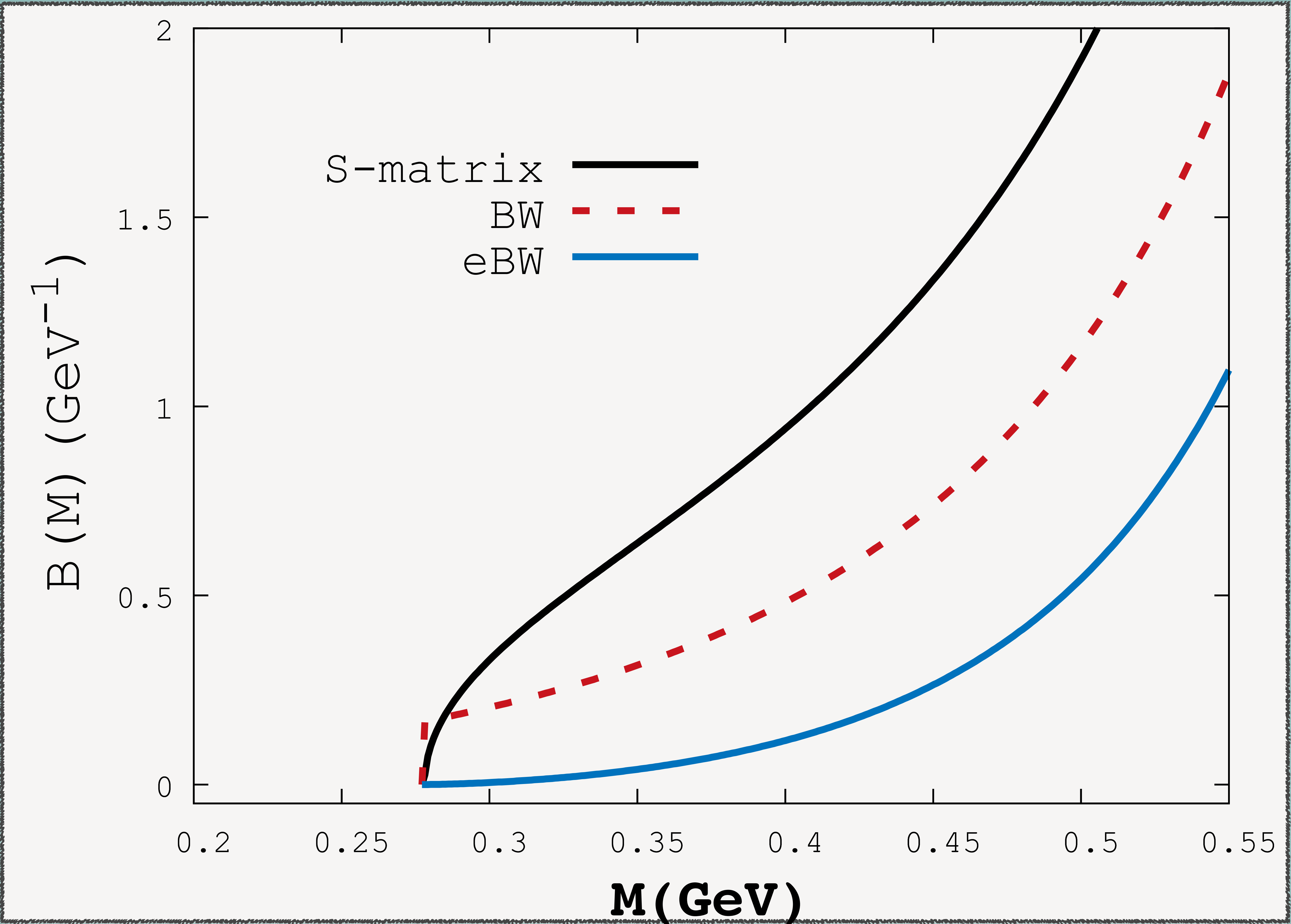
$\Gamma_{ee} = 7.04 \pm 0.06$ keV



δ(I=1 J=1) (degree)
M(GeV)
Estabrooks & Martin
Froggatt & Petersen
Protopopescu et al.
this work
$\delta \approx a_S \times q^3$
$a_S = 0.038\, m_\pi^{-3}$
0
20
40
60
80
100
120
140
160
180
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1



30
25
20
15
10
5
0
B(M) (GeV^{-1})
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
M(GeV)
S-matrix
BW
eBW



S-matrix
BW
eBW
B(M) (GeV^{-1})
M(GeV)
0
0.5
1
1.5
2
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

BETH-UHLENBECK APPROXIMATION

$$\delta = -\mathrm{Im}\,\mathrm{Tr}\ln G_\rho^{-1}$$

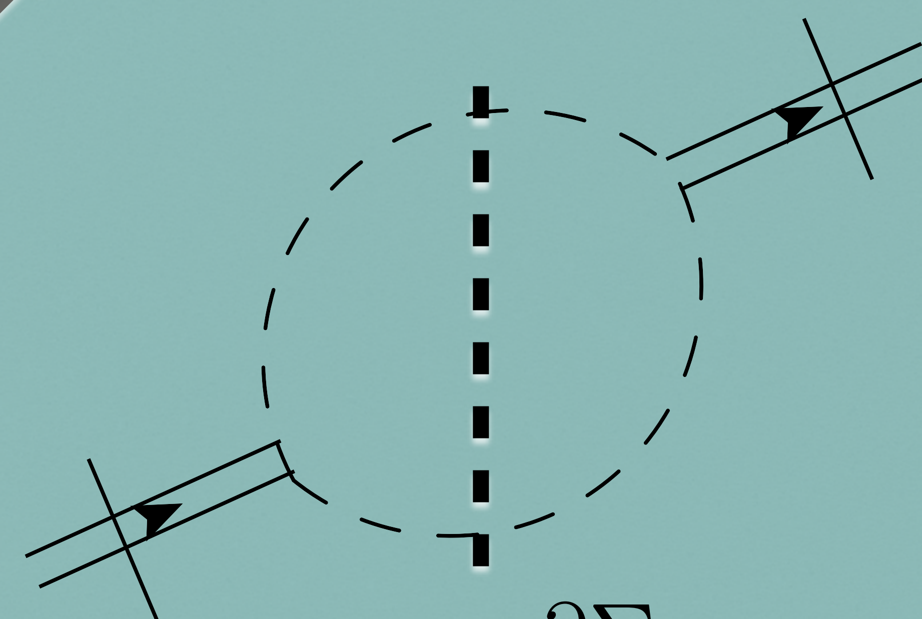
physical interpretation:

$$B = 2\,\frac{\partial}{\partial E}\delta$$

$$= -2\,\mathrm{Im}\frac{\partial}{\partial E}\ln G_\rho^{-1}$$

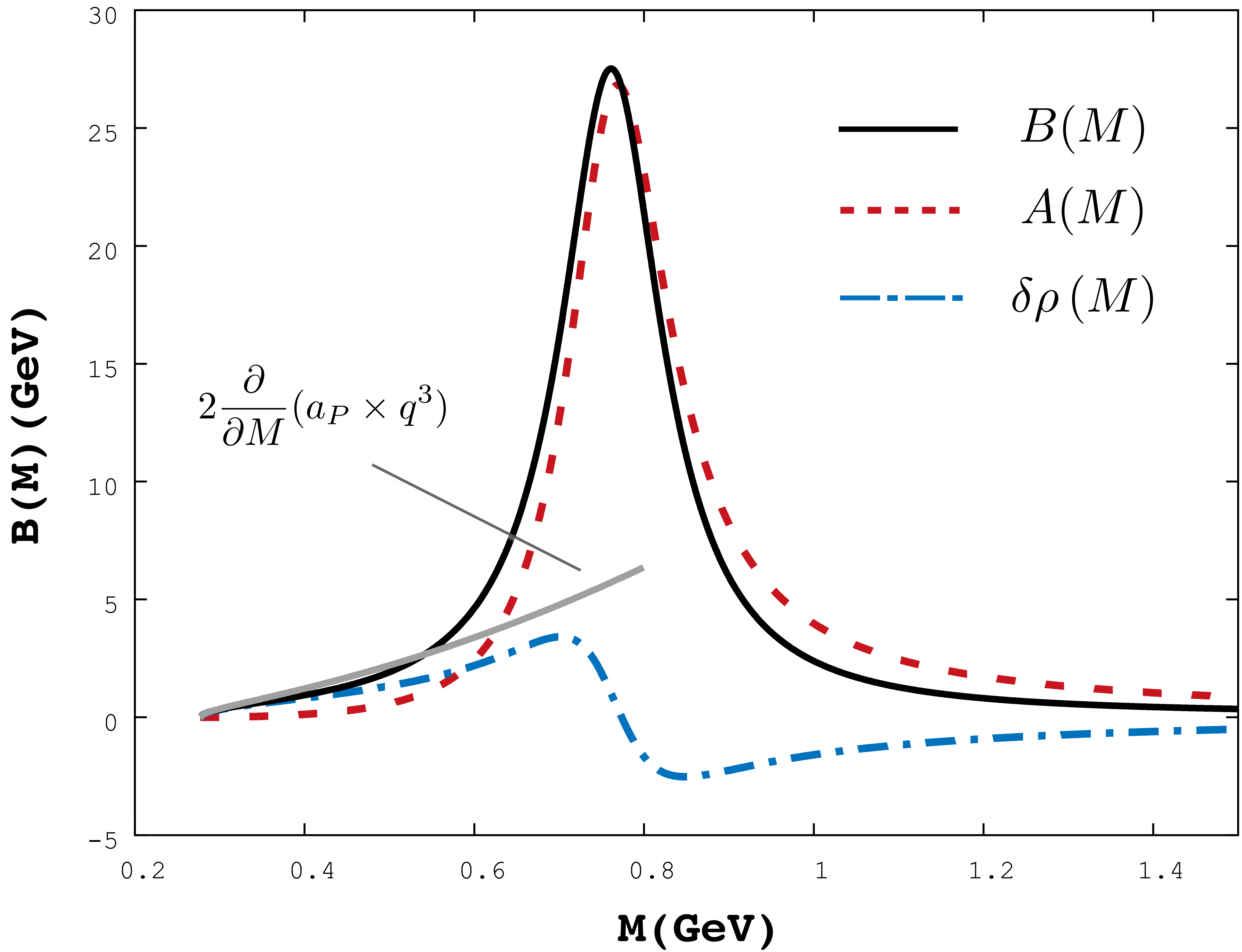
$$= -2\,\mathrm{Im}[G_\rho](2E) + 2\,\mathrm{Im}[\frac{\partial \Sigma_\rho}{\partial E}G_\rho]$$

contribution from correlated pi pi pair



$$\frac{\partial \Sigma_\rho}{\partial E}$$

$$=> \rho_\rho(E) + \delta\rho_\rho(E)$$

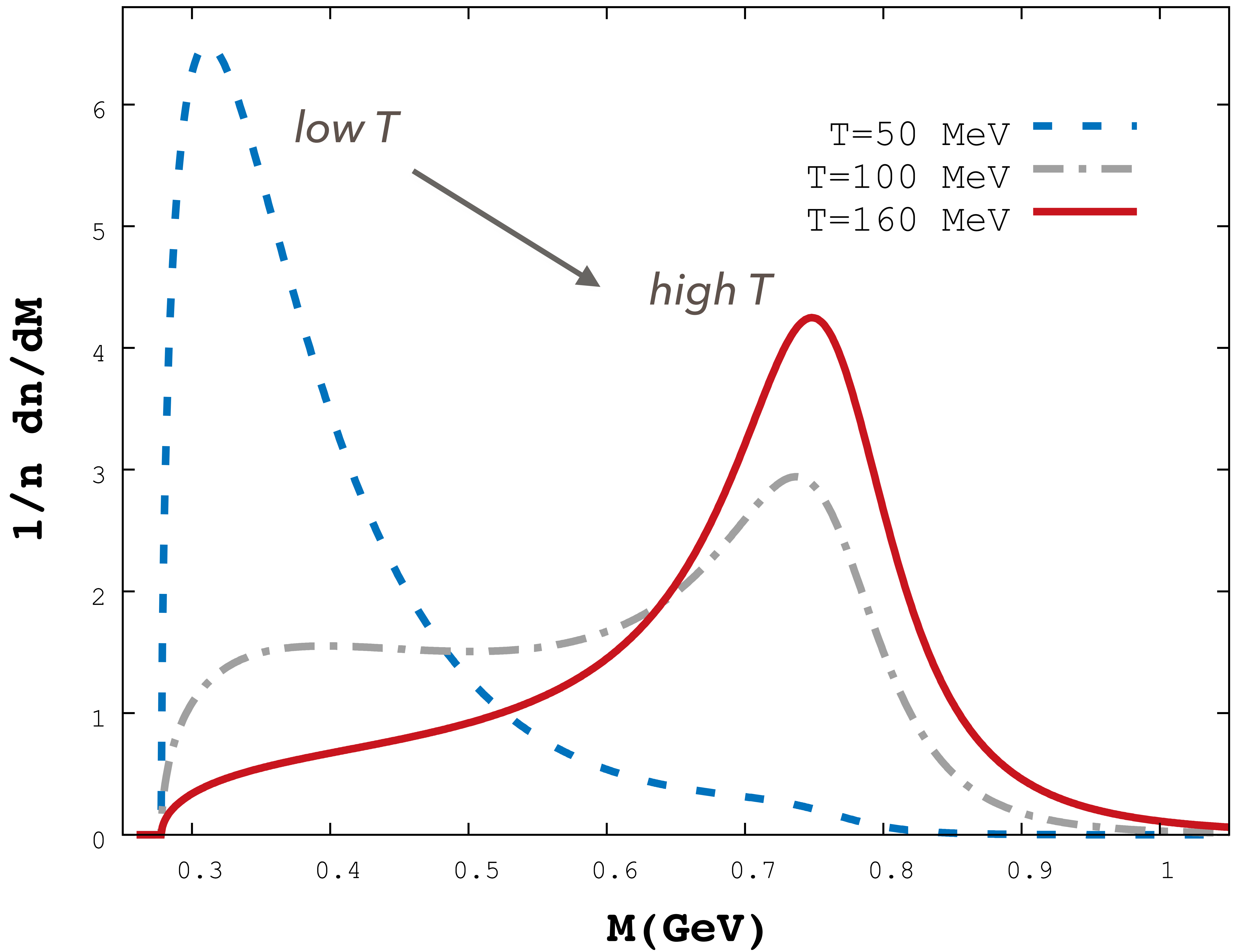


B(M) (GeV)
M(GeV)
$B(M)$
$A(M)$
$\delta\rho\,(M)$
$2\frac{\partial}{\partial M}(a_P \times q^3)$
30
25
20
15
10
5
0
-5
0.2
0.4
0.6
0.8
1
1.2
1.4

BOLTZMANN SUPPRESSION

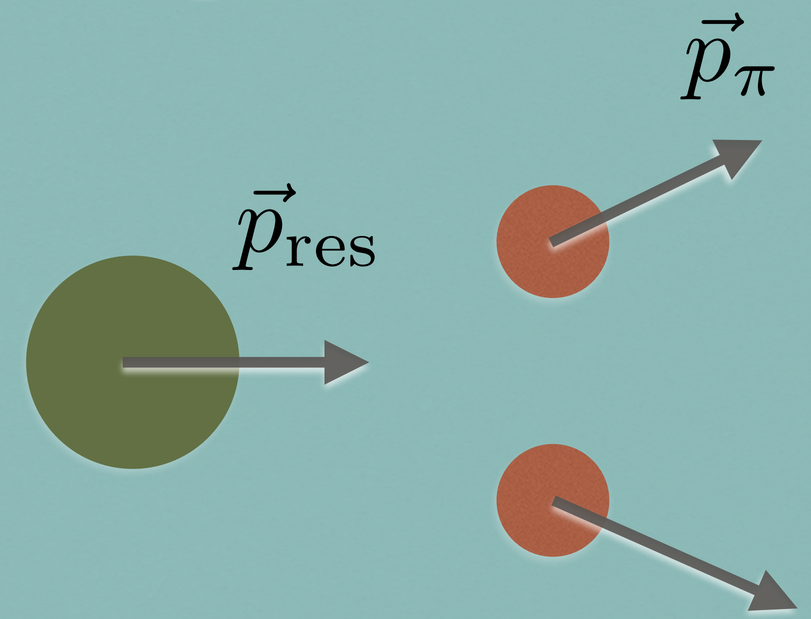
$$\Delta P \approx \frac{T^2}{2\pi^2} \int \frac{dM}{2\pi} B(M) \times \left(M^2 K_2(M/T)\right)$$

Boltzmann suppression



low T
high T
T=50 MeV
T=100 MeV
T=160 MeV
1/n dn/dM
M(GeV)
0
1
2
3
4
5
6
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

MOMENTUM SPECTRA OF DECAY PRODUCTS



$$E_\pi \frac{dn_\pi^{\rm dec}}{d^3p_\pi} = \int \frac{dM}{2\pi} B(M) \times$$

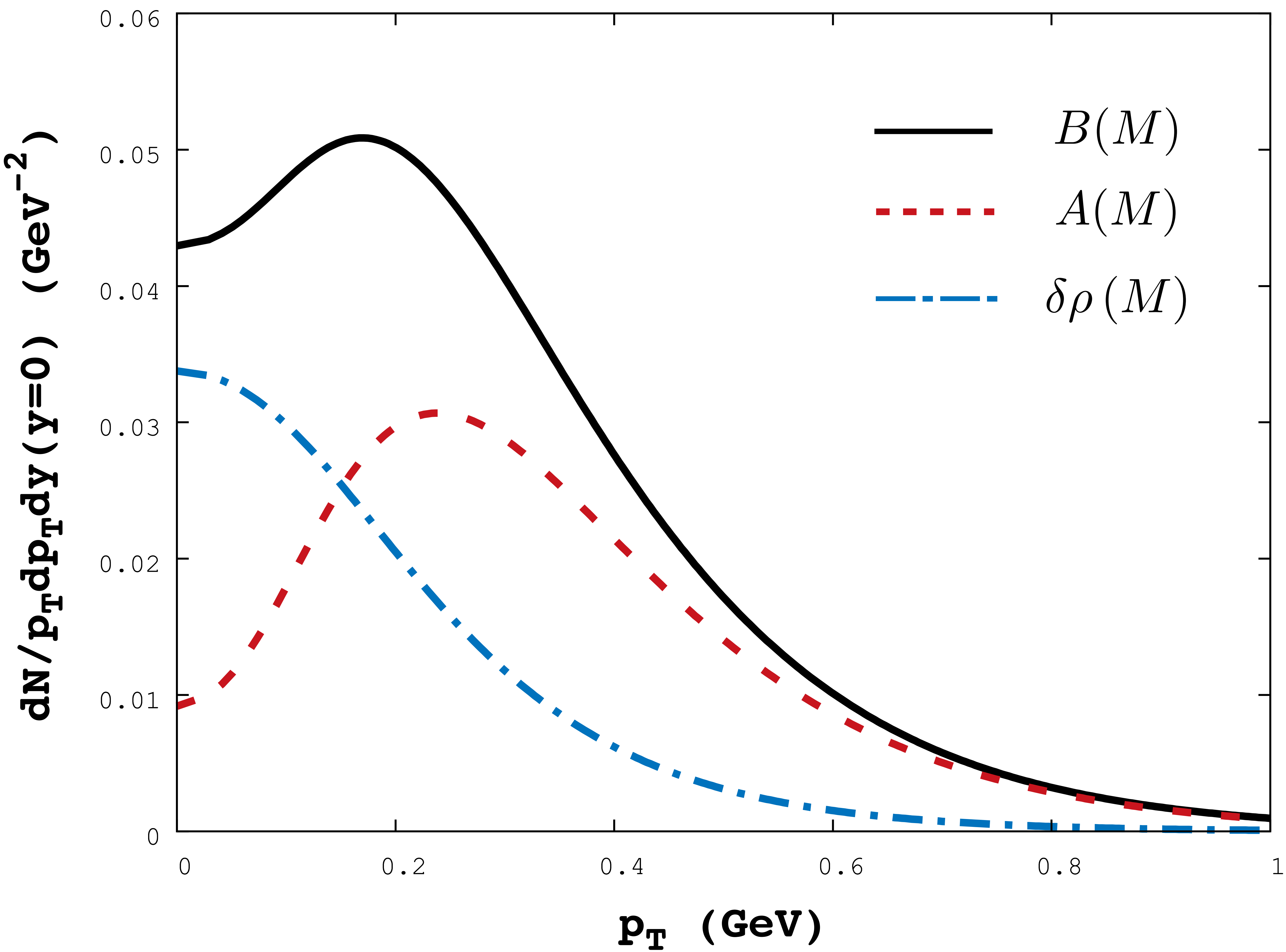
$$\int_{m_{\rm res} \to M} d^3p_{\rm res} \frac{dn_{\rm res}}{d^3p_{\rm res}} \times E_\pi^\star \times {\rm dPS}(\vec{p}_\pi^{\,\star})$$

decay kinematics

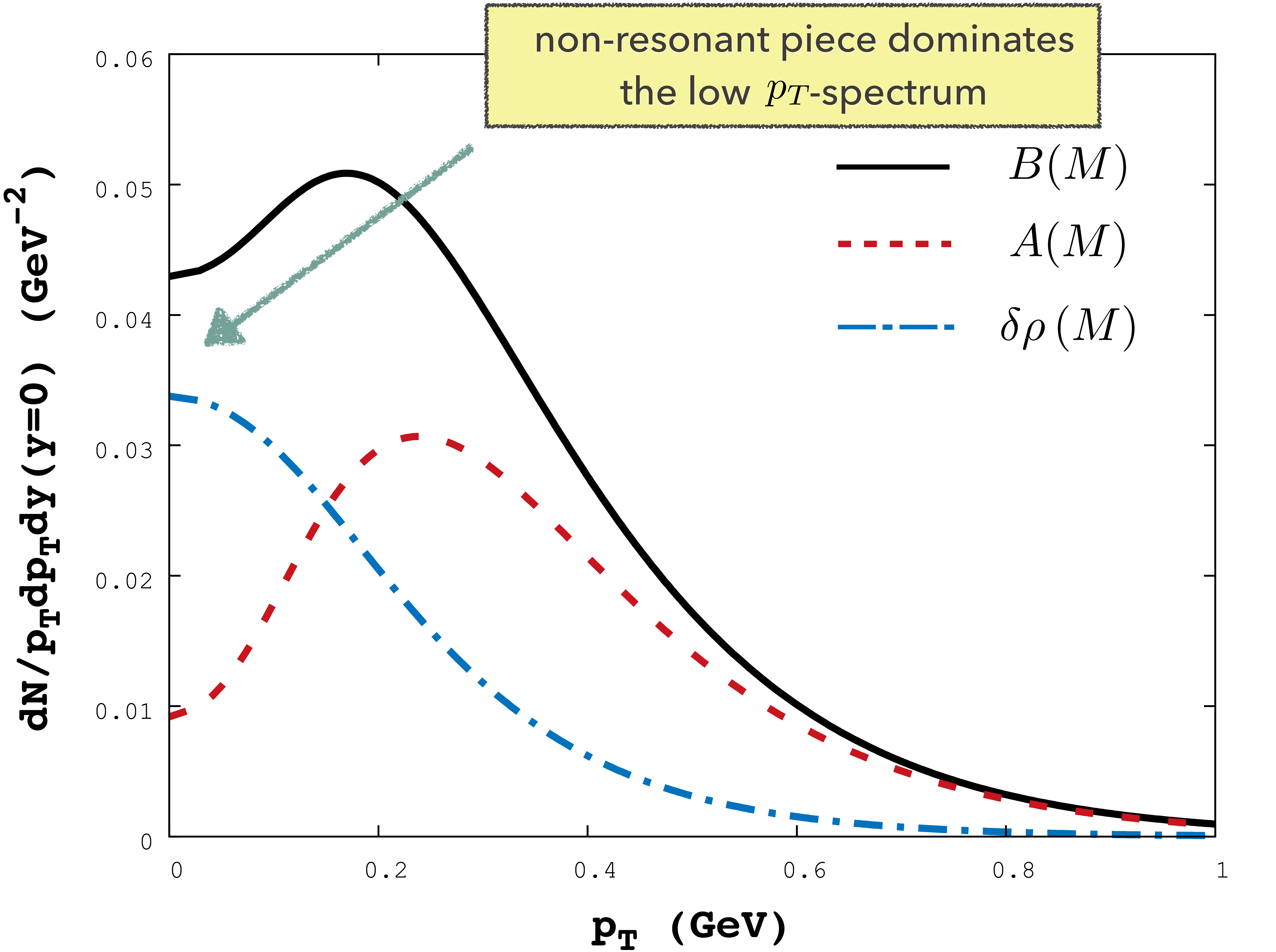
$$\mathcal{F}[p_\pi, M]$$

$$\frac{dn_{\rm res}}{d^3p_{\rm res}} \to \frac{g_{\rm res}}{(2\pi)^3} \frac{1}{e^{\beta E_{\rm res}} - 1}$$

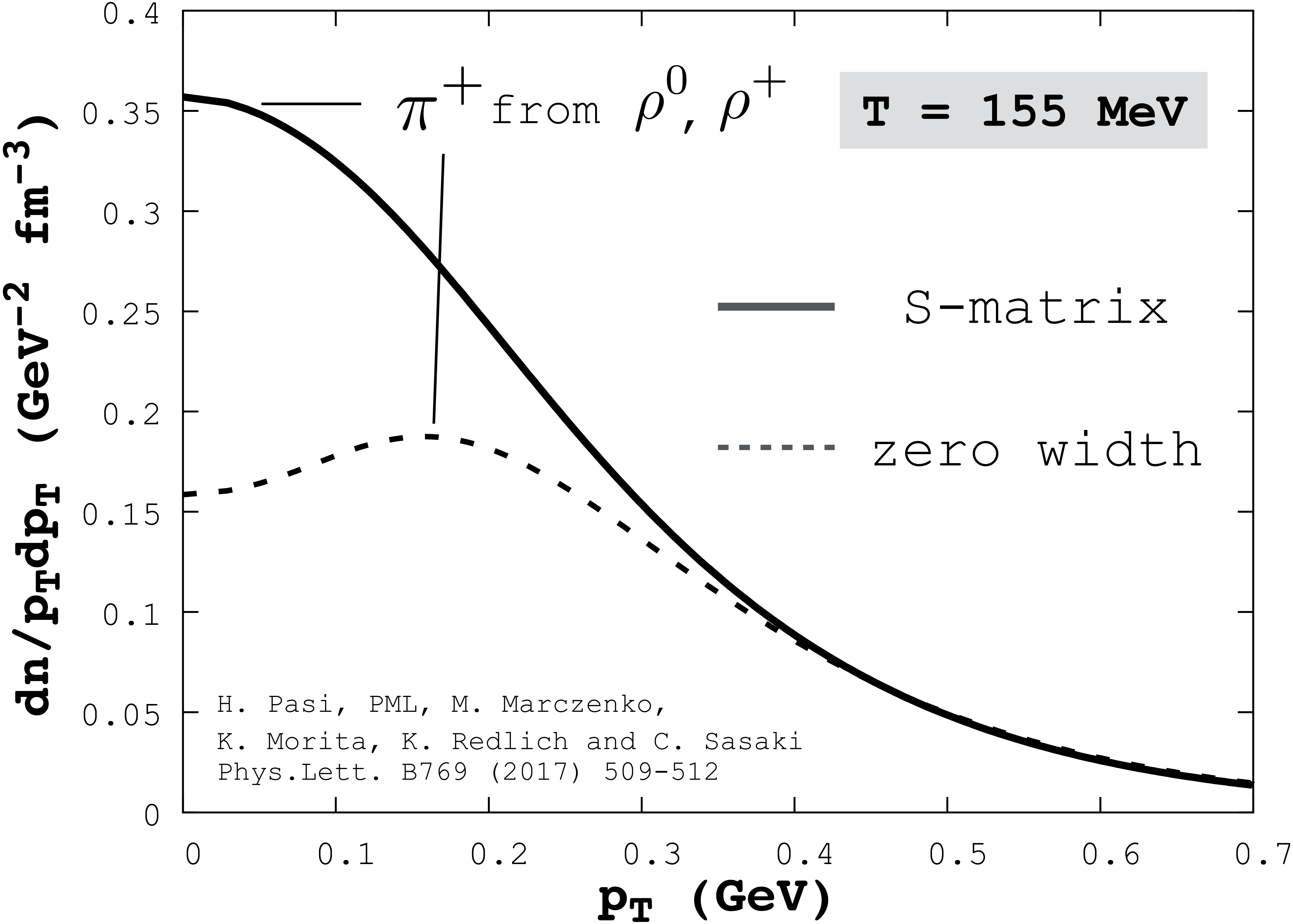
$${\rm dPS} \to \frac{1}{4\pi q^2} \delta(p_\pi^\star - q)$$



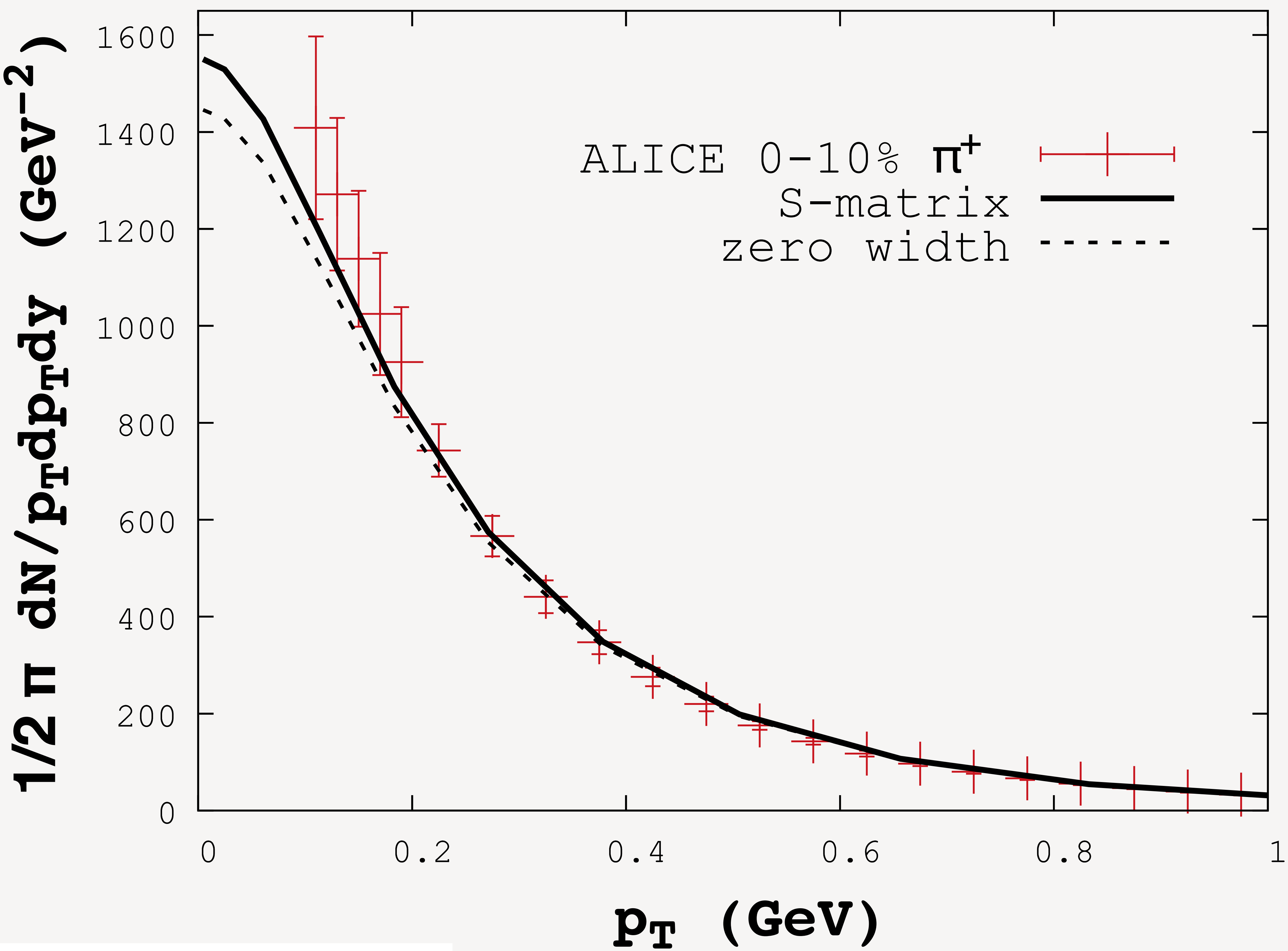
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
$dN/p_T dp_T dy(y=0)$ (GeV^{-2})
p_T (GeV)
$B(M)$
$A(M)$
$\delta\rho(M)$



non-resonant piece dominates
the low p_T-spectrum
$B(M)$
$A(M)$
$\delta\rho\,(M)$
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
$dN/p_T dp_T dy(y=0)$ (GeV^{-2})
p_T (GeV)



0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
π^+ from ρ^0, ρ^+
T = 155 MeV
S-matrix
zero width
dn/p$_T$dp$_T$ (GeV^{-2} fm^{-3})
p$_T$ (GeV)
H. Pasi, PML, M. Marczenko,
K. Morita, K. Redlich and C. Sasaki
Phys.Lett. B769 (2017) 509-512



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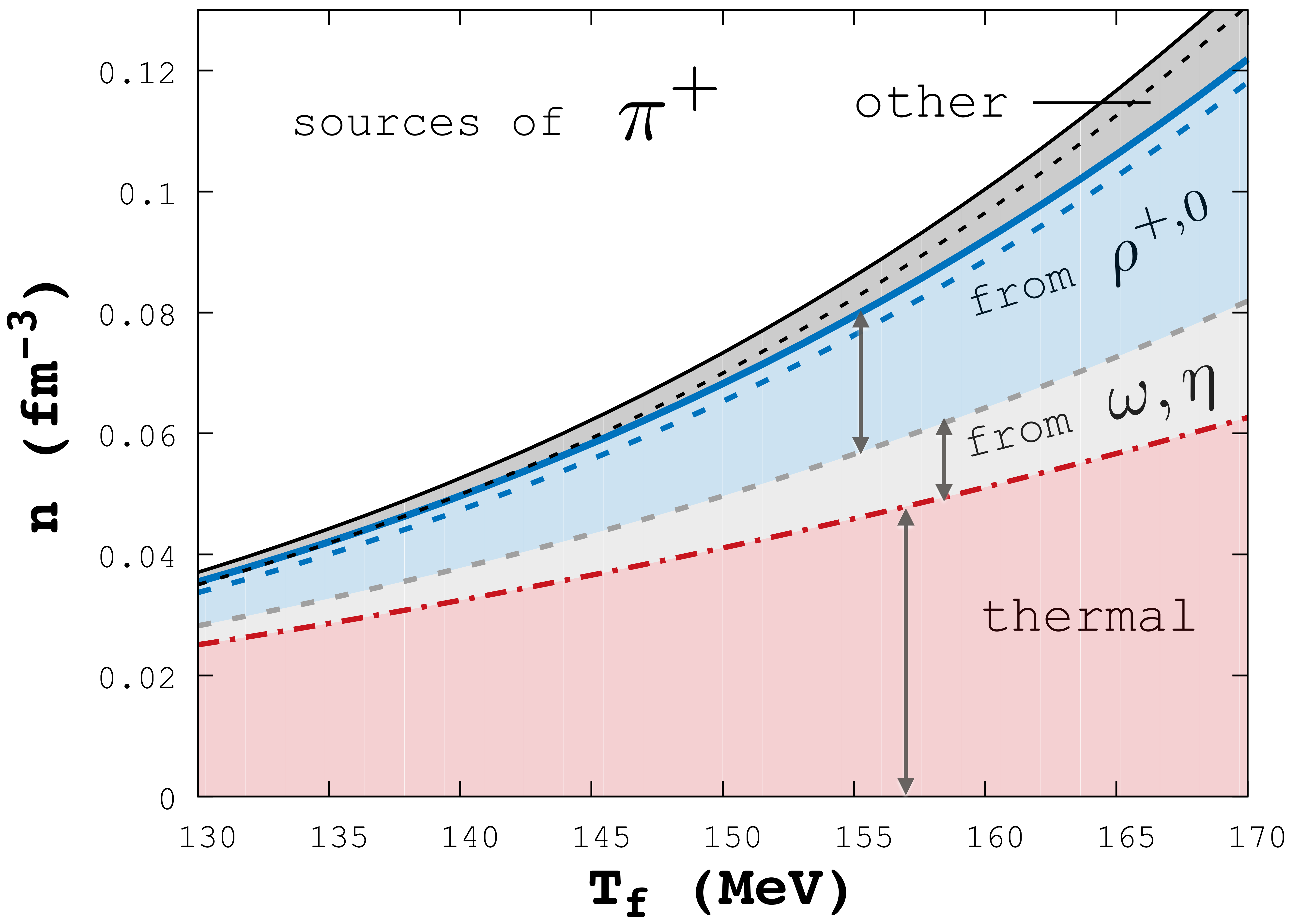
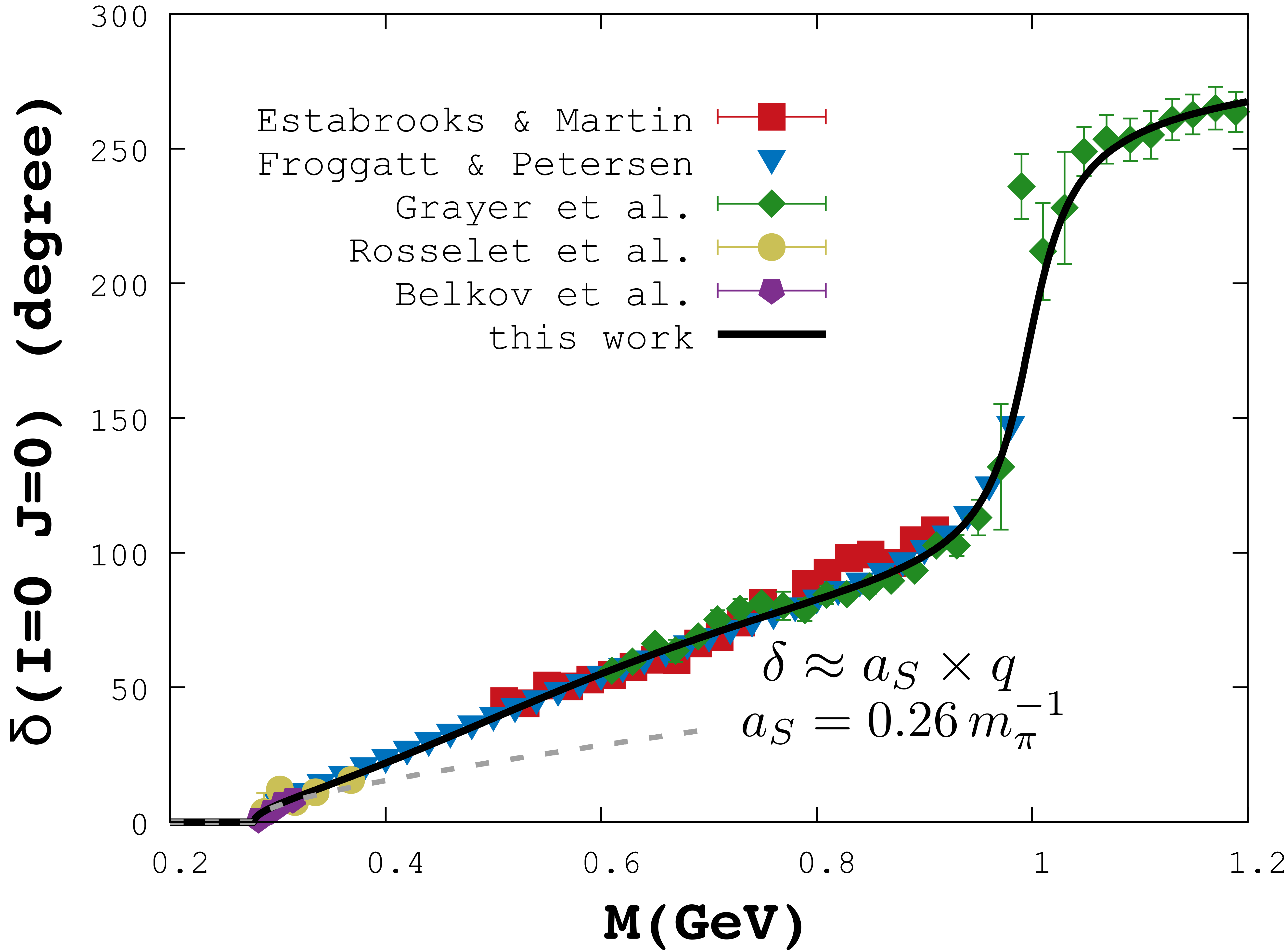
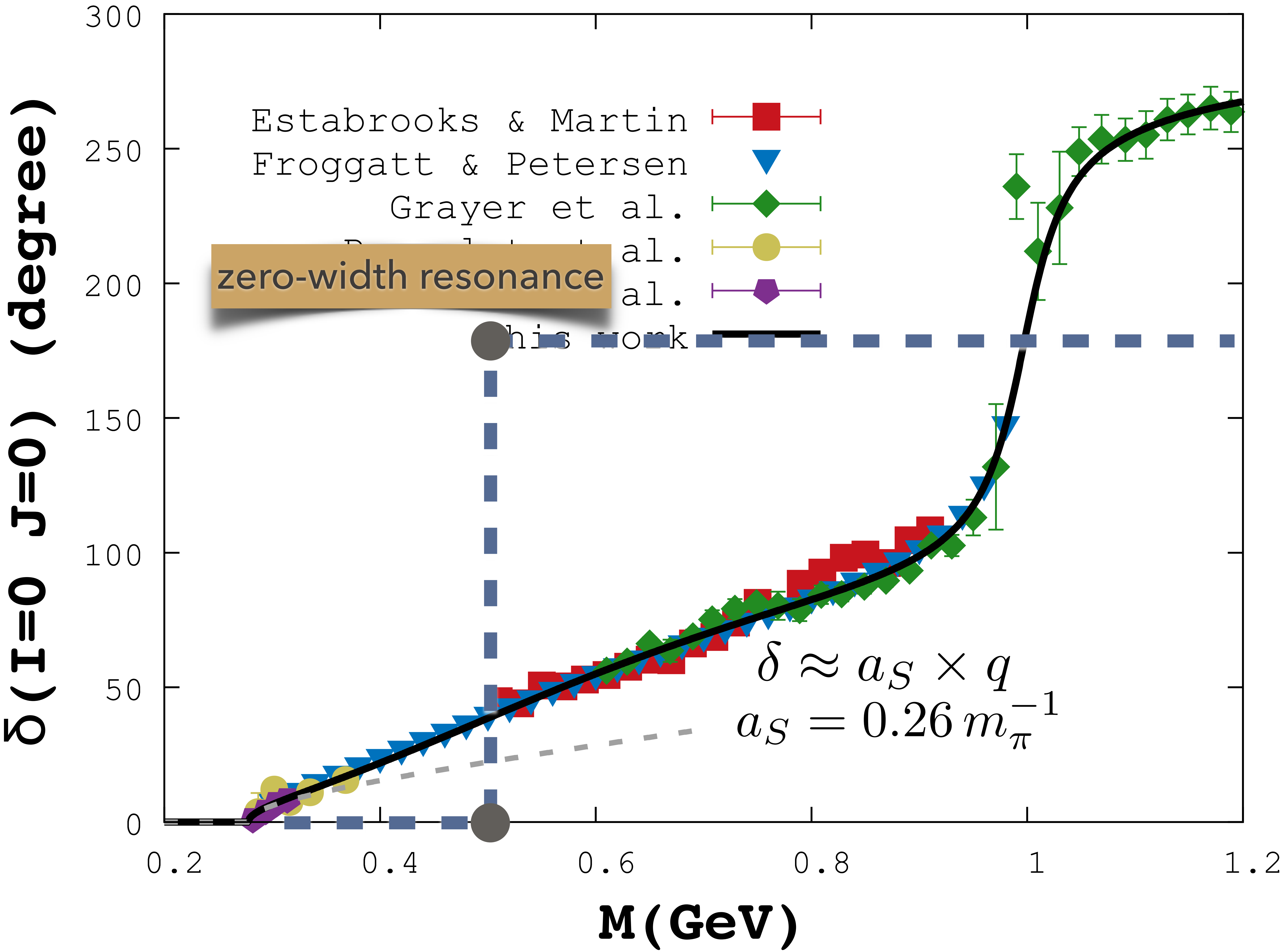


sources of π^+
other
from $\rho^{+,0}$
from ω, η
thermal
n (fm^{-3})
T_f (MeV)
0.12
0.1
0.08
0.06
0.04
0.02
0
130
135
140
145
150
155
160
165
170

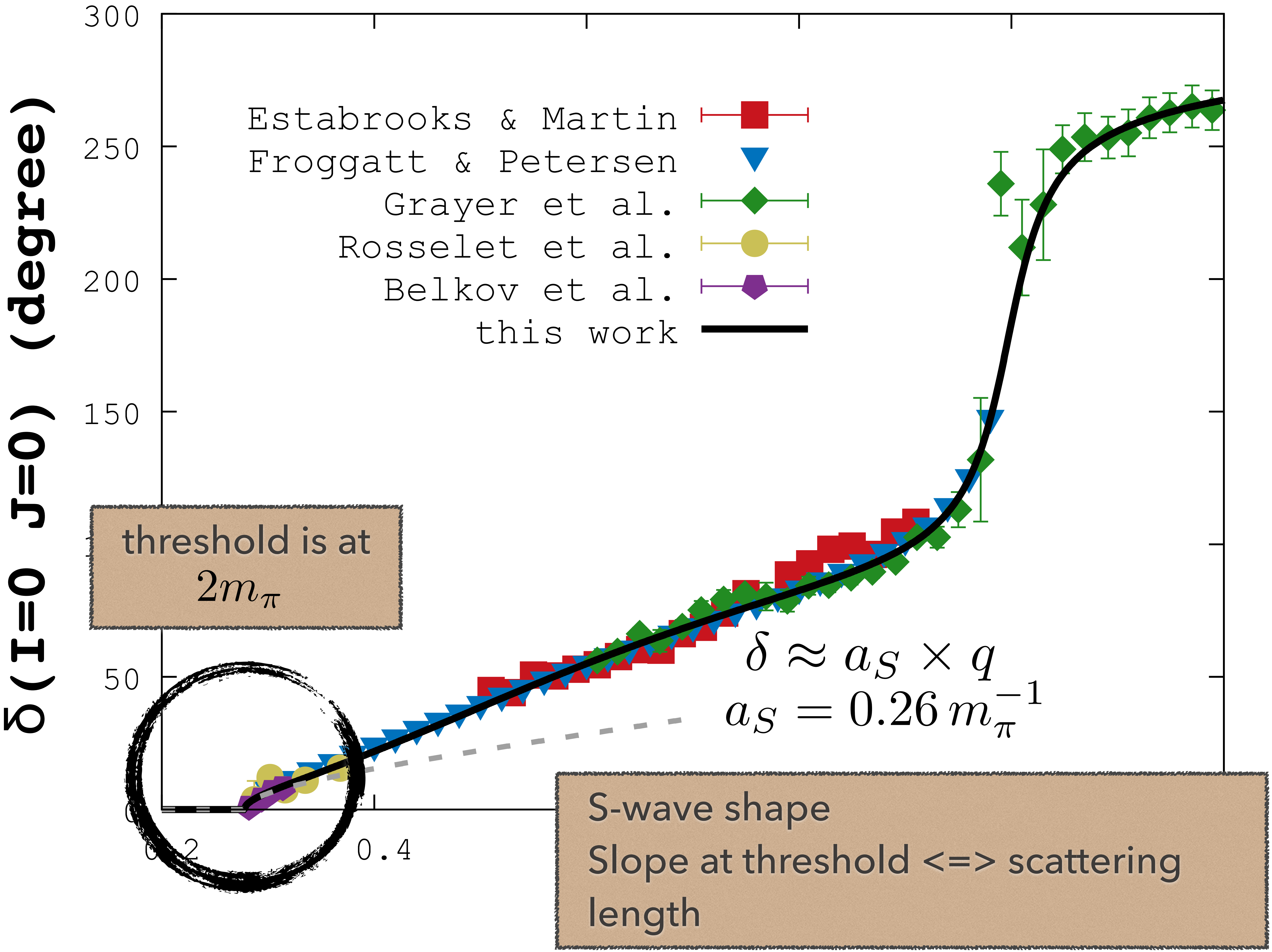
PI PI SCATTERING (S-WAVE)



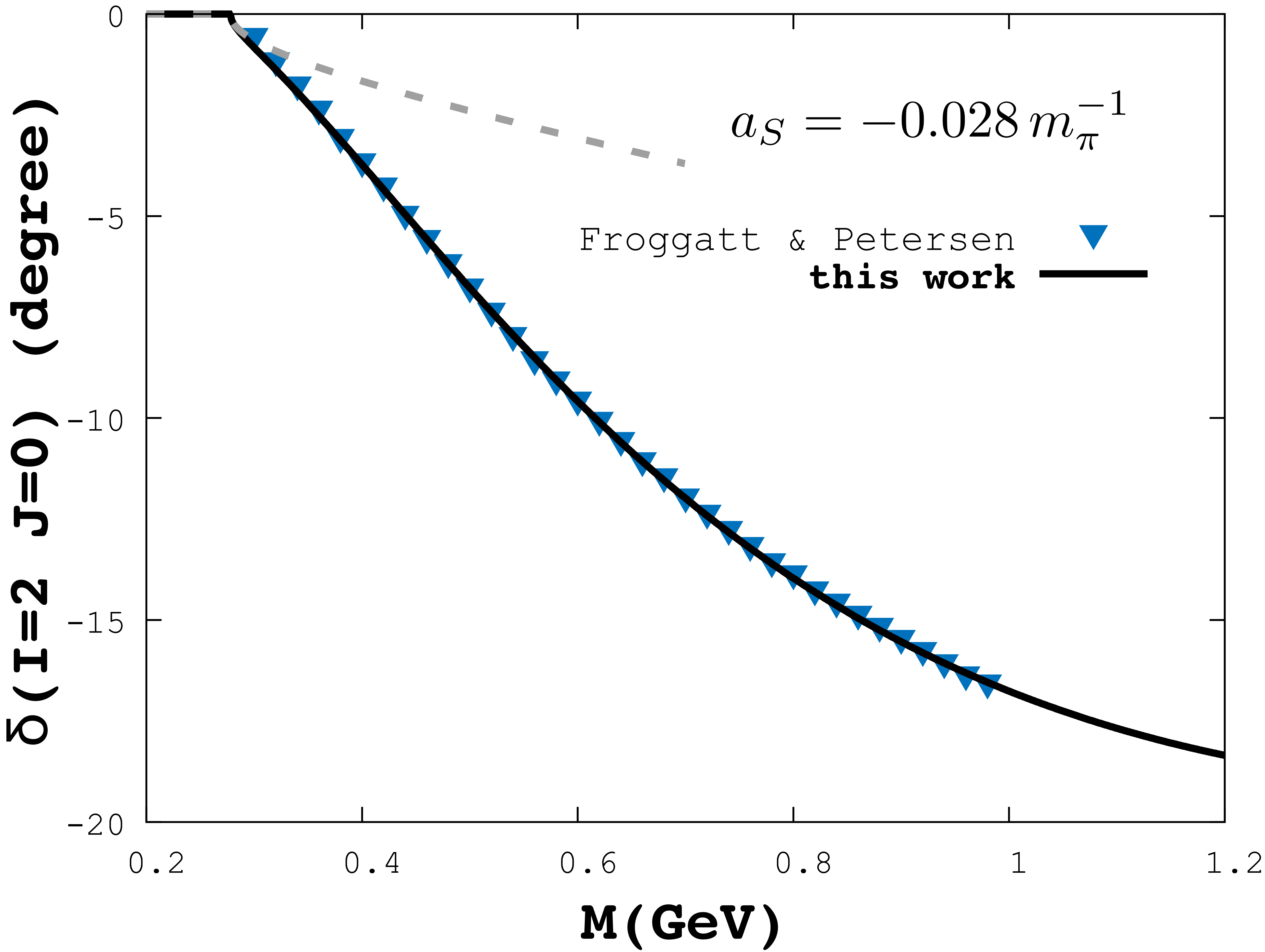
δ(I=0 J=0) (degree)
M(GeV)
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
Rosselet et al.
Belkov et al.
this work
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
0
50
100
150
200
250
300
0.2
0.4
0.6
0.8
1
1.2



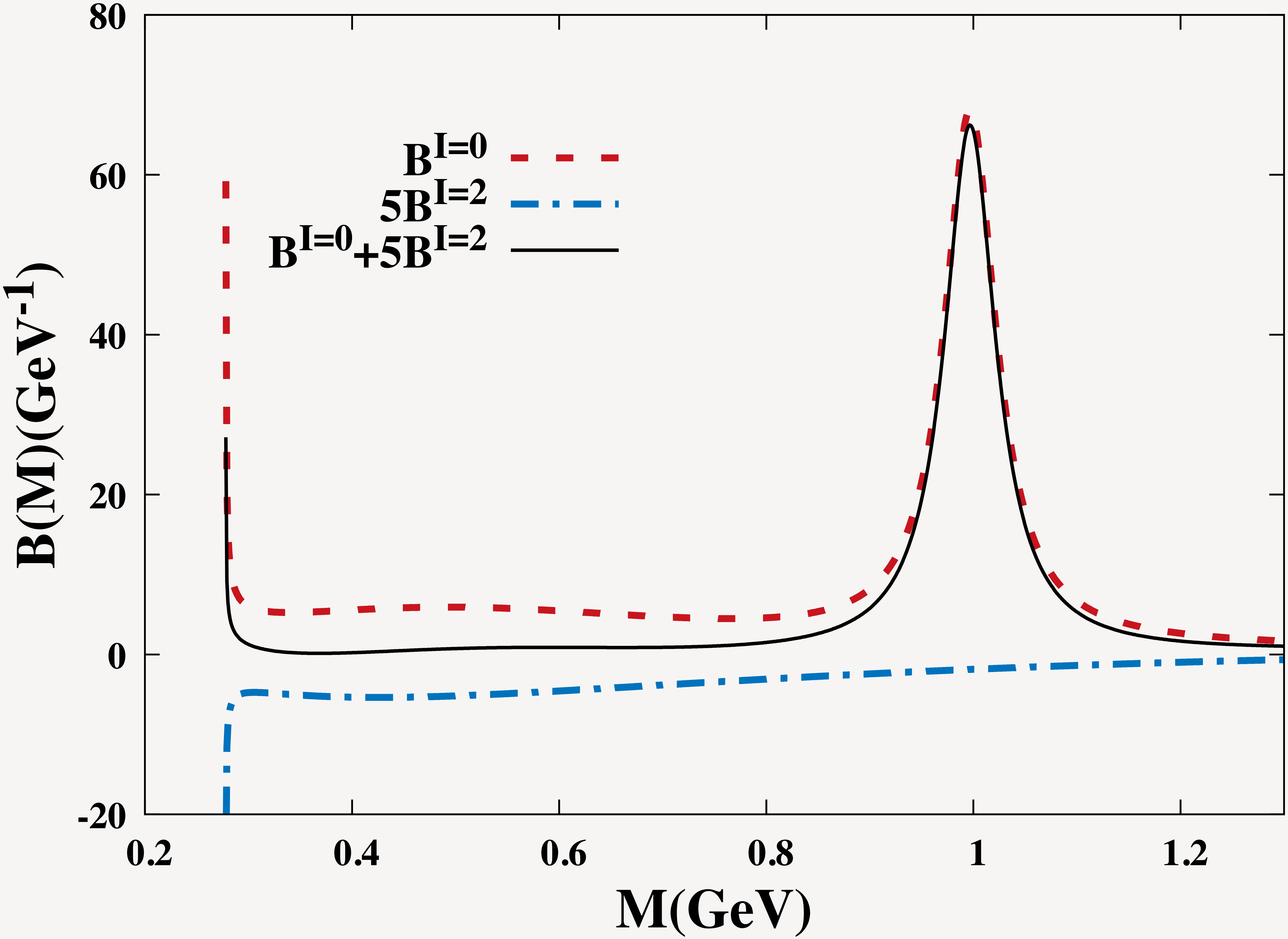
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
zero-width resonance
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
δ(I=0 J=0) (degree)
M(GeV)
0
50
100
150
200
250
300
0.2
0.4
0.6
0.8
1
1.2



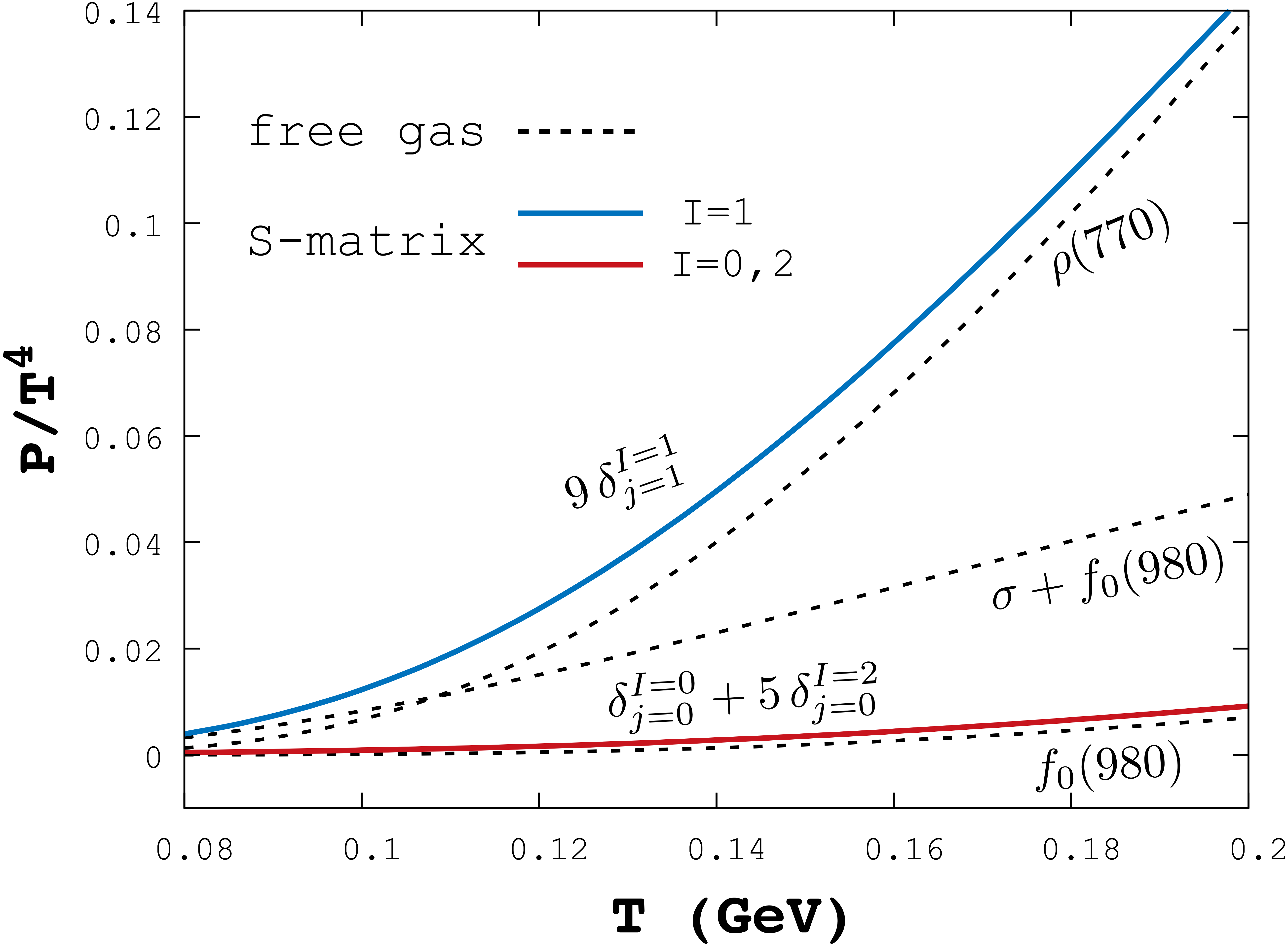
δ(I=0 J=0) (degree)
300
250
200
150
50
0.2
0.4
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
Rosselet et al.
Belkov et al.
this work
threshold is at
$2m_\pi$
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
S-wave shape
Slope at threshold <=> scattering length



$a_S = -0.028\, m_\pi^{-1}$
Froggatt & Petersen
this work
δ(I=2 J=0) (degree)
M(GeV)
0
-5
-10
-15
-20
0.2
0.4
0.6
0.8
1
1.2



80
60
40
20
0
-20
0.2
0.4
0.6
0.8
1
1.2
M(GeV)
B(M)(GeV-1)
$B^{I=0}$
$5B^{I=2}$
$B^{I=0}+5B^{I=2}$



P/T^4
T (GeV)
free gas
S-matrix
I=1
I=0,2
ρ(770)
$9\,\delta^{I=1}_{j=1}$
$\sigma + f_0(980)$
$\delta^{I=0}_{j=0} + 5\,\delta^{I=2}_{j=0}$
$f_0(980)$
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.08
0.1
0.12
0.14
0.16
0.18
0.2

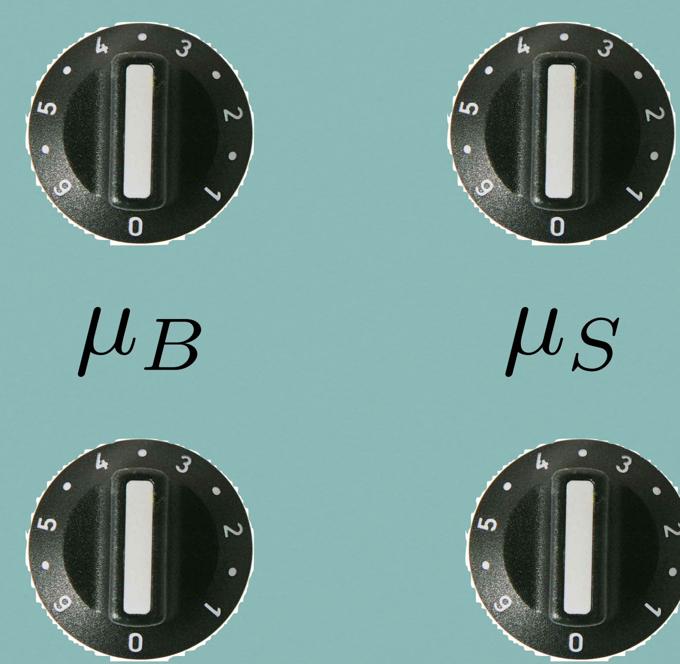
PI-N SYSTEM

FLUCTUATIONS

- studying the system by linear response

$$\mu = \mu_B B + \mu_Q Q + \mu_S S$$

$$\chi_{B,S,\ldots} = \frac{1}{\beta V} \frac{\partial^2}{\partial \bar{\mu}_B \partial \bar{\mu}_S \ldots} \ln Z$$



μ_B μ_S

μ_Q m_q

FLUCTUATIONS

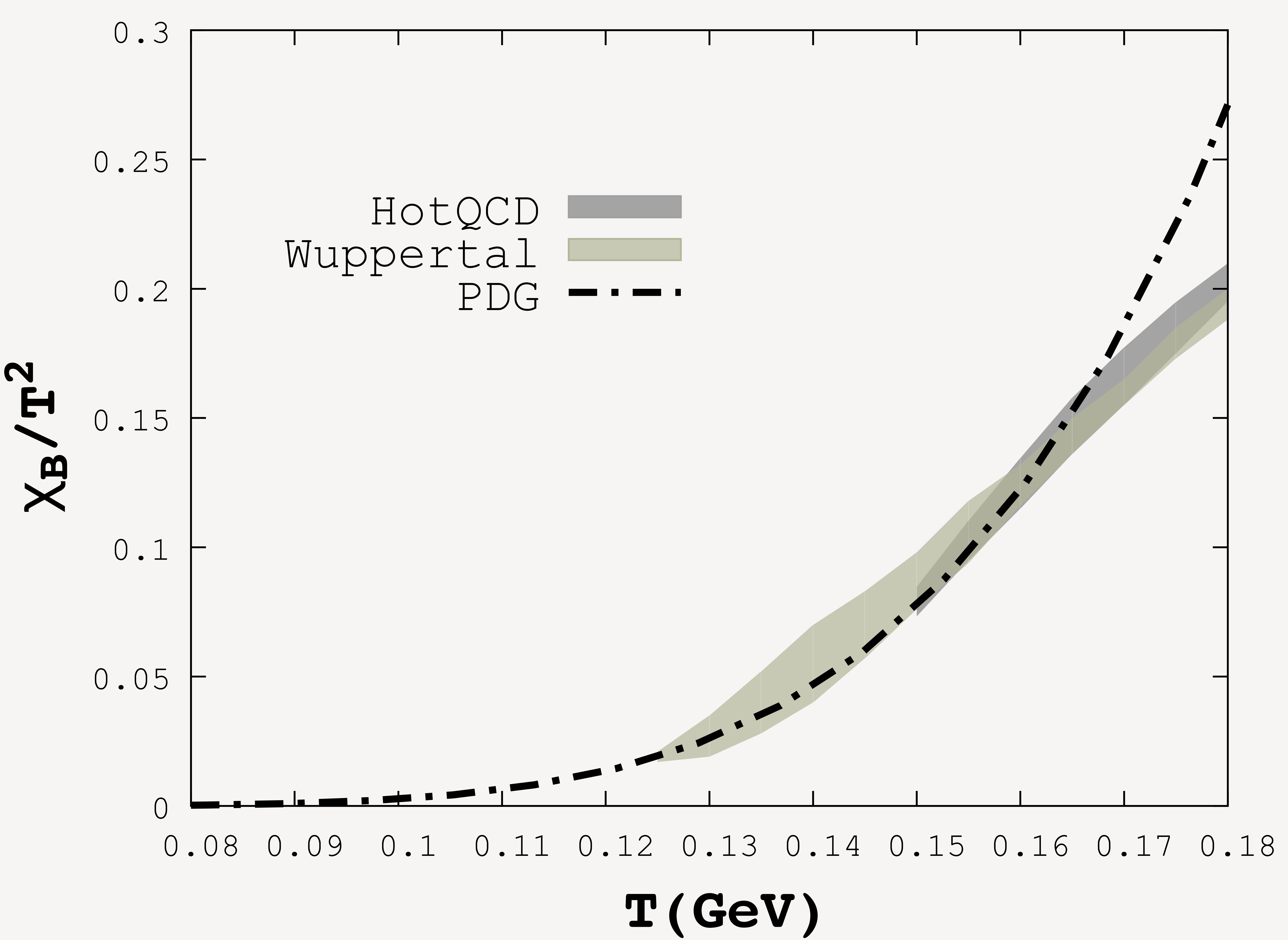
- taking derivative

$$\chi_B = \frac{\partial^2}{\partial \bar{\mu}_B \partial \bar{\mu}_B} P \quad \textit{at the limit} \quad \mu_B \to 0$$

probes fluctuations

$$\chi_B = \frac{1}{\beta V} \frac{\partial^2}{\partial \bar{\mu}_B \partial \bar{\mu}_B} \ln Z$$

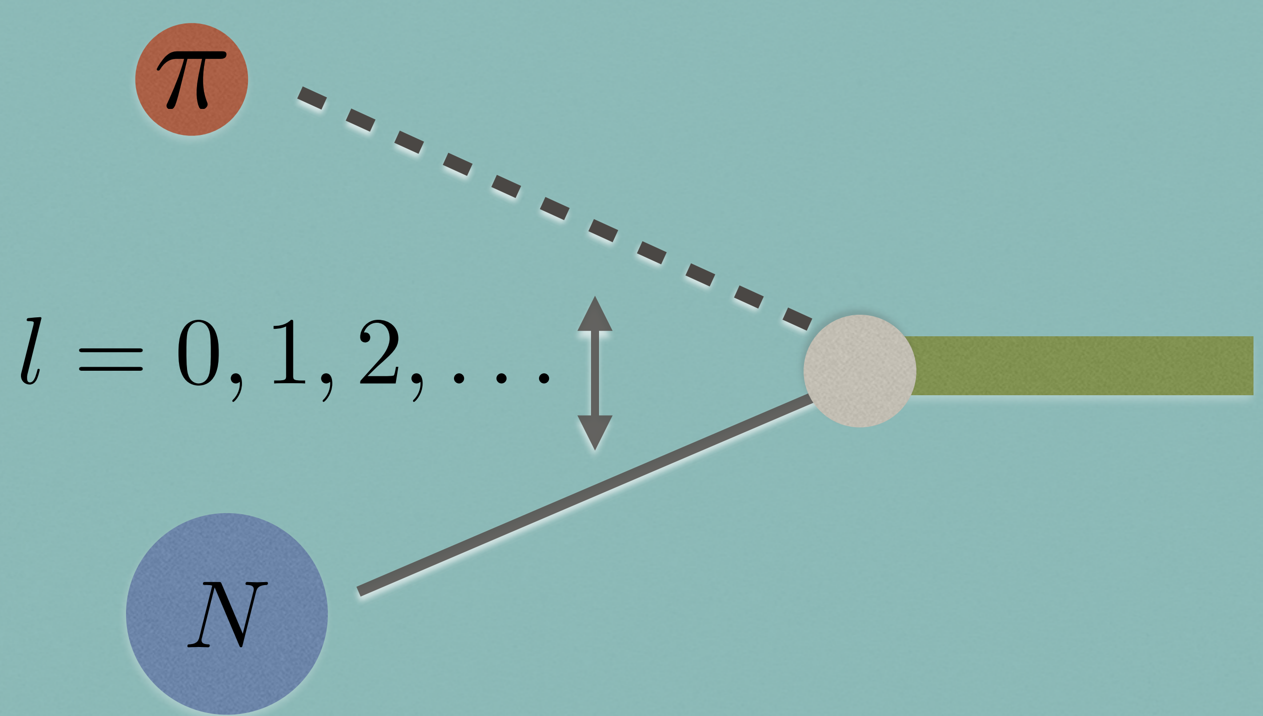
$$= T^2 \langle\langle \int d^4x \, \bar{\psi}(x) \gamma^0 \psi(x) \bar{\psi}(0) \gamma^0 \psi(0) \rangle\rangle_c$$



HotQCD
Wuppertal
PDG
χ_B/T^2
T(GeV)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18

N* AND DELTAS

$$I = 1, j = 0$$



$$I = 1/2, j = 1/2$$

N^* Δ

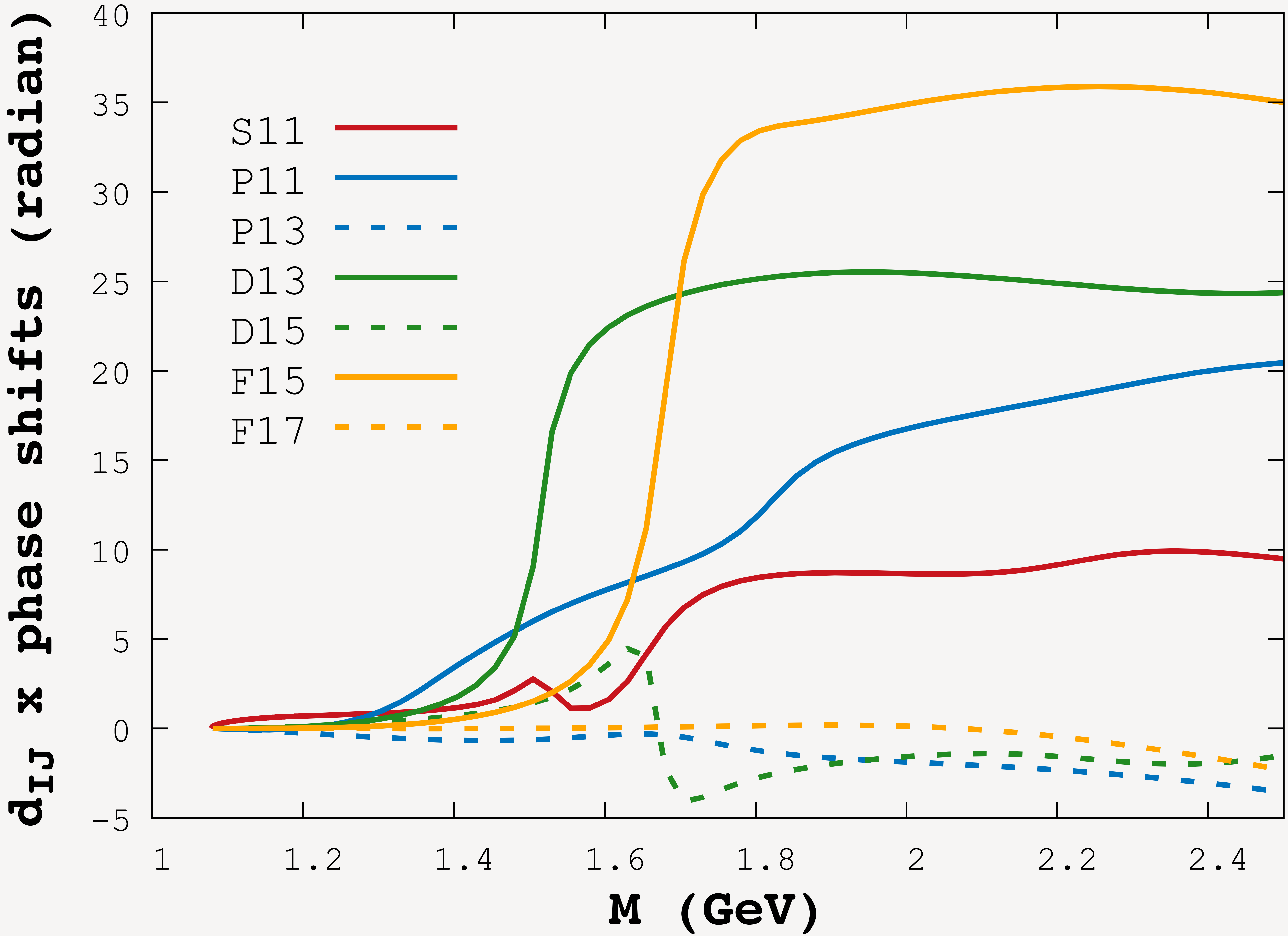
$$I = 1/2, 3/2$$

$$j = |\, l \pm 1/2 \,|,$$

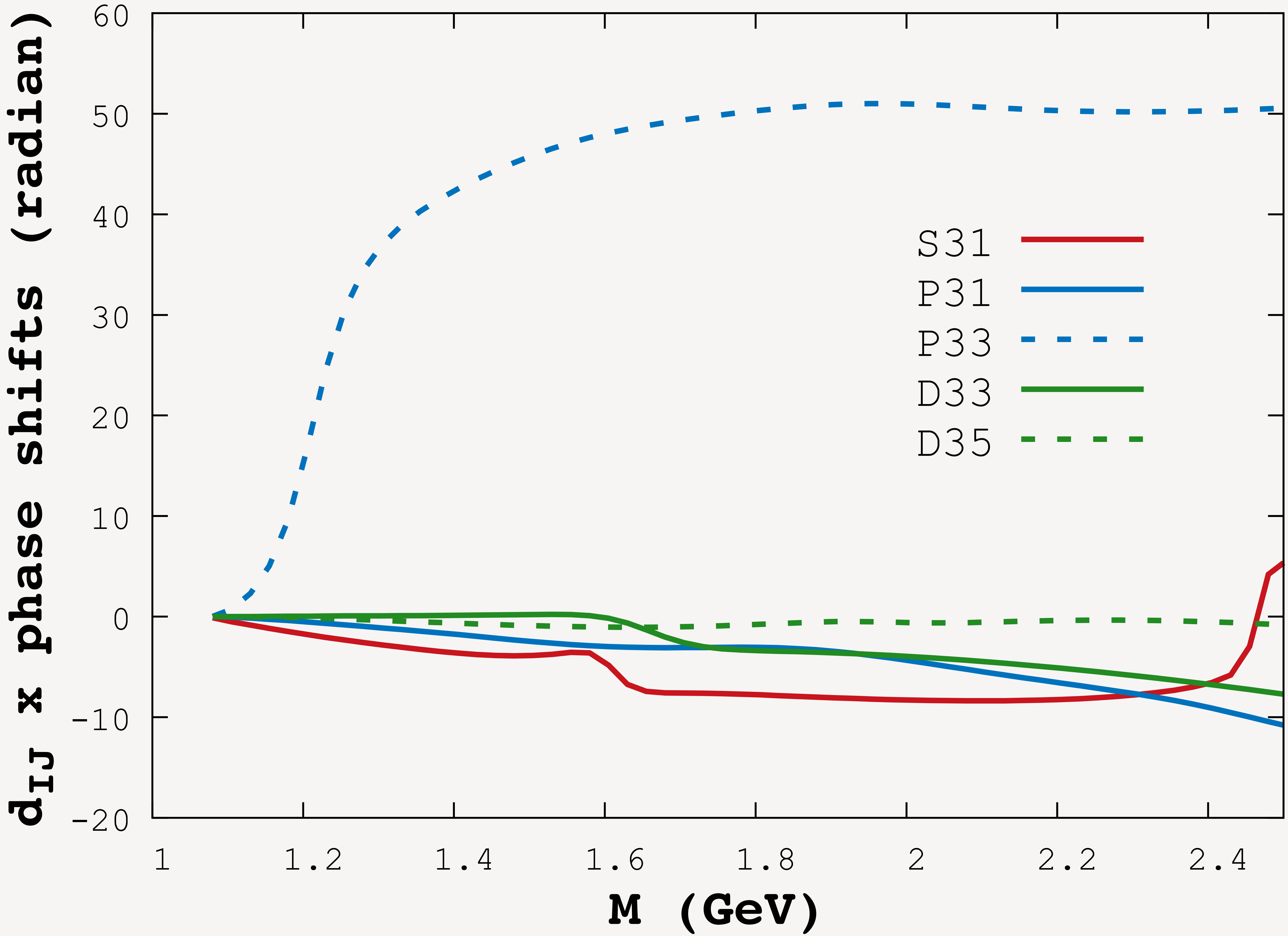
$$P = (-1)^{l+1}$$

N* AND DELTAS

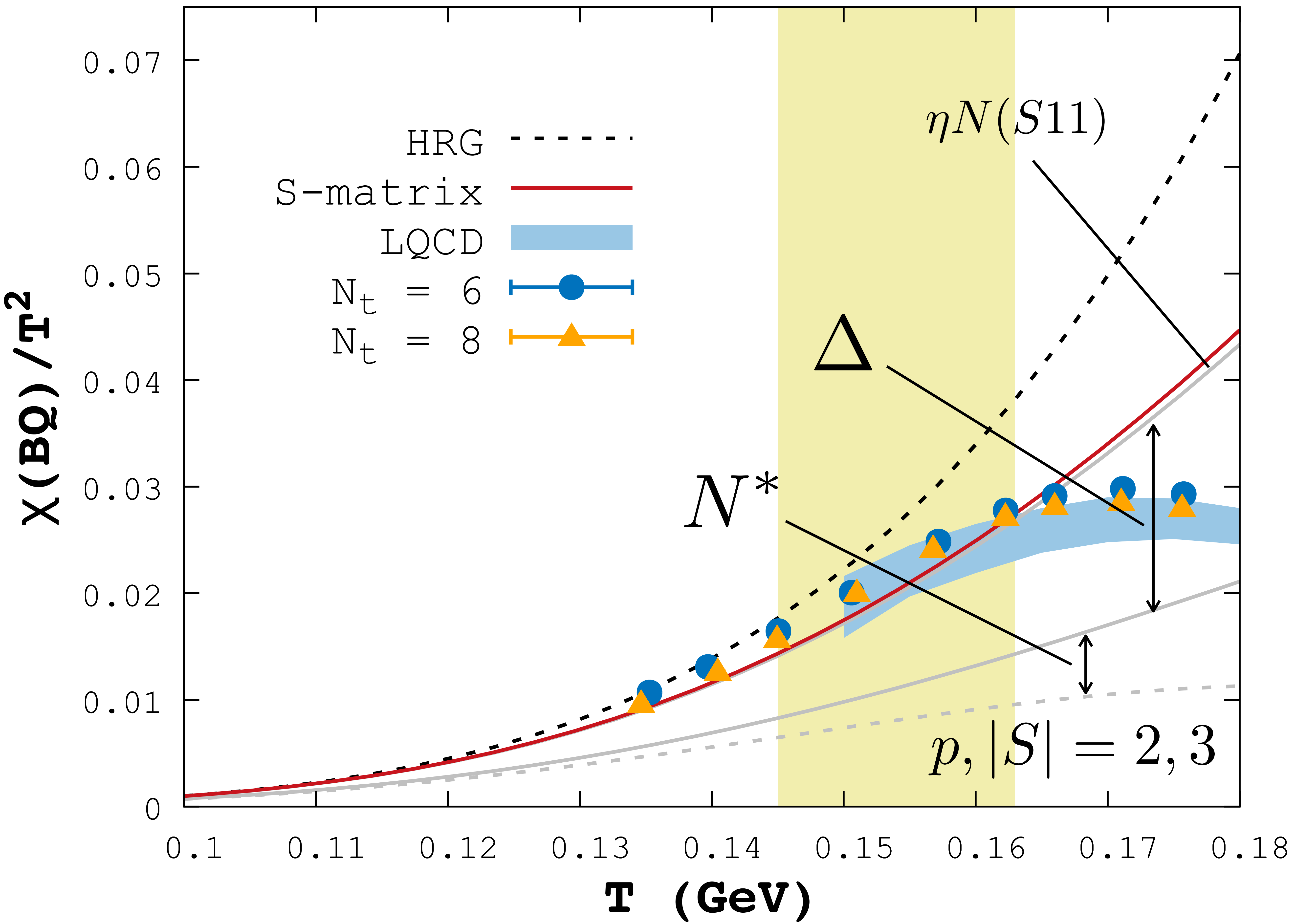
- N*: 1535 (S11), 1440 (P11), 1520 (D13) …
  Δ : 1232 (P33), 1620 (S31) …
- Repulsive forces between pions and nucleons
- BQ-correlation: S = -1 hyperons are excluded!



d_{IJ} x phase shifts (radian)
40
35
30
25
20
15
10
5
0
-5
1
1.2
1.4
1.6
1.8
2
2.2
2.4
M (GeV)
S11
P11
P13
D13
D15
F15
F17



d_{IJ} x phase shifts (radian)
M (GeV)
S31
P31
P33
D33
D35
60
50
40
30
20
10
0
-10
-20
1
1.2
1.4
1.6
1.8
2
2.2
2.4



HRG
S-matrix
LQCD
$N_t = 6$
$N_t = 8$
$\eta N(S11)$
Δ
N^*
$p, |S| = 2, 3$
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18
$\chi(BQ)/T^2$
T (GeV)

KNOWN UNKNOWNS ???

- Inelasticity:

  η production (*ok*)

  multi-pions states (*in progress*)

COUPLED-CHANNEL PROBLEM

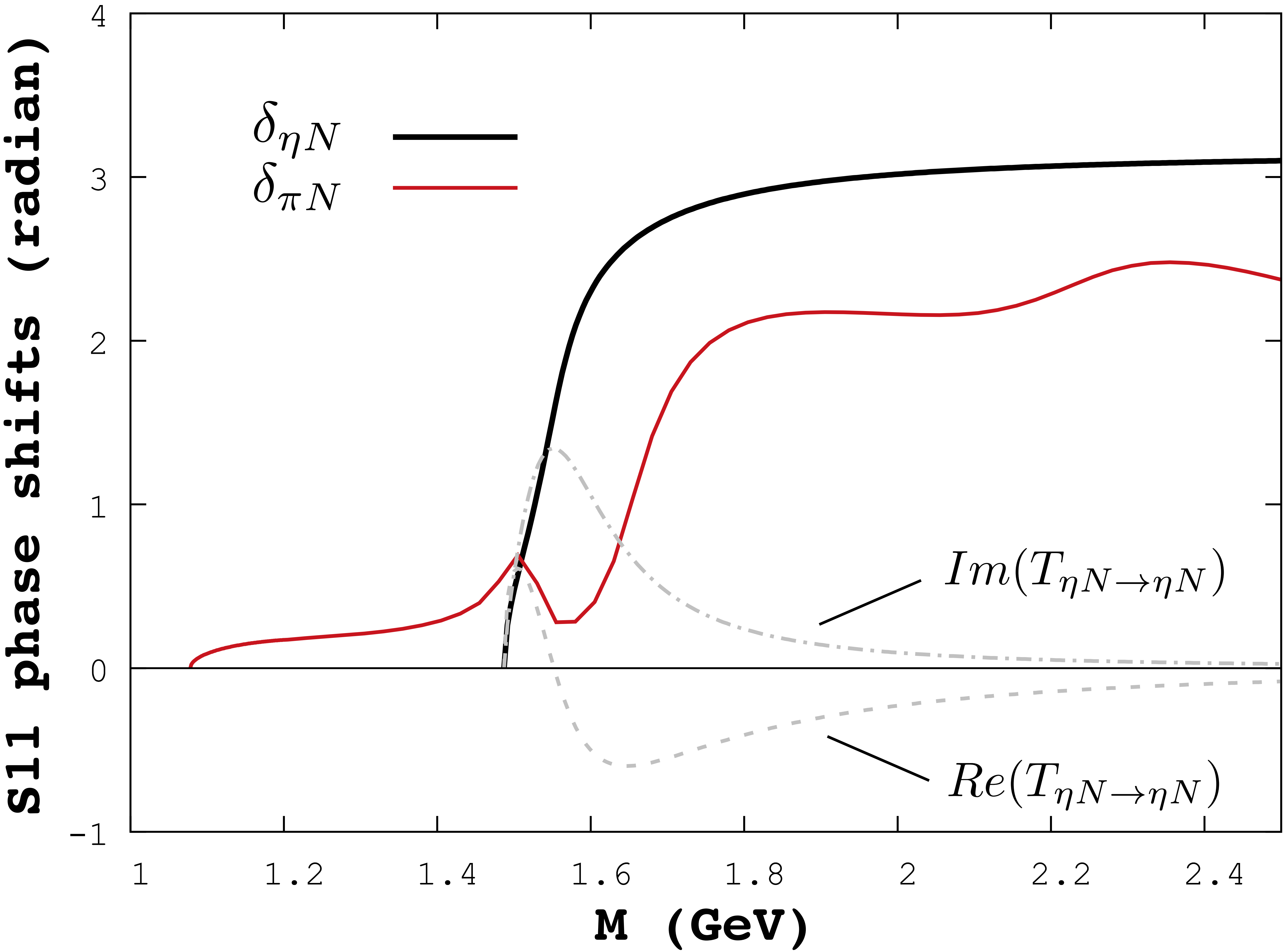
$$S = \begin{pmatrix} \eta\, e^{2\,i\,\delta_I} & i\sqrt{1-\eta^2}\, e^{i\,(\delta_I+\delta_{II})} \\ i\sqrt{1-\eta^2}\, e^{i\,(\delta_I+\delta_{II})} & \eta\, e^{2\,i\,\delta_{II}} \end{pmatrix}$$

$$\begin{aligned} \mathcal{Q}(M) &\equiv \frac{1}{2}\,\mathrm{Im}\,(\mathrm{tr}\ln\,\mathrm{S}) \\ &= \frac{1}{2}\,\mathrm{Im}\,(\ln\,\det\,[\mathrm{S}]) \\ &= \delta_I + \delta_{II}. \end{aligned}$$

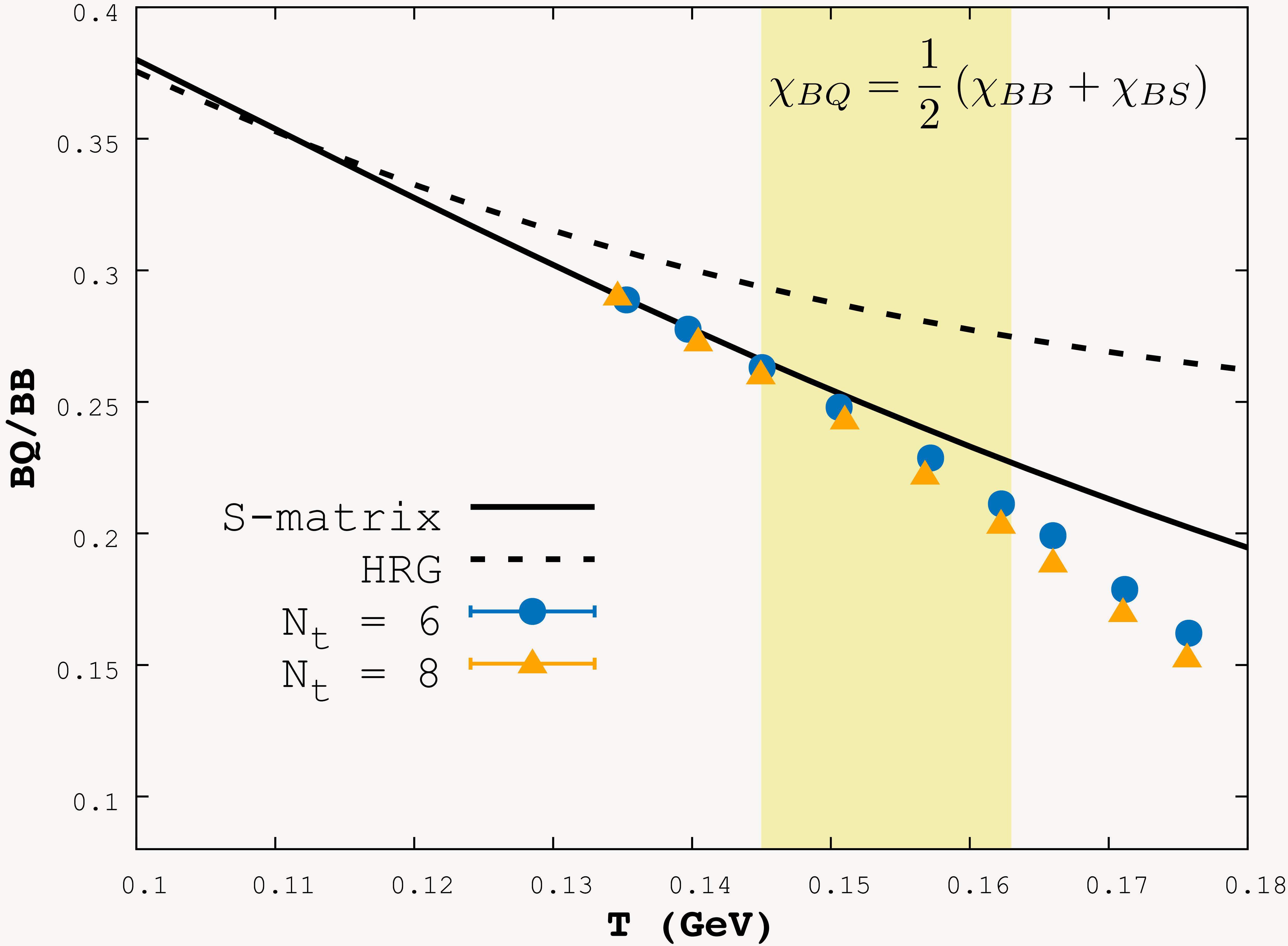
πN *system*

$$\pi N \to \begin{pmatrix} \pi N \\ \eta N \end{pmatrix} \to \pi N$$

$$\pi\eta \to \begin{pmatrix} \pi N \\ \eta N \end{pmatrix} \to \pi\eta$$



S11 phase shifts (radian)
M (GeV)
$\delta_{\eta N}$
$\delta_{\pi N}$
$Im(T_{\eta N\to\eta N})$
$Re(T_{\eta N\to\eta N})$
4
3
2
1
0
-1
1
1.2
1.4
1.6
1.8
2
2.2
2.4



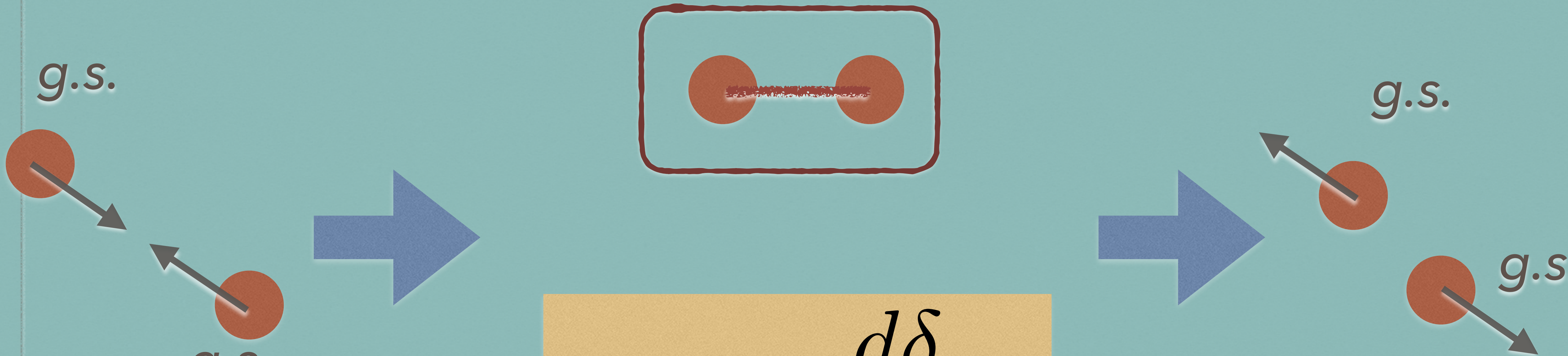
$\chi_{BQ} = \frac{1}{2}(\chi_{BB} + \chi_{BS})$
S-matrix
HRG
N_t = 6
N_t = 8
BQ/BB
T (GeV)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18

TIME DELAY

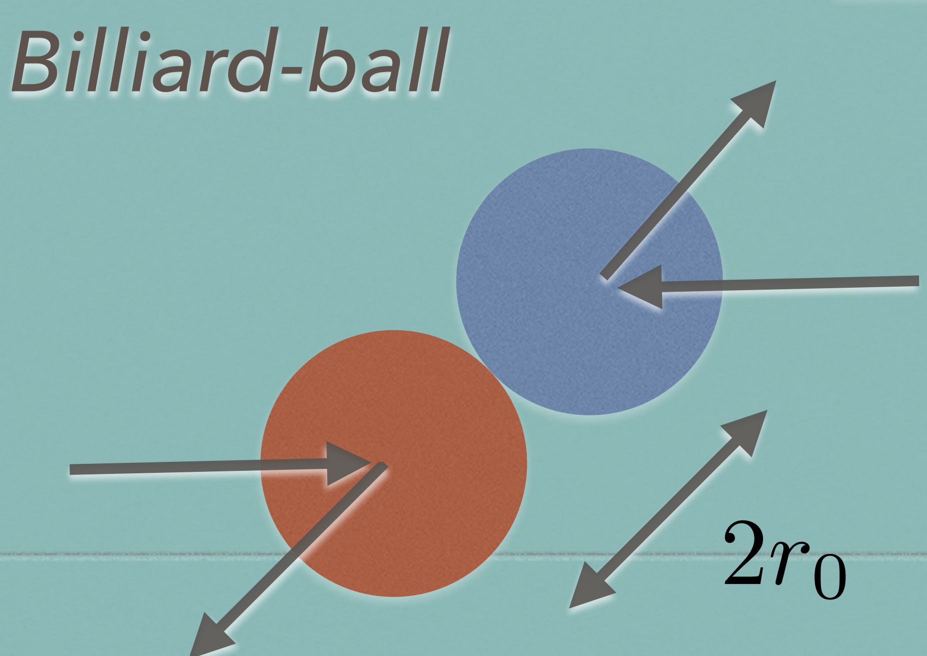
P. Danielewicz and S. Pratt
Phys.Rev. C53 (1996) 249-266

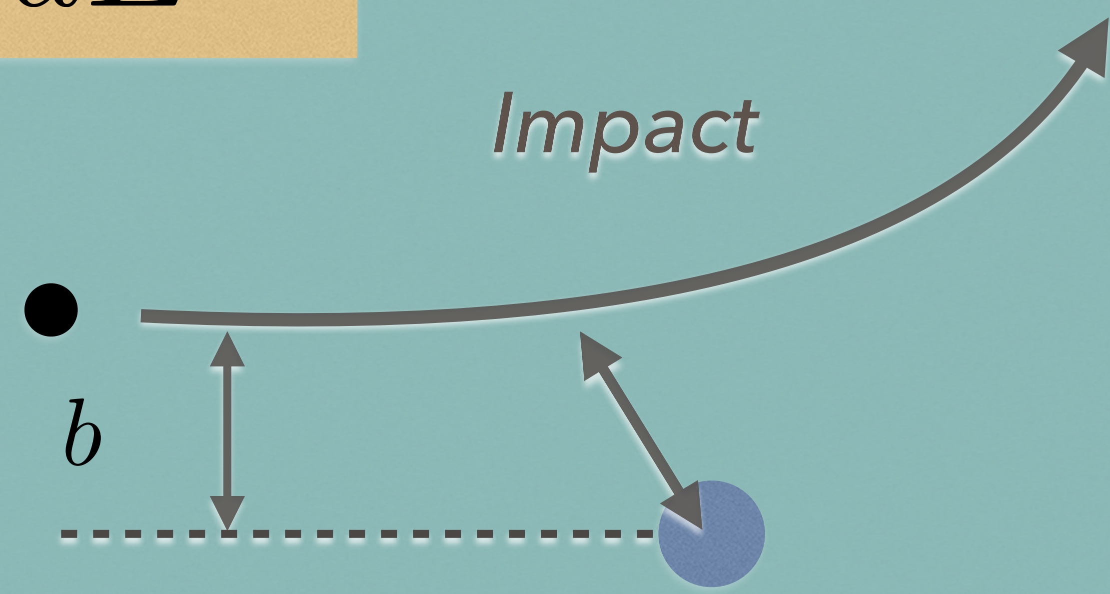
S. Leupold
Nucl.Phys. A695 (2001) 377-394

Yu. B. Ivanov et al
Phys.Atom.Nucl.64:652-669,2001



$$\tau \sim 2\frac{d\delta}{dE}$$





SUMMARY

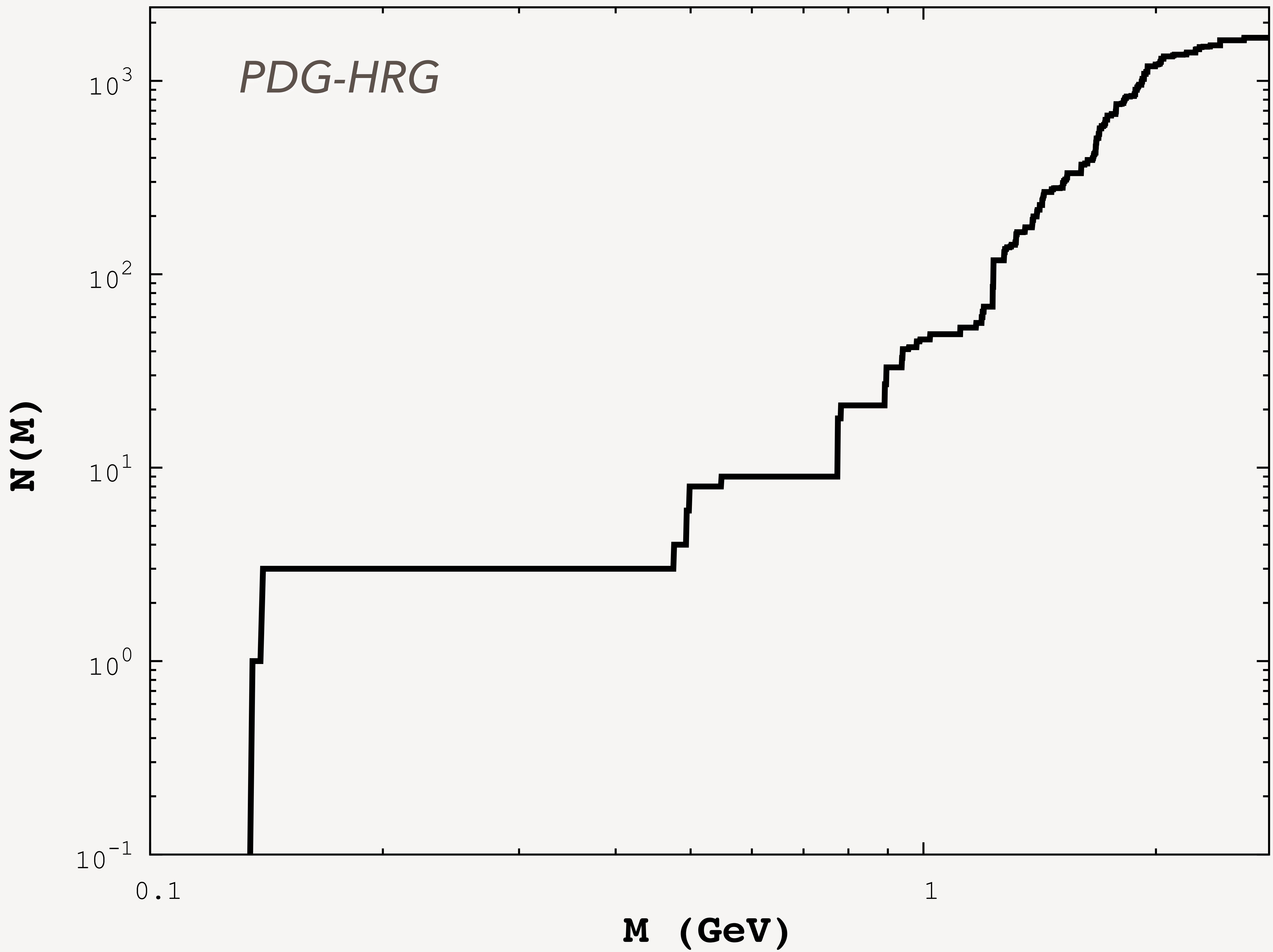
- change in density of state / time delay

$$2\frac{d\delta}{dE}$$

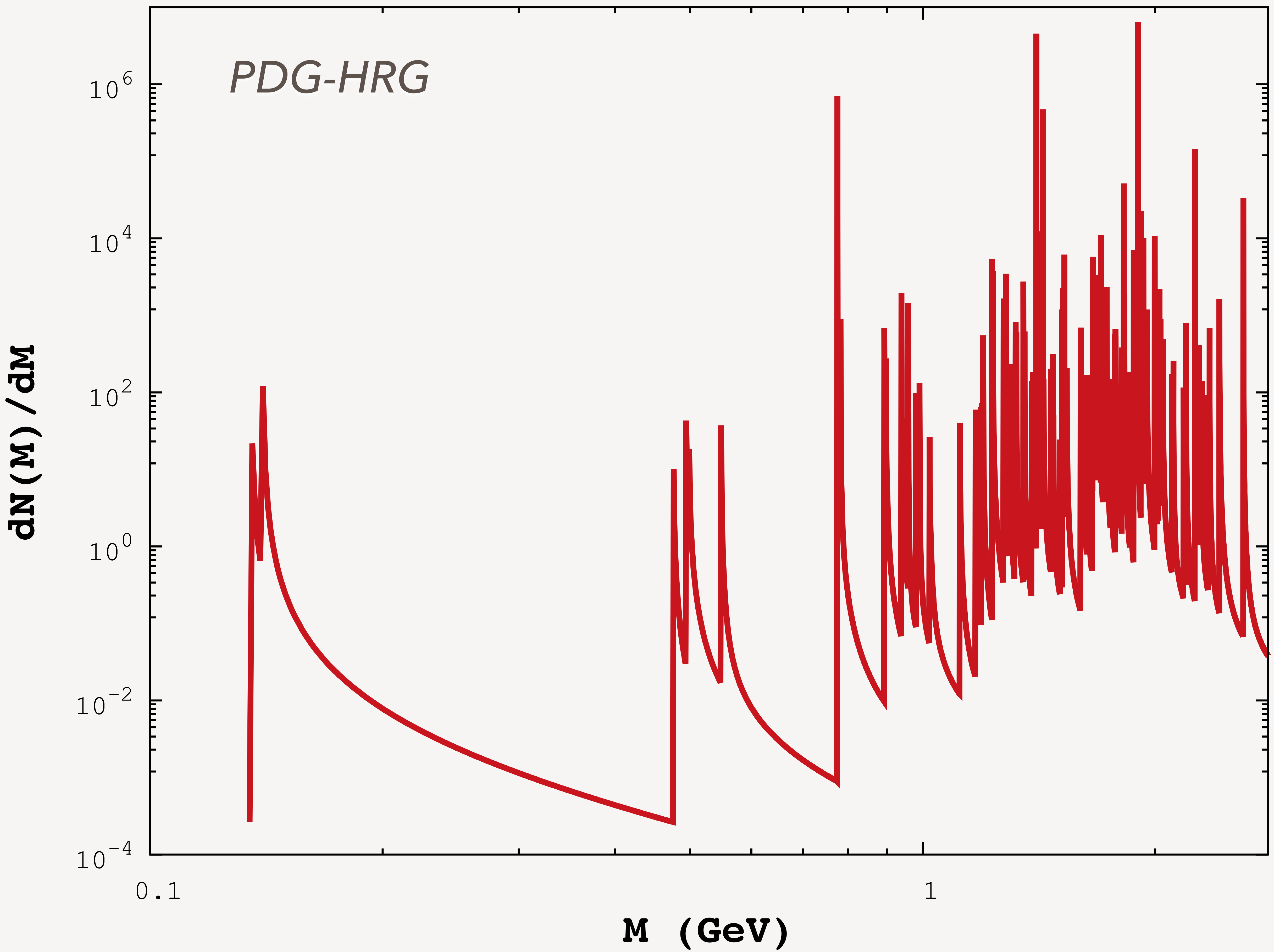
- S-matrix approach to thermodynamics

Broad resonances

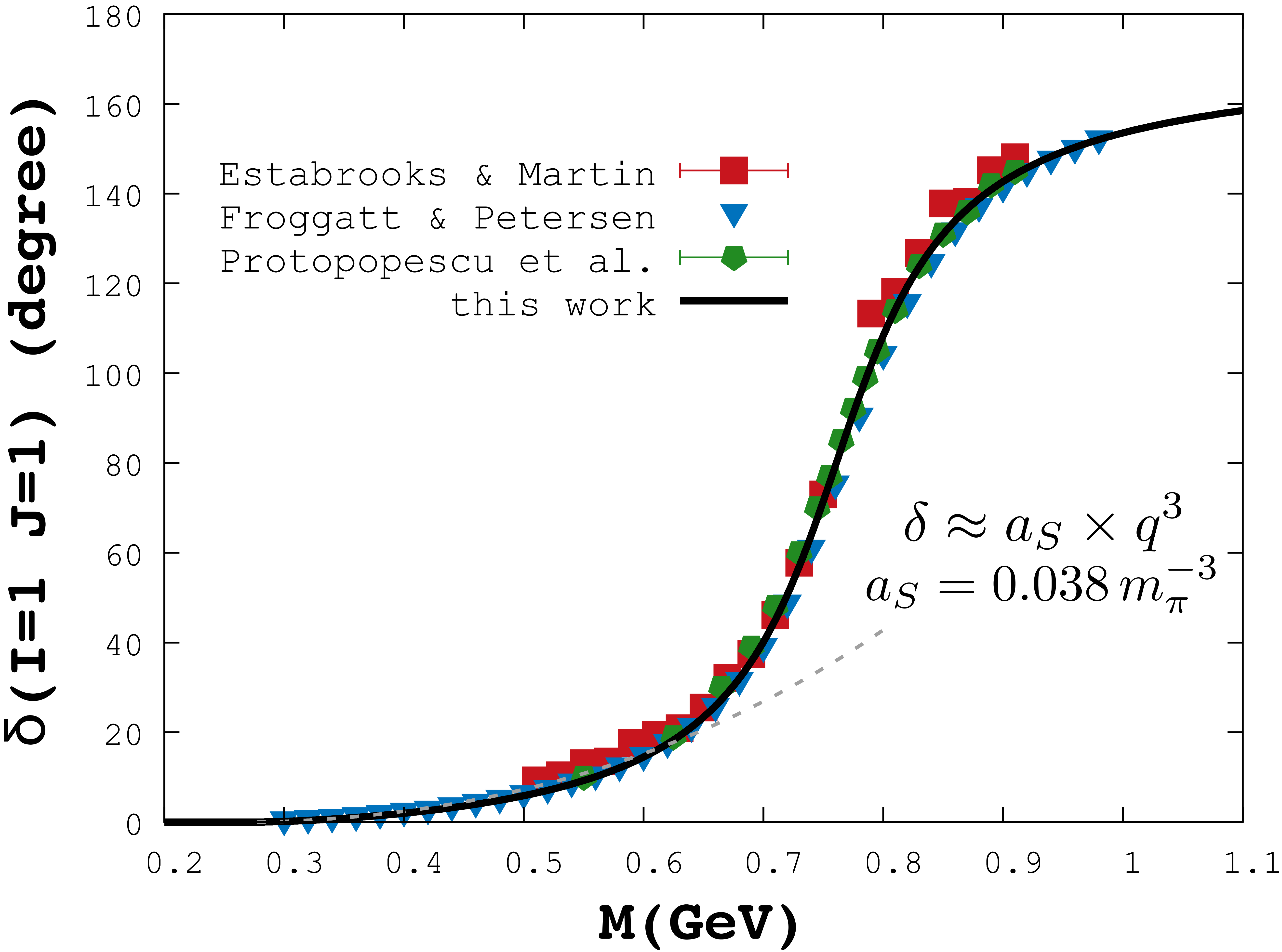
Repulsive channels



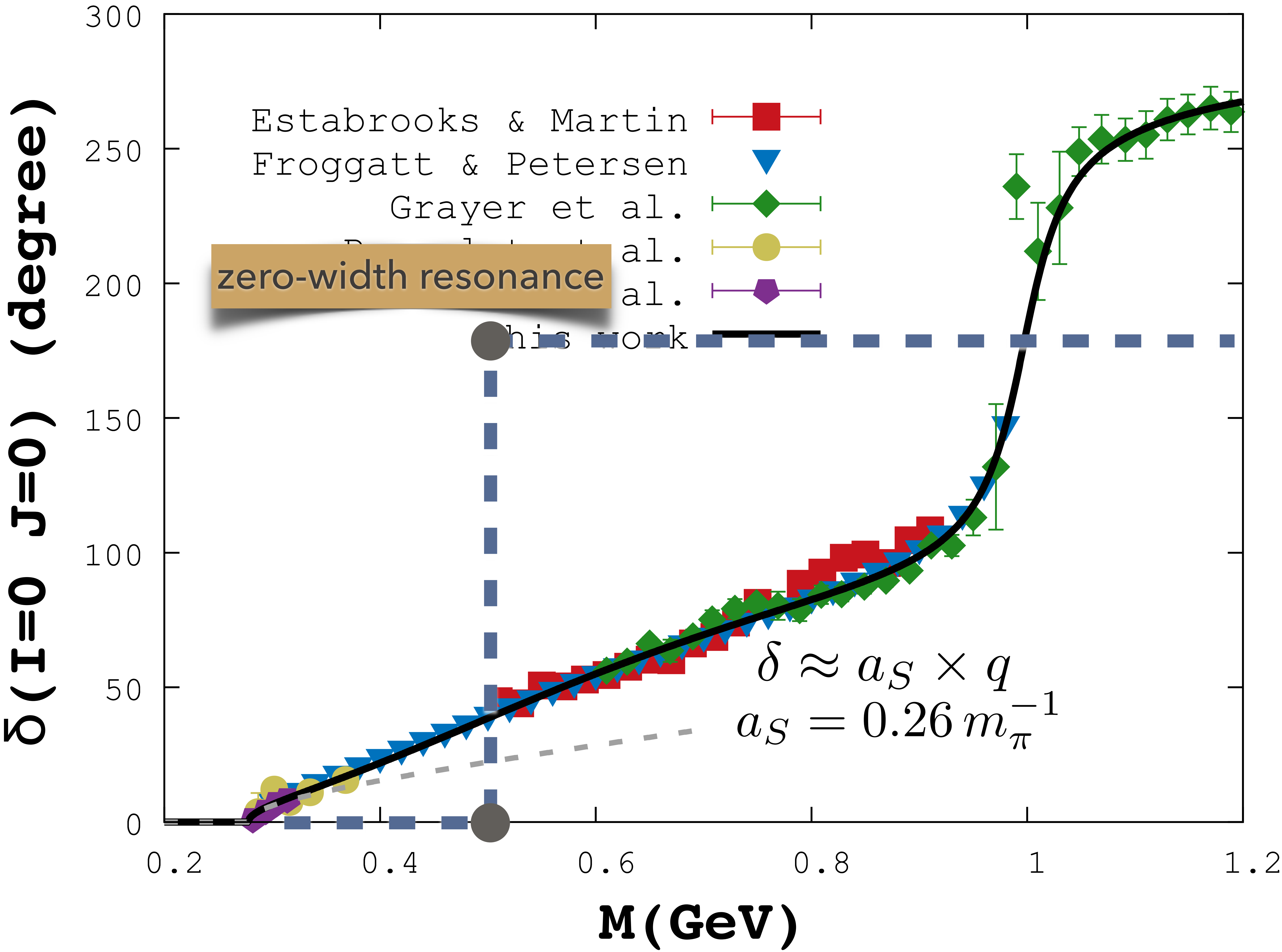
PDG-HRG
10^3
10^2
10^1
10^0
10^-1
N(M)
0.1
1
M (GeV)



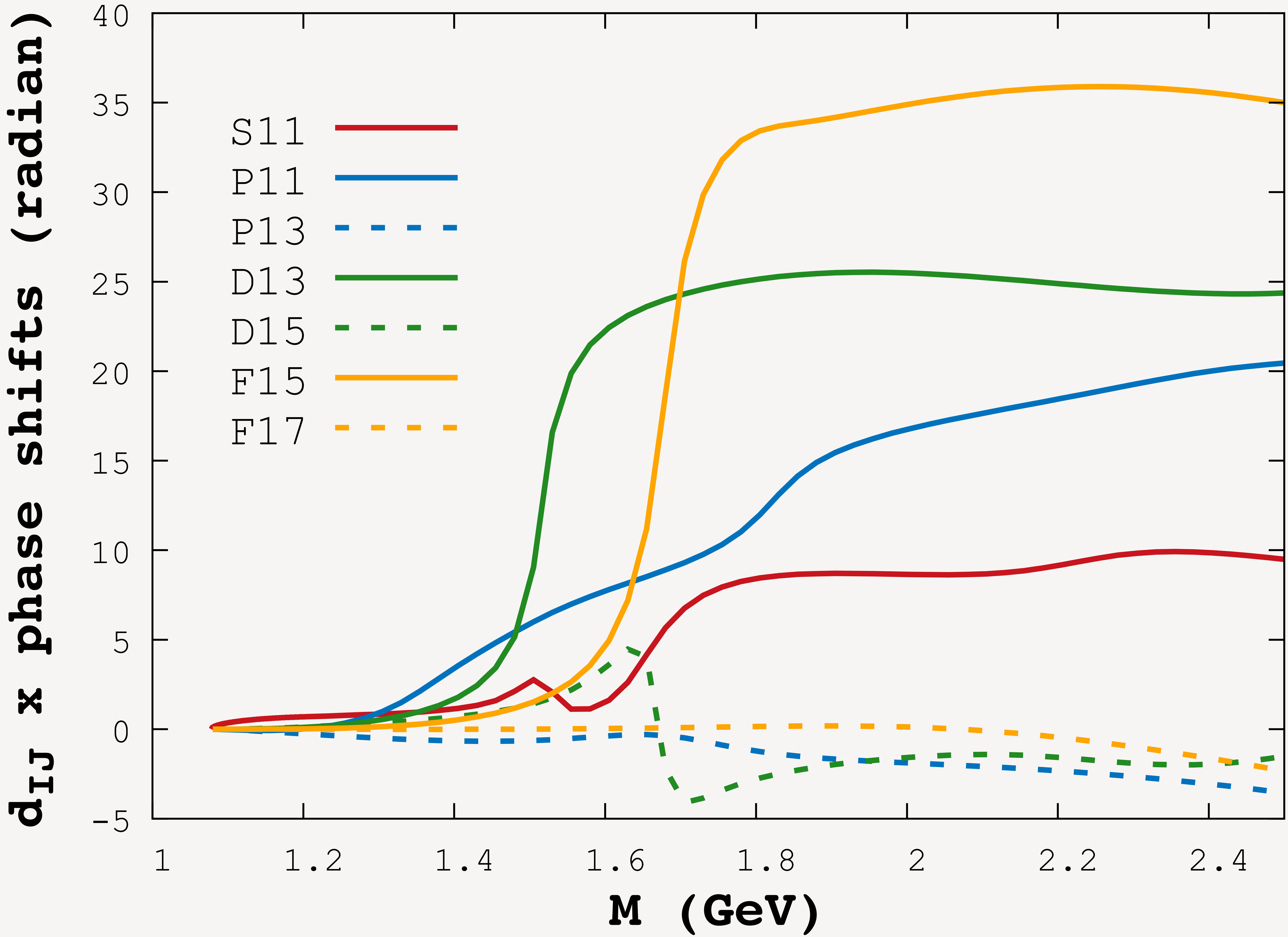
PDG-HRG
10^6
10^4
10^2
10^0
10^-2
10^-4
0.1
1
dN(M)/dM
M (GeV)



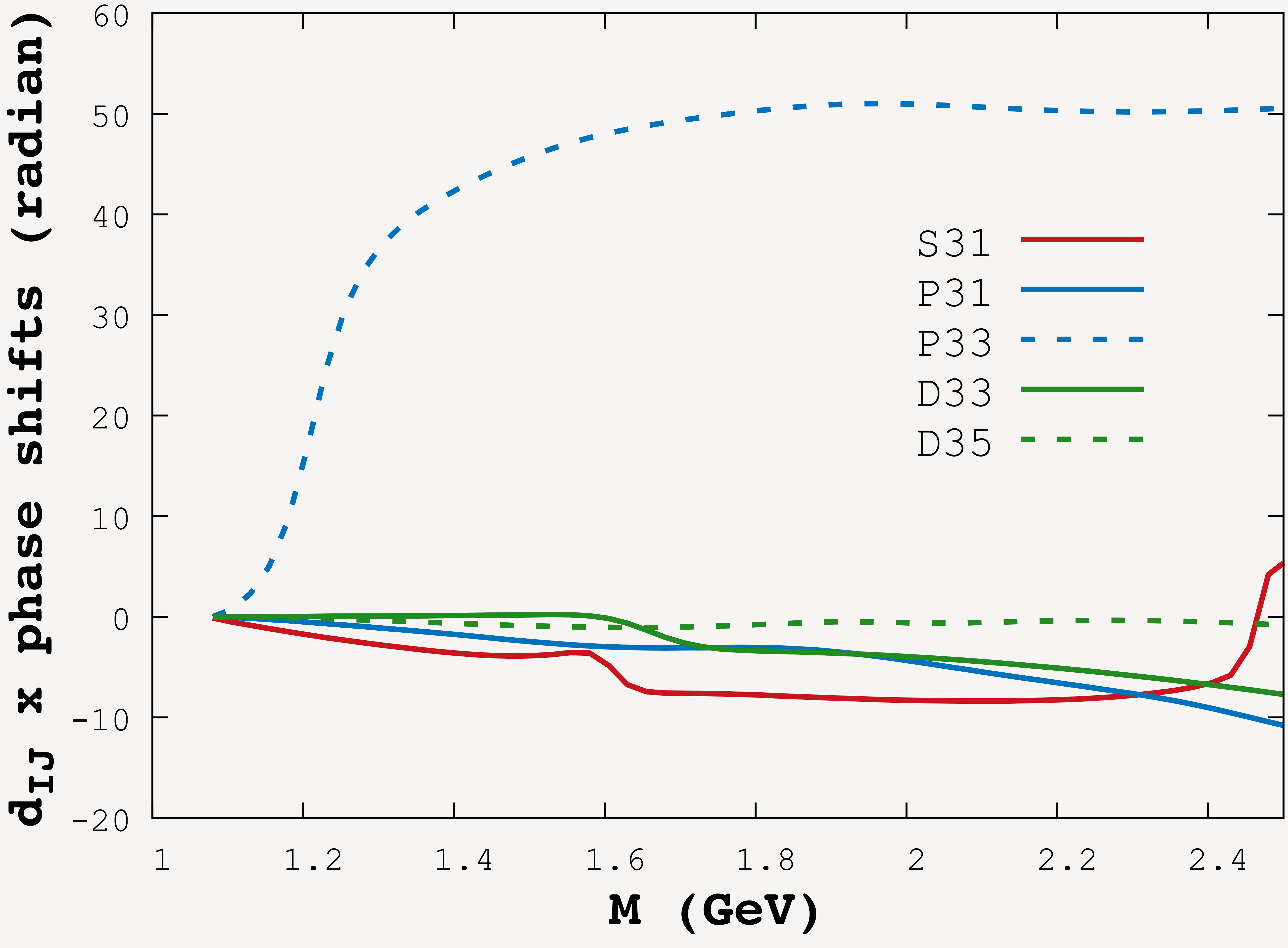
Estabrooks & Martin
Froggatt & Petersen
Protopopescu et al.
this work
$\delta \approx a_S \times q^3$
$a_S = 0.038\, m_\pi^{-3}$
δ(I=1 J=1) (degree)
M(GeV)
0
20
40
60
80
100
120
140
160
180
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1



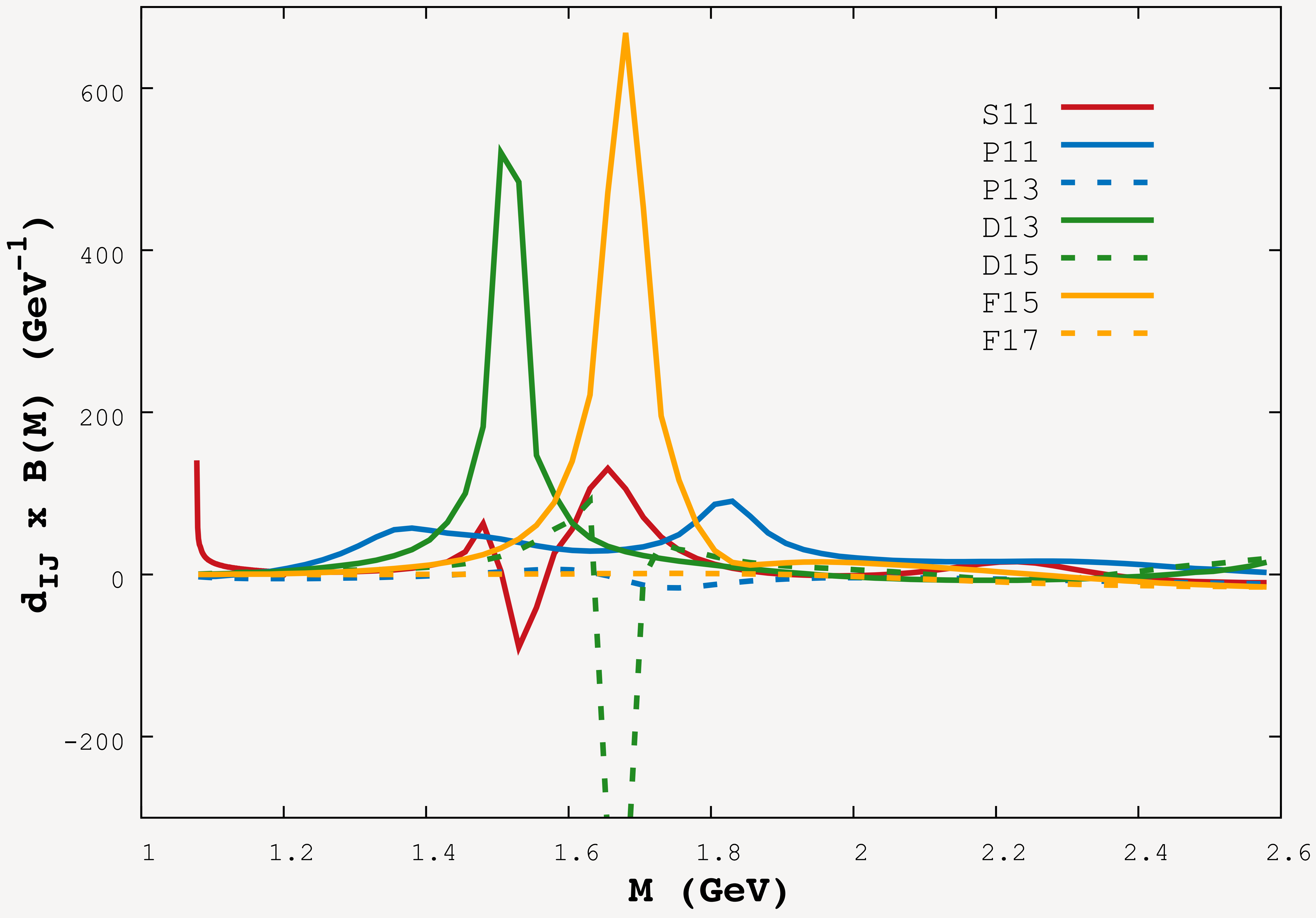
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
al.
al.
zero-width resonance
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
δ(I=0 J=0) (degree)
M(GeV)
300
250
200
150
100
50
0
0.2
0.4
0.6
0.8
1
1.2



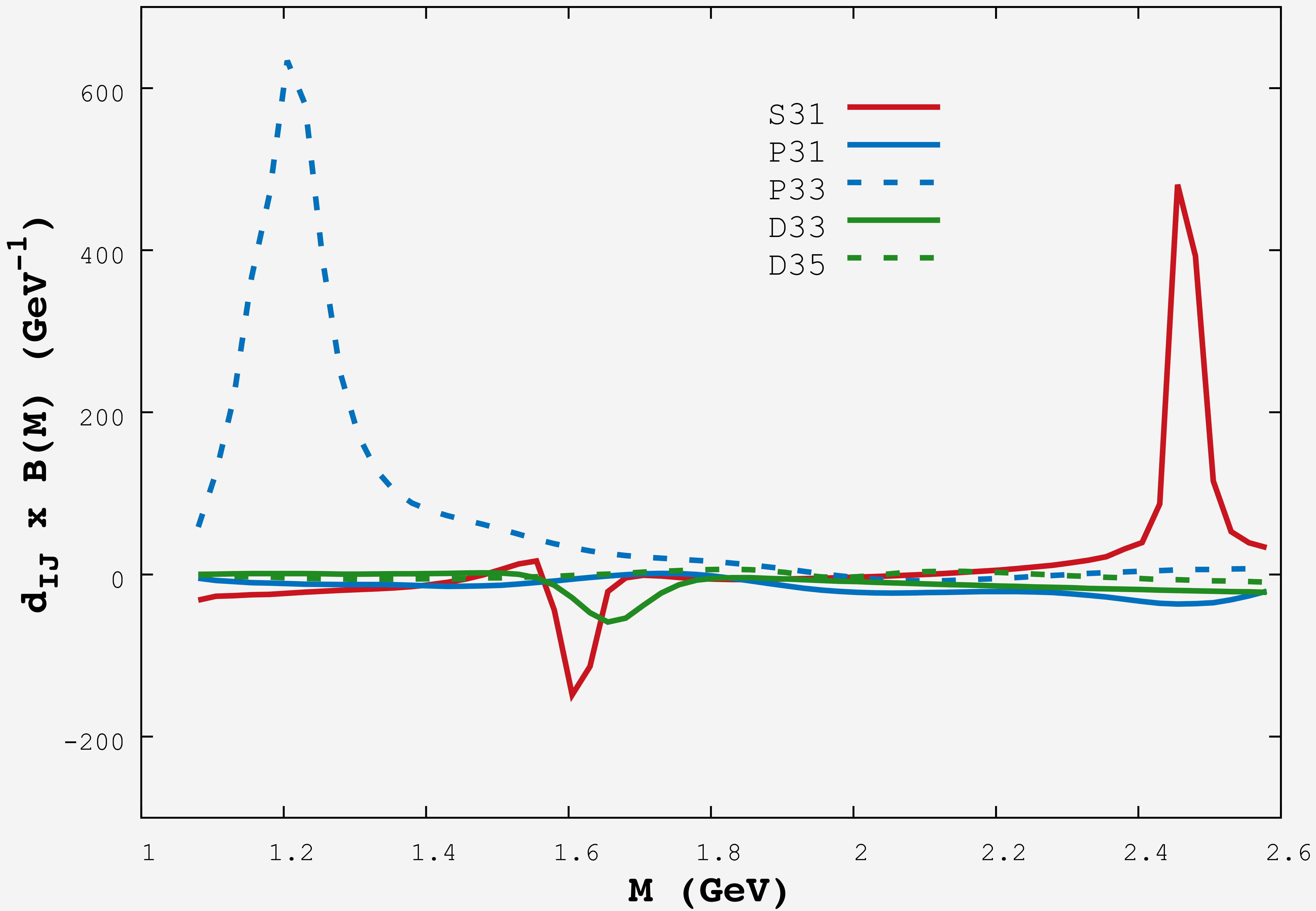
d_{IJ} x phase shifts (radian)
M (GeV)
S11
P11
P13
D13
D15
F15
F17
40
35
30
25
20
15
10
5
0
-5
1
1.2
1.4
1.6
1.8
2
2.2
2.4



60
50
40
30
20
10
0
−10
−20
1
1.2
1.4
1.6
1.8
2
2.2
2.4
M (GeV)
d_{IJ} x phase shifts (radian)
S31
P31
P33
D33
D35



S11
P11
P13
D13
D15
F15
F17
600
400
200
0
-200
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
M (GeV)
d_{IJ} x B(M) (GeV^{-1})



S31
P31
P33
D33
D35
600
400
200
0
-200
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
M (GeV)
d_{IJ} x B(M) (GeV^{-1})

N-BODY SCATTERING

PML, Eur. Phys. J. C **77** no.8 533 (2017)

WHY N-BODY?

- EOS for dense system
  -> need higher coefficients of
  quantum cluster / virial expansion
  (three-body forces, etc.)
- Explore the influence of N-body scatterings on heavy ion collision observables:
  pT-spectra, flow etc.
- phenomenology
  -> model S-matrix element instead…

RECIPE

Feynman amplitude

- generalized phase shift

$$\mathcal{Q}_N(M) = \frac{1}{2}\,\mathrm{Im}\left[\ln\left(1 + \int d\phi_N\, i\mathcal{M}\right)\right]$$

$$d\phi_N = \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}\cdots\frac{d^3p_N}{(2\pi)^3}\frac{1}{2E_N}\times$$
$$(2\pi)^4\,\delta^4(P - \sum_i p_i).$$

phase space approach

PHASE SPACE DOMINANCE

$$\mathcal{Q}_N(M) = \frac{1}{2}\,\mathrm{Im}\left[\ln\left(1 + \int d\phi_N\, i\mathcal{M}\right)\right]$$

- structureless scattering

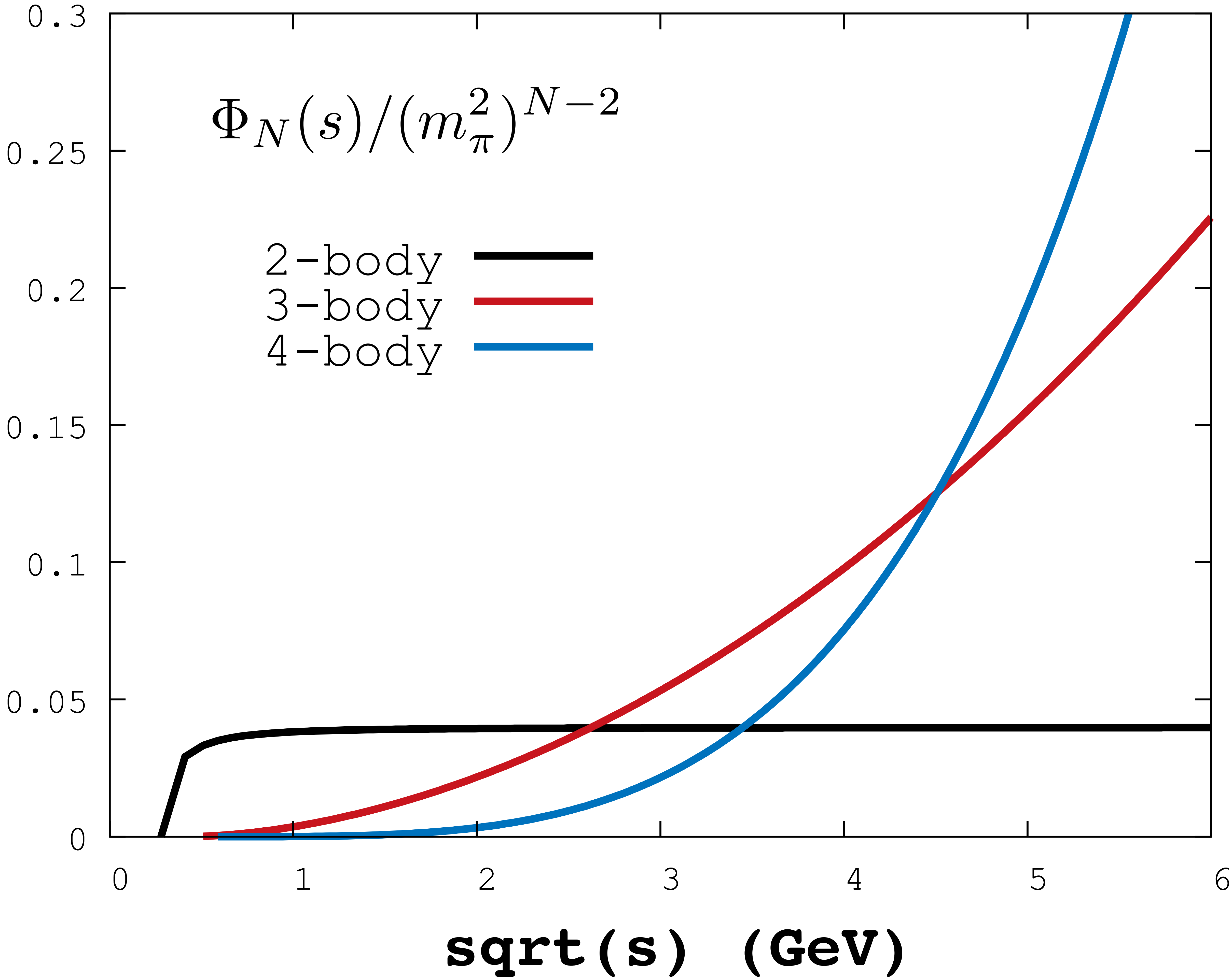
Dimension: $\sim E^{2N-4}$

$$i\mathcal{M} = i\,\lambda_N$$

Källén triangle function

$$\phi_N(s) = \frac{1}{16\,\pi^2 s}\int_{s'_-}^{s'_+} ds'\,\sqrt{\lambda(s, s', m_N^2)}\times$$

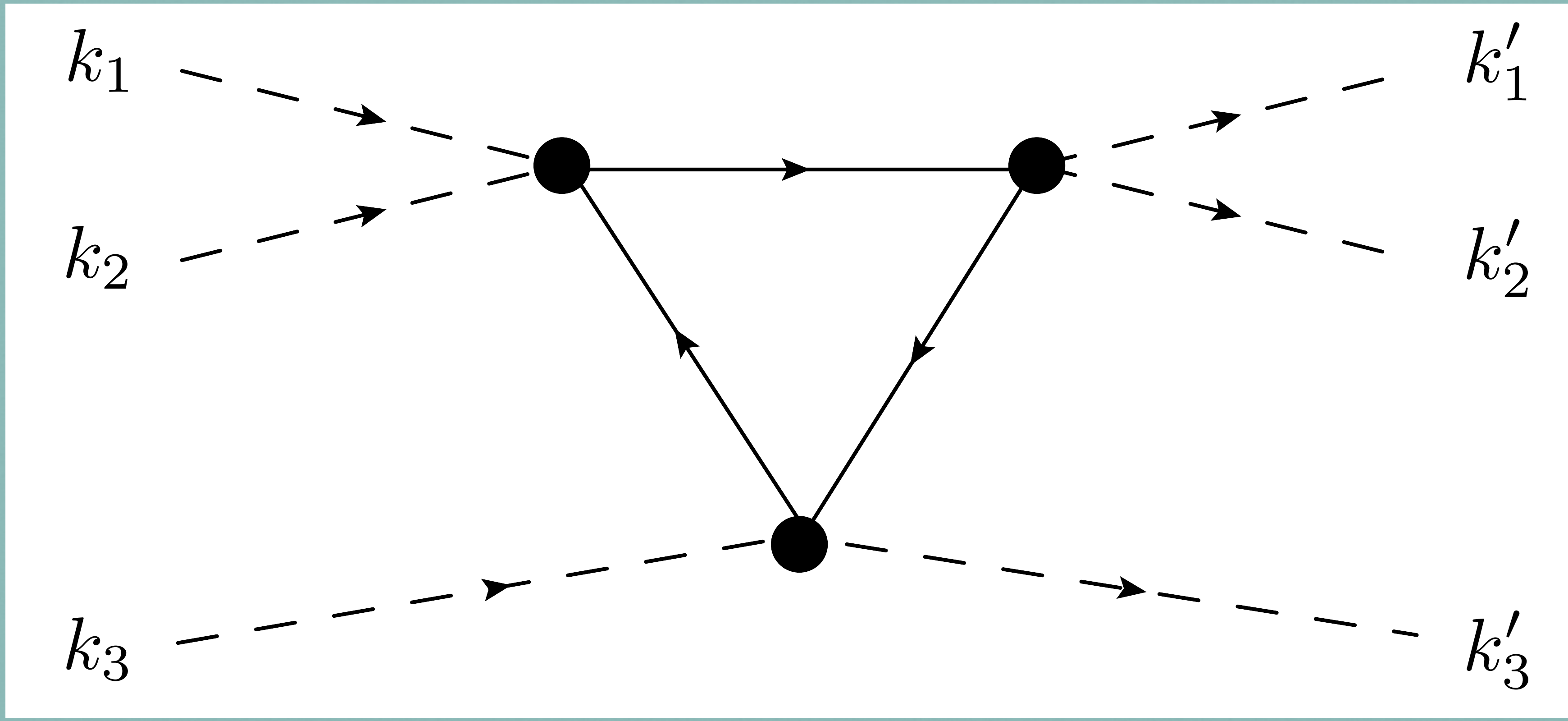
$$\phi_{N-1}(s', m_1^2, m_2^2, ..., m_{N-1}^2)$$



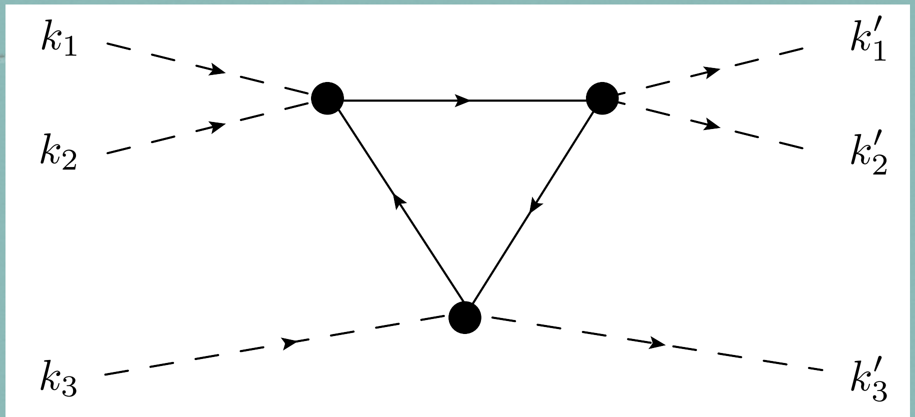
$\Phi_N(s)/(m_\pi^2)^{N-2}$
2-body
3-body
4-body
0
0.05
0.1
0.15
0.2
0.25
0.3
0
1
2
3
4
5
6
sqrt(s) (GeV)

TRIANGLE DIAGRAM

- 3-body diagram



Explicit calculation



$$i\mathcal{M}^{\triangle}(Q_1, Q_2, Q_3) = \int \frac{d^4 l}{(2\pi)^4} \times (-i\,\lambda)^3 \times i\,G(l) \times i\,G(l+Q_1) \times i\,G(l-Q_2)$$

↓ *Feynman's trick + dim reg.*

$$i\mathcal{M}^{\triangle}(Q_1^2, Q_2^2, s = P_I^2) = -i\frac{\lambda^3}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \frac{1}{\Delta(x,y)}$$

$$\Delta(x,y) = m_\pi^2 - x(1-x)\,Q_1^2 - y(1-y)\,Q_2^2 \\ - 2\,xy\,Q_1 \cdot Q_2 - i\epsilon.$$

- to lowest order $\mathcal{Q}(s) \approx \frac{1}{2}\,\mathrm{Im}\left[\int d\phi_3\, i\mathcal{M}^{\mathrm{triangle}}\right],$

=> only need to deal with on-shell condition

$$k_i' = k_i$$

analytic result:

$$i\mathcal{M}^{\triangle,o.s.}(Q_1^2, s) = -i\,\frac{\lambda^3}{16\,\pi^2}\,\frac{z}{Q_1^2}\,\ln\frac{1-z}{1+z}$$

$$z = \frac{1}{\sqrt{1-\frac{4m_\pi^2}{Q_1^2}}}.$$

$$\mathcal{Q}(s) \approx \frac{1}{2}\,\mathrm{Im}\left[\int d\phi_3\, i\mathcal{M}^{\text{triangle}}\right],$$

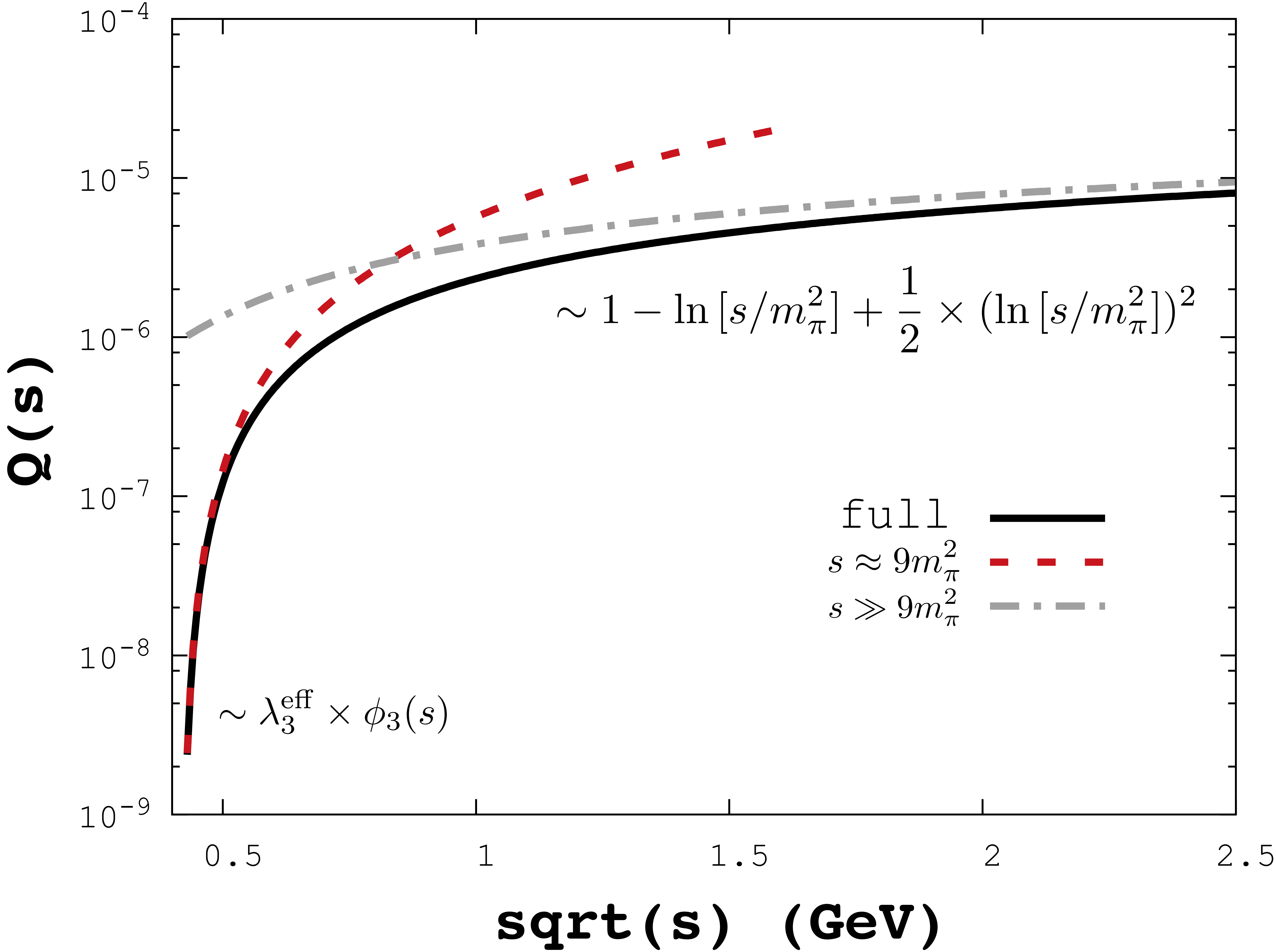
Limits:

$s \to 9m_\pi^2$ $$\mathcal{Q}(s) \approx \frac{1}{2} \times \lambda_3^{\text{eff}} \times \phi_3(s).$$

$s \gg 9m_\pi^2$ $$\mathcal{Q}(s) \approx \frac{\lambda^3}{8192\,\pi^5} \int_{\xi_0}^{1} d\xi\,(\frac{1}{\xi} - 1)\,\left[-z\,\ln\left|\frac{1-z}{1+z}\right|\right]$$

$$\approx \frac{\lambda^3}{4096\,\pi^5} \times \left[1 + \ln\frac{\xi_0}{4} + (\ln\frac{\xi_0}{4})^2\right]$$

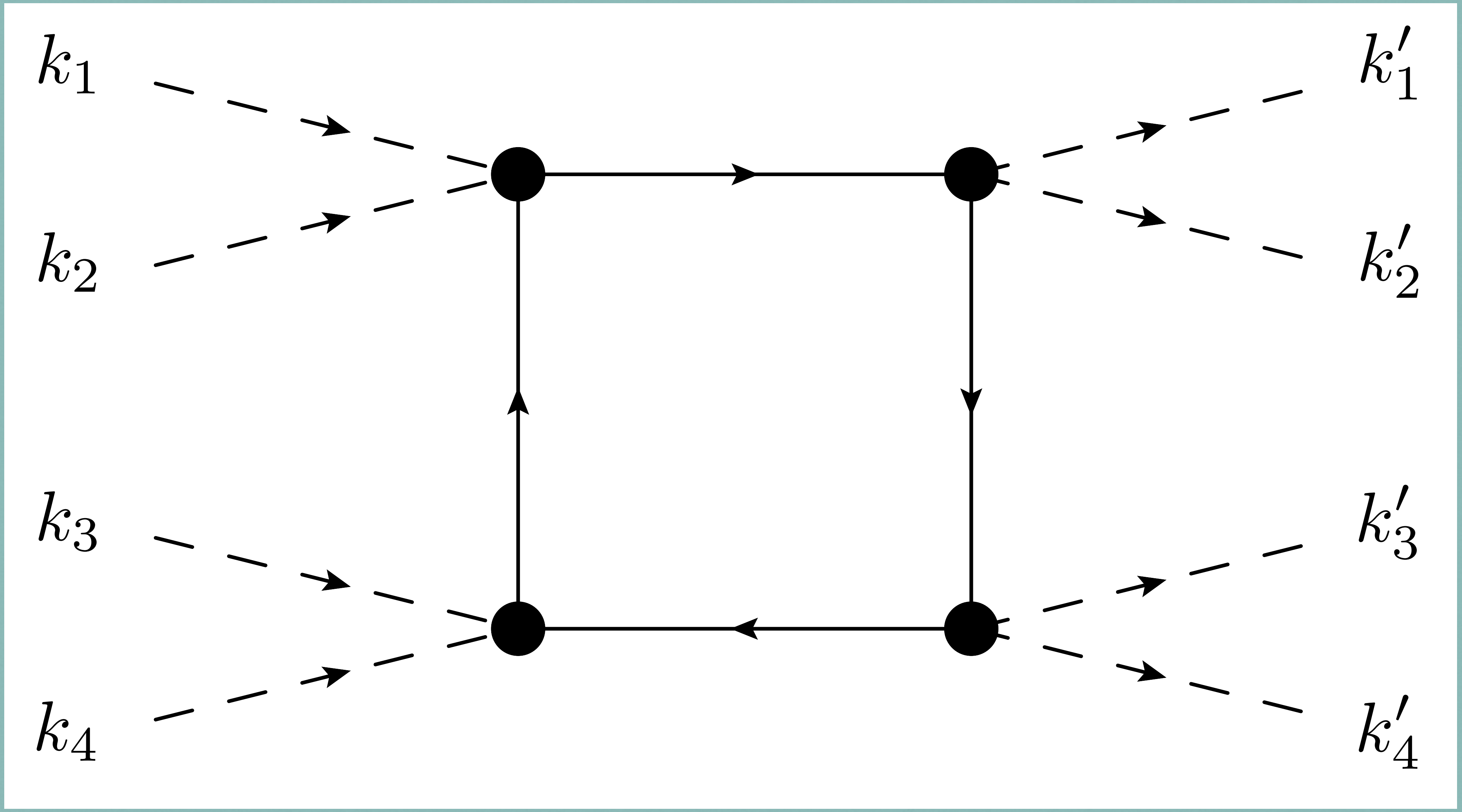
where $$\xi_0 = \frac{4m_\pi^2}{s}.$$



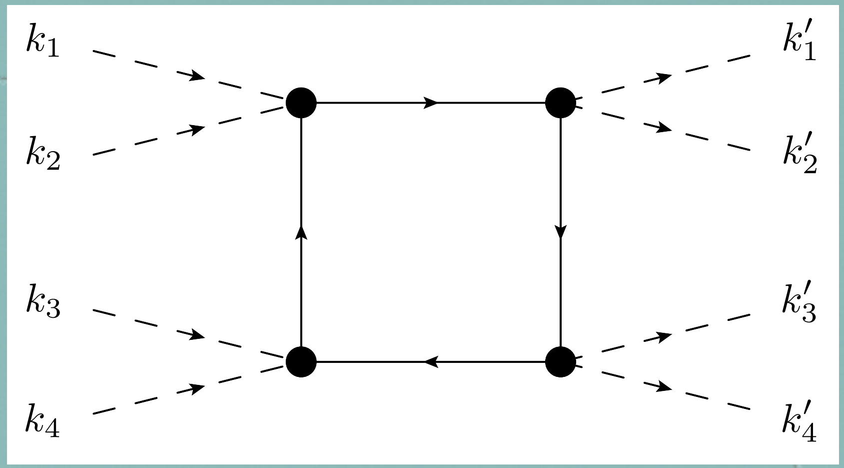
Q(s)
sqrt(s) (GeV)
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
0.5
1
1.5
2
2.5
$\sim 1 - \ln[s/m_\pi^2] + \frac{1}{2} \times (\ln[s/m_\pi^2])^2$
$\sim \lambda_3^{\rm eff} \times \phi_3(s)$
full
$s \approx 9m_\pi^2$
$s \gg 9m_\pi^2$

BOX DIAGRAM

- 4-body diagram



Explicit calculation



$$i\mathcal{M}^{\rm box}(Q_1,Q_2,Q_3,Q_4) = \int \frac{d^4l}{(2\pi)^4}\,(-i\,\lambda)^4 \times i\,G(l) \times i\,G(l+Q_1)$$
$$\times\, i\,G(l+Q_1-Q_3) \times i\,G(l-Q_2)$$

Feynman's trick + dim reg.

$$i\mathcal{M}^{\rm box}(Q_1,Q_2,Q_3,Q_4) = i\frac{\lambda^4}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\,\times$$
$$\int_0^{1-x-y} dz \,\times \left(\frac{1}{\Delta(x,y,z)}\right)^2$$

$$\mathcal{Q}(s) \approx \frac{1}{2}\,\mathrm{Im}\left[\int d\phi_4\, i\mathcal{M}^{\mathrm{box,o.s.}}\right].$$

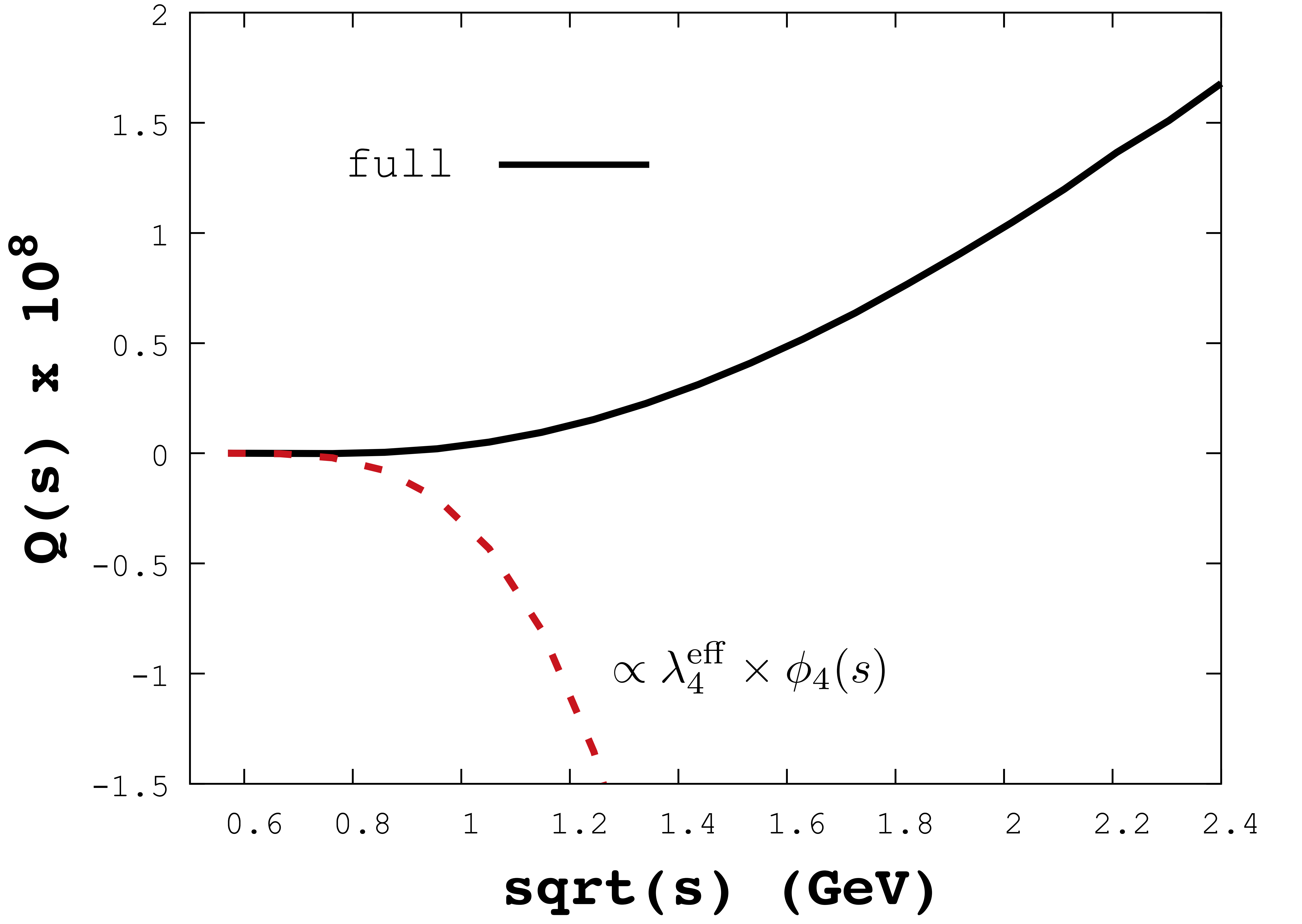
Limits: $s \to 16\, m_\pi^2$

$$\mathrm{Im}\,\left(i\mathcal{M}^{\mathrm{box,o.s.}}(q_1^2, q_2^2, s)\right) \approx \lambda_4^{\mathrm{eff}}$$

$$\lambda_4^{\mathrm{eff}} = \frac{\lambda^4}{256\pi^2}\frac{1}{m_\pi^4} \times (\frac{\sqrt{3}}{2}\,\ln(7 - 4\sqrt{3}) + 2)$$ *Negative!*

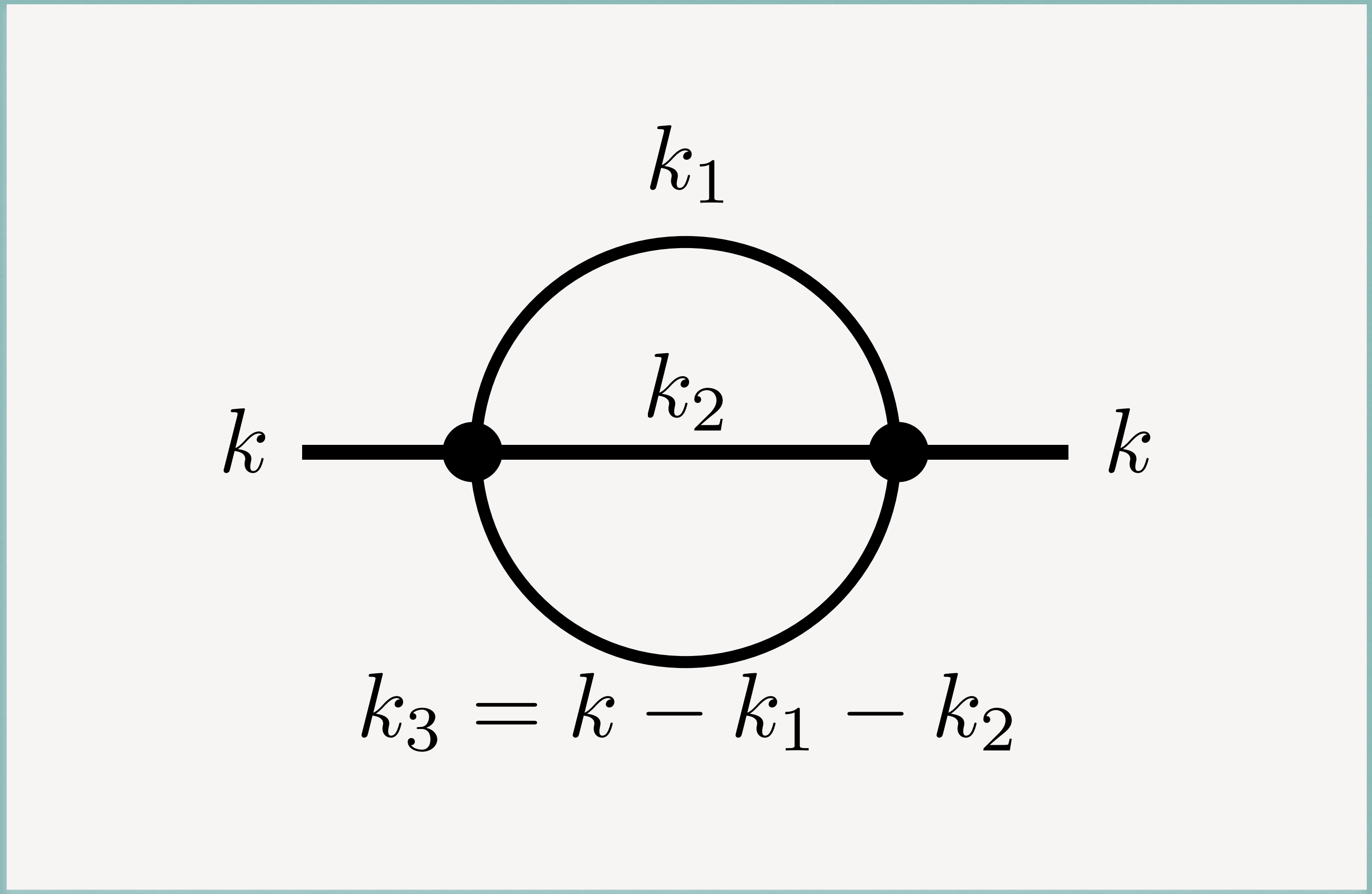
$$\mathcal{Q}(s) \approx \frac{1}{2} \times \lambda_4^{\mathrm{eff}} \times \phi_4(s).$$

$s >> 16\, m_\pi^2$???



Q(s) x 10^8
sqrt(s) (GeV)
full
$\propto \lambda_4^{\rm eff} \times \phi_4(s)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4

SUNSET DIAGRAM



$$\mathrm{Im}\, I \propto \frac{1}{2} \int d\phi_3 \left|\Gamma_{s\to\pi\pi\pi}\right|^2$$

T-MATRIX REPRESENTATION

$$\frac{1}{4\,i}\,\mathrm{tr}\,\left[S^{-1}\,\frac{\overleftrightarrow{\partial}}{\partial E}\,S\right]_c \longleftrightarrow \frac{\partial\delta_E}{\partial E}$$

$$\frac{1}{4}\,\frac{\partial}{\partial E}\,\mathrm{tr}\,\left[T+T^\dagger\right]_c \longleftrightarrow (1-2\sin^2\delta_E)\times\frac{\partial\delta_E}{\partial E}$$

$$\frac{1}{4\,i}\,\mathrm{tr}\,\left(T^\dagger\,\frac{\overleftrightarrow{\partial}}{\partial E}\,T\right)_c \longleftrightarrow 2\sin^2\delta_E\times\frac{\partial\delta_E}{\partial E}.$$

Landau Lifshitz classification

GOING FURTHER

- Inelasticity & higher virial terms

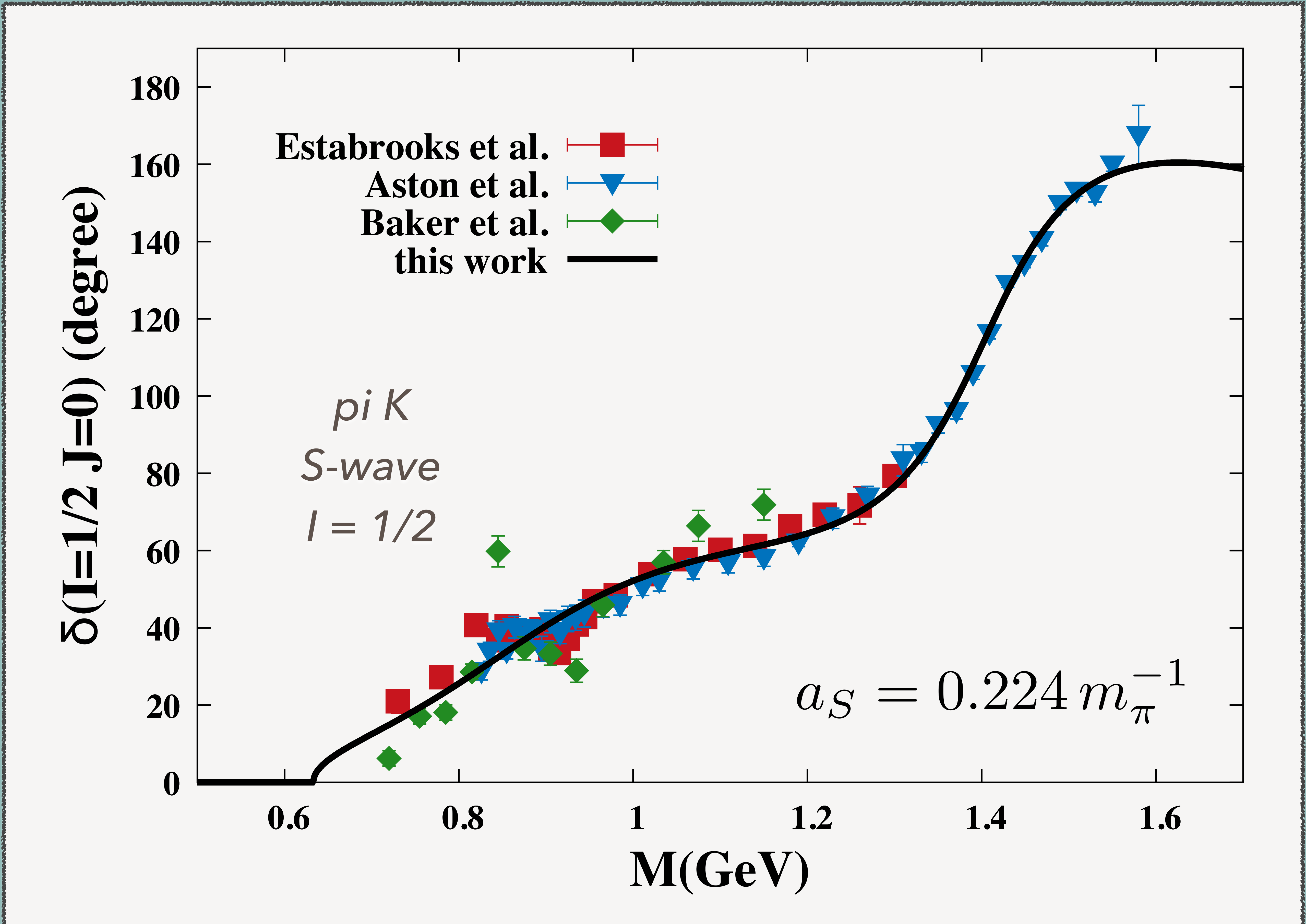
  => isobar approach

- S-matrix as a theoretical framework:

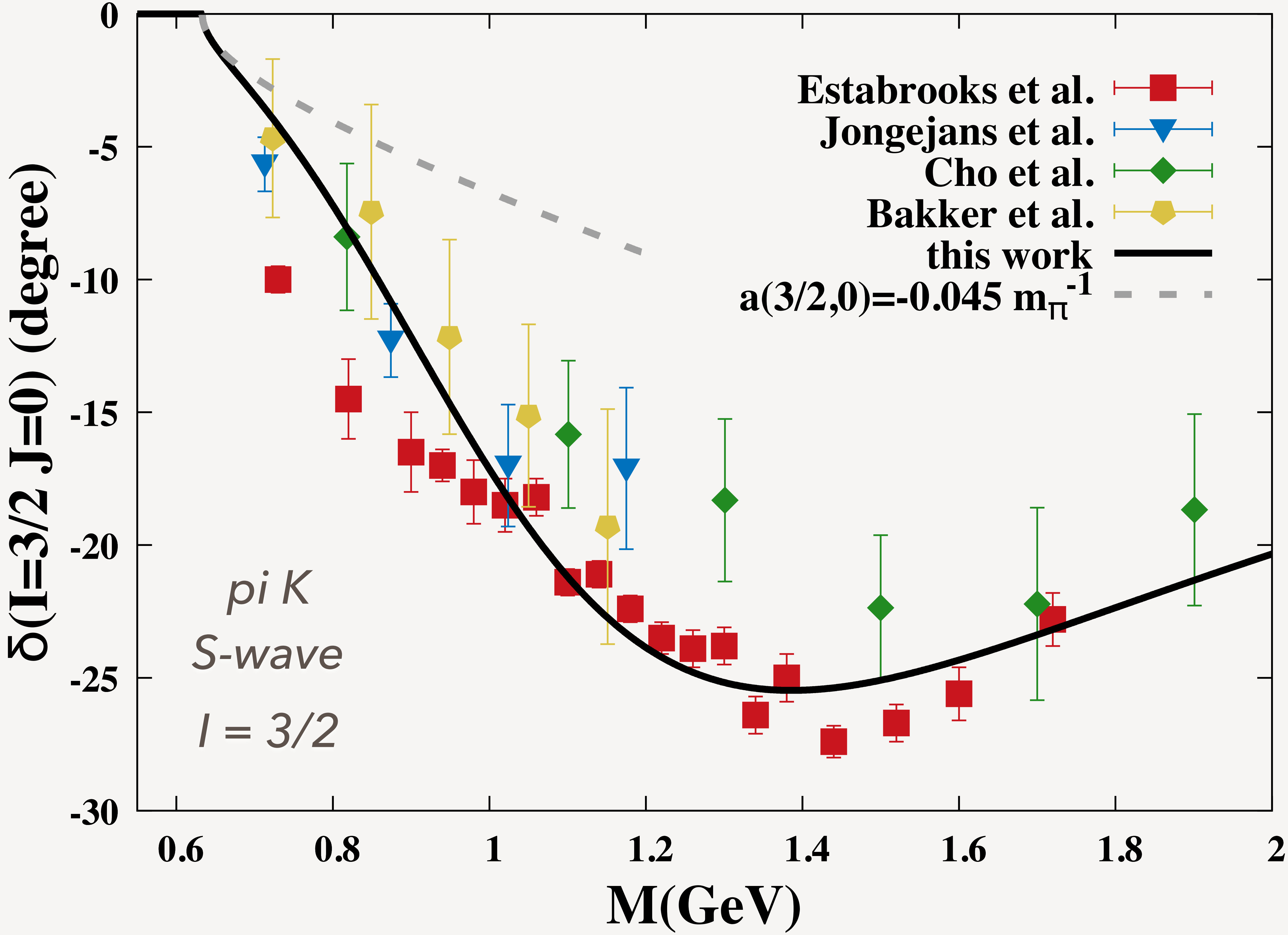
  B versus A, kinetic theory
  CJT / NPI
  optical potential, in-medium properties,
  thermal amplitudes & all that

THANK YOU

PI K SCATTERING



Estabrooks et al.
Aston et al.
Baker et al.
this work
pi K
S-wave
I = 1/2
$a_S = 0.224\, m_\pi^{-1}$
δ(I=1/2 J=0) (degree)
M(GeV)
0
20
40
60
80
100
120
140
160
180
0.6
0.8
1
1.2
1.4
1.6



Estabrooks et al.
Jongejans et al.
Cho et al.
Bakker et al.
this work
a(3/2,0)=-0.045 m_π^-1
δ(I=3/2 J=0) (degree)
M(GeV)
pi K
S-wave
I = 3/2
0
-5
-10
-15
-20
-25
-30
0.6
0.8
1
1.2
1.4
1.6
1.8
2