Hydrodynamic Fluctuations in Heavy-Ion Collisions

in collaboration with Joseph Kapusta
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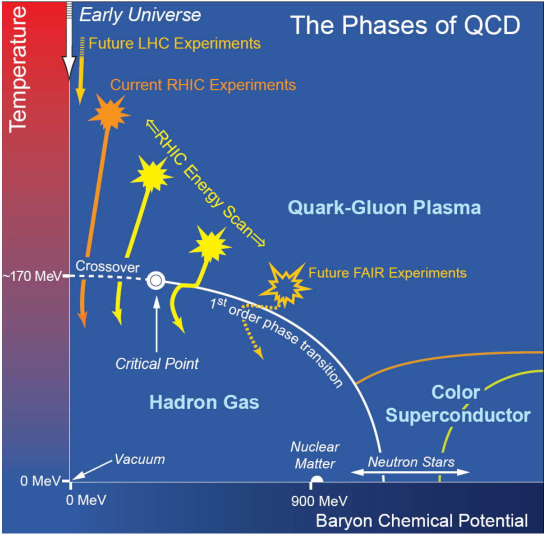
Outline

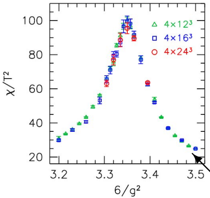
- Introduction: QCD phase diagram and critical point
- A way to detect the critical point: transport coefficients
- What are the hydrodynamic fluctuations?
- Correlation of hydrodynamic fluctuations
- How to measure them?
- Illustrative model
- Conclusions

Introduction: QCD phase diagram and critical point

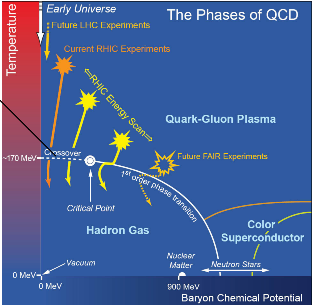


Introduction: QCD phase diagram and critical point

Crossover at $\mu_B \simeq 0$: well-known by lattice-QCD calculations



Y. Aoki *et al.* (2006)



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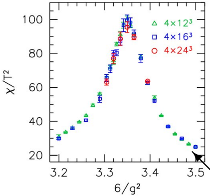

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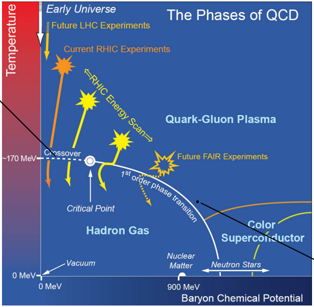
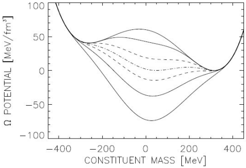


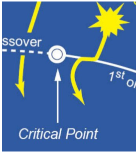
First order transition at $\mu_B > T$.
Effective field theory models: Nambu-Jona-Lasinio model, linear sigma model, effective potential model...



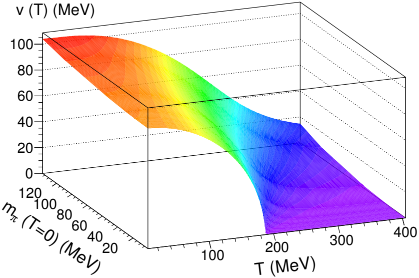
O. Scavenius *et al.* (2000)

Introduction: QCD phase diagram and critical point

Non-zero baryonic chemical potential



Second-order phase transition: characterized by a continuous order parameter

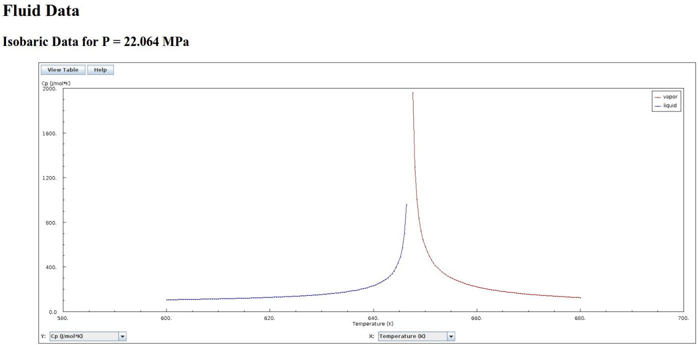


A. Dobado and JMT-R (2012)

Current and future experimental programmes:

- RHIC (New York) working at low energy
- FAIR (Darmstadt)
- NA61/SHINE (CERN)
- NICA (Dubna)

Critical behavior



Water at the liquid-gas critical point.
Taken from the National Institute of Standards and Technology

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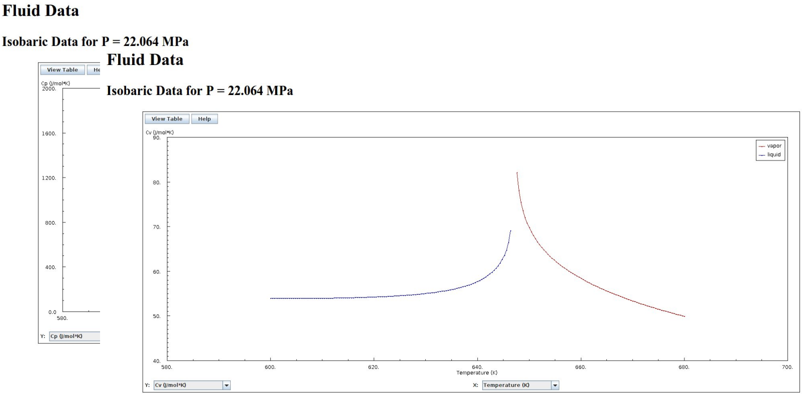

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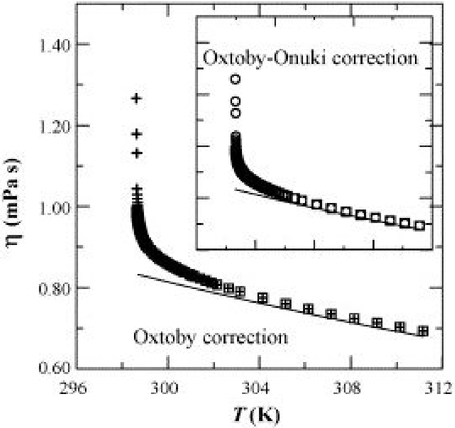
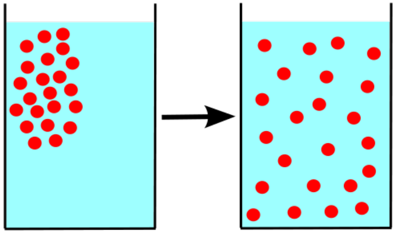


Souto-Carideet *al.* (2006) Binary mixture of dimethyl carbonate + dodecane

A way to detect the critical point: transport coefficients

$$\text{Response Flow} = -\text{Transport Coeff.} \times \text{Gradient of Hydro. Field}$$

A transport coefficient relates the gradient of some hydrodynamic field (velocity, temperature, chemical potential...) with the flux of a conserved current which tries to restore the equilibrium in the system.



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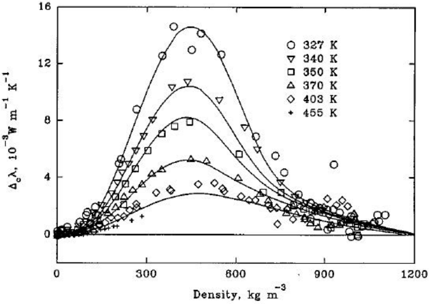
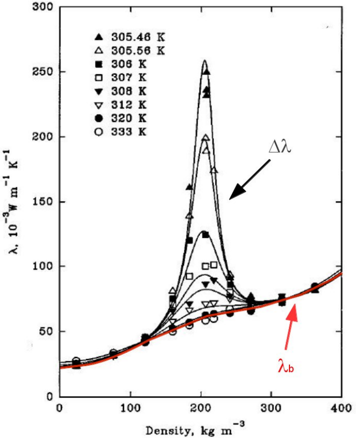


Carbon dioxide

Theoretical curves using mode-coupling theory. J. Luettmer-Strathmann *et al.* (1995)



Ethane $\lambda = \lambda_b + \Delta\lambda$

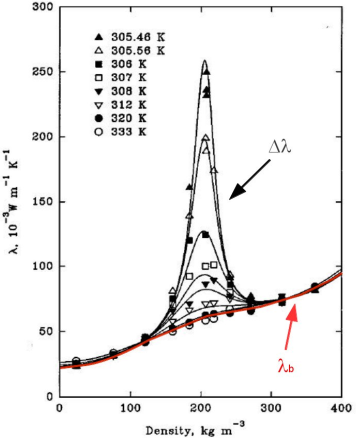
A way to detect the critical point: thermal conductivity

$$\text{Heat Flux} = -\lambda \times \text{Gradient of temperature}$$

$$\lambda = \lambda_b + \Delta\lambda$$

λ_b: Boltzmann equation or Green-Kubo relation (microscopic details)

$\Delta\lambda$: (Extended) mode-coupling theory (no microscopic details, universality)



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Transport coefficients near a critical point

We will focus on transport coefficients as indicators of critical behavior.

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How to study the transport coefficients?

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How to study the transport coefficients?

HYDRODYNAMIC FLUCTUATIONS !

(They basically are fluctuations of the energy-momentum tensor $T^{\mu\nu}$ and the baryonic current J_B^μ)

Relativistic Hydrodynamics

Hydrodynamic fields in $d+1$ dimensions

$$T^{\mu\nu}(t, x_1, x_2, ..., x_d) \qquad J^\mu_B(x)$$

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Equations of motion

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu J^\mu_B = 0$$

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This is true on average. Quantities can fluctuate due to microscopical evolution in the phase space. $P = \langle P \rangle + \delta P, ...$

Zeroth order hydrodynamics: Ideal terms

Energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{ideal}$$

$$T^{\mu\nu}_{ideal} = (P + \epsilon) u^\mu u^\nu - P g^{\mu\nu}$$

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Baryon current

$$J^\mu_B = J^\mu_{B\ ideal}$$

$$J^\mu_{B\ ideal} = n u^\mu$$

Dissipative terms

Energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{ideal} + \tau^{\mu\nu}$$

$$\tau^{\mu\nu} = \eta_S(\Delta^\mu u^\nu + \Delta^\nu u^\mu) + \left(\zeta - \frac{2}{3}\eta_S\right) h^{\mu\nu} \partial_\rho u^\rho$$

$$\Delta^\mu = -h^{\mu\nu}\partial_\nu \qquad h^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}$$

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Baryon current

$$J^\mu_B = J^\mu_{B\ \ ideal} + \Delta J^\mu_B$$

$$\Delta J^\mu_B = \lambda \left(\frac{nT}{w}\right)^2 \Delta^\mu(\mu_B/T)$$

Fluctuating terms

Energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{ideal} + \tau^{\mu\nu} + S^{\mu\nu}$$

$$\langle S^{\mu\nu}(x) \rangle = 0$$

$$\langle S^{\mu\nu}(x_1) S^{\alpha\beta}(x_2) \rangle = 2T \left[\eta_S (h^{\mu\alpha} h^{\nu\beta} + h^{\mu\beta} h^{\nu\alpha}) + (\zeta - \frac{2}{3}\eta_S) h^{\mu\nu} h^{\alpha\beta} \right] \delta(x_1 - x_2)$$

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Baryon current

$$J^{\mu}_{B} = J^{\mu}_{B\ ideal} + \Delta J^{\mu}_{B} + I^{\mu}$$

$$\langle I^{\mu}(x) \rangle = 0$$

$$\langle I^{\mu}(x_1) I^{\nu}(x_2) \rangle = 2\lambda \left(\frac{nT}{w} \right)^2 h^{\mu\nu} \delta(x_1 - x_2)$$

Hydrodynamic fluctuations

The study of $S^{\mu\nu}$ including the shear and bulk viscosities at $\mu_B = 0$ was made in Kapusta, Mueller, Stephanov (2012).
In this talk I will consider $\mu_B \neq 0$ and focus on the study of I^{μ}.

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$$\langle I^\mu(x_1) I^\nu(x_2) \rangle = 2\lambda \left(\frac{nT}{w} \right)^2 (u^\mu u^\nu - g^{\mu\nu})\, \delta(x_1 - x_2)$$

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FLUCTUATION-DISSIPATION THEOREM

Close to the critical point, the correlation becomes important due to the critical behavior of λ. Our aim in to quantify this critical behavior and the enhancement of the correlation function in relativistic heavy ion collisions.

Summary so far

1. Close to a critical point fluctuations become important and certain quantities diverge with critical exponents.
2. Fluctuation-dissipation theorem: the hydrodynamic fluctuations are related to transport coefficients.
3. Close to the critical point, transport coefficients may have critical divergencies.
4. Correlation of hydrodynamic fluctuation can help to locate the critical point, as they are enhanced close to it.
5. The correlation function can be used to constraint the values of the transport coefficients.

Theoretical derivation of the correlation function

Fluctuation-Dissipation theorem

$$\langle I^\mu(x_1) I^\nu(x_2) \rangle = 2\lambda \left(\frac{nT}{w} \right)^2 (u^\mu u^\nu - g^{\mu\nu})\, \delta(x_1 - x_2)$$

Theoretical derivation of the correlation function

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$$\langle I^\mu(x_1) I^\nu(x_2)\rangle = 2\lambda \left(\frac{nT}{w}\right)^2 (u^\mu u^\nu - g^{\mu\nu})\, \delta(x_1 - x_2)$$

We want to express this relation in terms of more physical variables. We use the equations of motion

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu J^\mu_B = 0$$

with the simplification

$$\partial_\mu T^{\mu\nu}_{ideal} = 0; \quad \partial_\mu (J^\mu_{B,ideal} + I^\mu) = 0$$

And solve them for a 1+1 dimensional Bjorken expansion

Bjorken model

- 1+1 dimensional expansion, we forget about transverse plane
- $t, z \to \tau = \sqrt{t^2 - z^2}, \quad \xi = \tanh^{-1}(z/t)$

Flow velocity fluctuations

$$u^0 = \cosh(\xi + \omega(\tau, \xi))$$

$$u^3 = \sinh(\xi + \omega(\tau, \xi))$$

Entropy and particle density fluctuations

$$s(\tau, \xi) = \frac{s_i \tau_i}{\tau} + \delta s(\tau, \xi)$$

$$n(\tau, \xi) = \frac{n_i \tau_i}{\tau} + \delta n(\tau, \xi)$$

Linearized equations of motion

$$\partial_\mu T^{\mu\nu} = 0 \ , \qquad \partial_\mu J^\mu = 0$$

We have chosen $\delta s, \delta n, \omega$ as independent variables. To first order in fluctuations:

$$\tau \frac{\partial \delta s}{\partial \tau} + \delta s + s \frac{\partial \omega}{\partial \xi} - \frac{\mu s}{T} \frac{\partial f}{\partial \xi} = 0$$

$$\tau \frac{\partial \delta n}{\partial \tau} + \delta n + n \frac{\partial \omega}{\partial \xi} + s \frac{\partial f}{\partial \xi} = 0$$

$$\tau \frac{\partial \omega}{\partial \tau} + (1 - v_\sigma^2)\omega + \frac{v_n^2 T}{w} \frac{\partial \delta s}{\partial \xi} + \frac{v_s^2 \mu_B}{w} \frac{\partial \delta n}{\partial \xi} = 0$$

where the noise is given by $I^0 = s(\tau) f(\tau, \xi) \sinh \xi, I^3 = s(\tau) f(\tau, \xi) \cosh \xi$

Legend: T=temperature; μ_B=chemical potential; w=enthalpy density; v_σ, v_n, v_s=adiabatic,isochoric and isoentropic speeds of sound

Langevin Equation

$$X \equiv \{\delta s, \delta n, \omega\}$$

In Fourier space they form a Langevin equation

$$\tau \frac{\partial \tilde{X}}{\partial \tau} + \mathbf{D}\tilde{X} + \tilde{\mathbf{f}} = 0$$

The solution is

$$\tilde{X}(k,\tau) = -\int_{\tau_i}^{\tau} \frac{d\tau'}{\tau'} \; \tilde{G}_X(k,\tau,\tau')\tilde{\mathbf{f}}(k,\tau')$$

Where the function $\tilde{G}_X(k,\tau,\tau')$ is the solution of the homogeneous equation.

$$\tilde{G}_X(k,\tau,\tau') = \mathcal{T} \exp\left[-\int_{\tau'}^{\tau} \frac{d\tau''}{\tau''} \mathbf{D}(k,\tau'') \right]$$

Correlation function

The solution is

$$\tilde{X}(k, \tau - \tau_i) = - \int_{\tau_i}^{\tau} \frac{d\tau'}{\tau'} \, \tilde{G}_X(k, \tau, \tau') \tilde{\mathbf{f}}(k, \tau')$$

Note that X represents $\delta s, \delta n, \omega$ but it can be also used for any other thermodynamical variable $\delta T, \delta \mu, \delta P, \delta \epsilon$... using the adequate thermodynamical relations.

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This expression relates the correlation function of any thermodynamical quantity with the correlation function of the hydrodynamic fluctuations, and eventually with the thermal conductivity.

$$\langle \tilde{X}_1 \tilde{X}_2 \rangle \leftrightarrow \langle f_1 f_2 \rangle \leftrightarrow \langle I_1^\mu I_2^\nu \rangle \leftrightarrow \lambda$$

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...after some algebra...

Quantifying the hydrodynamical correlations

Equal-time correlation function of fluctuations

$$\langle \tilde{X}(k,\tau_f)\tilde{Y}(k,\tau_f)\rangle = \frac{2}{A}\int_{\tau_i}^{\tau_f} \frac{d\tau}{\tau^3}\lambda(\tau)\left[\frac{n(\tau)T(\tau)}{s(\tau)w(\tau)}\right]^2 \tilde{G}_X(k,\tau_f,\tau)\tilde{G}_Y(k,\tau_f,\tau)$$

- The thermal conductivity $\lambda(\tau)$ enhances the correlation function near the critical point.
- The Green functions $\tilde{G}_X$ are solution of the homogeneous equation.
- $\int_{\tau_i}^{\tau_f}$ represents adiabatic evolution of the system in the phase diagram with a given equation of state (to be defined in the next slide), starting at $\tau_i = 0.5$ fm and finishing at the freeze-out time τ_f.

Legend: A =nucleus transverse area, n =baryonic density; T =temperature, s =entropy density, w =enthalpy density.

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Legend: A =nucleus transverse area, n =baryonic density;
T =temperature, s =entropy density, w =enthalpy density.
However $\delta s, \delta n$ and ω are not directly measured in a heavy-ion collision.

How to measure it? Experimental observable

1. Nucleus-nucleus collision
2. Local thermalization ($\tau_i \simeq 0.5$ fm): Quark-Gluon Plasma
3. Expansion, cooling down and hadronization ($\tau \simeq 5$ fm): pions (•), kaons (•), protons (•)...
4. Freeze-out: no more collisions, frozen spectra which is detected ($\tau_f \simeq 5 - 10$ fm)

http://nuclear.ucdavis.edu/ calderon/Research/physicsResearch.html

How to measure it?

What is measured in heavy-ion collider?

Number of particles within some kinematical cut $(\Delta p, \Delta E, \Delta\varphi, \Delta y, \Delta\eta)$

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$$\frac{dN}{dy} \qquad \text{(with } y = \frac{1}{2} \log \frac{E+p}{E-p} \text{ the particle's rapidity)}$$

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Hydrodynamic fluctuations will show up through multiplicity fluctuations

$$\frac{dN}{dy} = \left\langle \frac{dN}{dy} \right\rangle + \delta\frac{dN}{dy}$$

Construct the correlation function:

$$\left\langle \delta\frac{dN}{dy_1}\delta\frac{dN}{dy_2} \right\rangle$$

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Number of particles within some kinematical cut $(\Delta p, \Delta E, \Delta\varphi, \Delta y, \Delta\eta)$

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Construct the correlation function:

$$\left\langle \delta\frac{dN}{dy_1}\delta\frac{dN}{dy_2} \right\rangle = \left\langle \frac{dN(y_1)}{dy_1}\frac{dN(y_2)}{dy_2} - \left\langle \frac{dN}{dy} \right\rangle^2 \right\rangle$$

Matching

- Cooper-Frye formula

$$E_p \frac{dN}{d^3p} = \frac{dN}{dy d^2 p_\perp} = d \int_{\Sigma_f} \frac{d^3\sigma_\mu}{(2\pi)^3} p^\mu \ f(\mathbf{x}, \mathbf{p})$$

particles detected in the final state = distribution of particles at freeze-out (last-scattering) surface

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Fluctuations on dN/dy are induced by fluctuations of $f(s, n, \omega)$

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- Cooper-Frye formula

$$E_p \frac{dN}{d^3p} = \frac{dN}{dyd^2p_\perp} = d \int_{\Sigma_f} \frac{d^3\sigma_\mu}{(2\pi)^3} p^\mu \; f(\mathbf{x}, \mathbf{p})$$

particles detected in the final state = distribution of particles at freeze-out (last-scattering) surface

Fluctuations on dN/dy are induced by fluctuations of $f(s, n, \omega)$

$$\left\langle \delta \frac{dN}{dy_1} \delta \frac{dN}{dy_2} \right\rangle$$

Matching

- Cooper-Frye formula

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where

$$\langle \tilde{X}(k,\tau_f) \tilde{Y}(k,\tau_f) \rangle = \frac{2}{A} \int_{\tau_i}^{\tau_f} \frac{d\tau}{\tau^3} \lambda(\tau) \left[\frac{n(\tau) T(\tau)}{s(\tau) w(\tau)} \right]^2 \tilde{G}_X(k,\tau_f,\tau) \tilde{G}_Y(-k,\tau_f,\tau)$$

Two-point correlation function

Following the previous steps more carefully and performing the transverse momentum integral we finally get

Correlation function

$$\left\langle \delta \frac{dN(y_1)}{dy_1} \delta \frac{dN(y_2)}{dy_2} \right\rangle \left\langle \frac{dN}{dy} \right\rangle^{-1} = \int dk \ e^{ik\Delta y} \int_{\tau_i}^{\tau_f} \frac{d\tau}{\tau^3} \ \lambda(\tau) \ \tilde{W}(k,\tau,\tau_f) \ ,$$

where $\Delta y = y_1 - y_2$ and $\tilde{W}$ certain complicated weight
(note that it contains nine terms, one for each independent correlation function).

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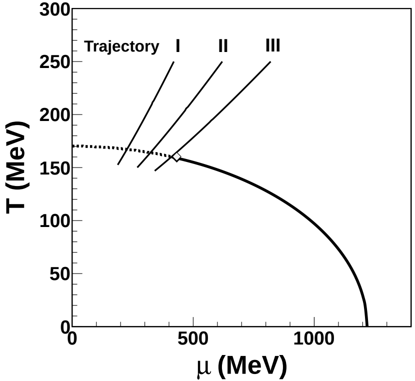
where $\Delta y = y_1 - y_2$ and $\tilde{W}$ certain complicated weight
(note that it contains nine terms, one for each independent correlation function).
Notice that in the generalized viscous case one would have

$$\lambda(\tau)\tilde{W}(k,\tau,\tau_f) \to \lambda(\tau)\tilde{W}_\lambda(k,\tau,\tau_f) + \eta(\tau)\tilde{W}_\eta(k,\tau,\tau_f) + \zeta(\tau)\tilde{W}_\zeta(k,\tau,\tau_f)$$

Illustrative model

We finally provide a simple model to see what to expect from these correlations.

- 1+1 dimensional Bjorken expansion (we forget about transverse plane)
- QCD input → phenomelogical EoS for massless gluons and (N_f) quarks



I: $s/n = 37.98$
II: $s/n = 26.08$
III: $s/n = 20.06$

$T_c = 160$ MeV
$\mu_c = 411.7$ MeV

Equation of State (Kapusta, 2010)

Background EoS (Ideal gas of massless gluons and quarks)

$$P(T,\mu) = A_4 T^4 + A_2 T^2 \mu^2 + A_0 \mu^4 - CT^2 - B$$

with

$$A_4 = \frac{\pi^2}{90}\left(16 + \frac{21 N_f}{2}\right) , \qquad A_2 = N_f/18$$

$$A_0 = N_f/324\pi^2 , \qquad B = 0.8 \times 170^2 \ \mathrm{MeV}^2$$

and C fixed so that the pressure is constant along the crossover line.

Free energy near the critical region

$$f = f_0(t) + f_1(t)\eta + f_2(t)\eta^2 + f_\sigma(t)|\eta|^\sigma$$

$f_i(t)$ parametrized to have the expected critical exponents and to reproduce the lattice QCD results at $\mu \to 0$.
$t = (T - T_c)/T_c, \eta = (n - n_c)/n_c$

QCD Thermal conductivity

Phenomenological QCD EoS near the critical point which takes the correct critical exponents of 3D Ising model (Kapusta, 2010)
We further assume that QCD critical point belongs to Model H (Son and Stephanov, 2004)

Correlation length

$$\xi = 0.69 \left[\frac{1}{3} \left(\frac{\delta - 1}{2 - \gamma} \right) |t|^{\gamma} + 5\delta |\eta|^{\delta - 1} \right]^{-\nu/\gamma} \text{fm}$$

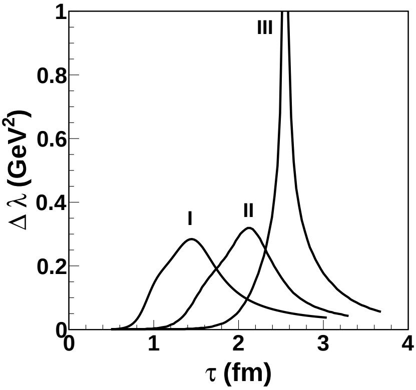
$$t = (T - T_c)/T_c; \quad \eta = (n - n_c)/n_c$$

Singular thermal conductivity (neglecting $\Delta\eta_S$)

$$\Delta\lambda \sim |t|^{\nu - \gamma} \simeq |t|^{-0.6}$$

QCD Thermal Conductivity

- Thermal conductivity: we use mode-coupling theory
- QCD universality class: 3D Ising model



I: s/n =37.98
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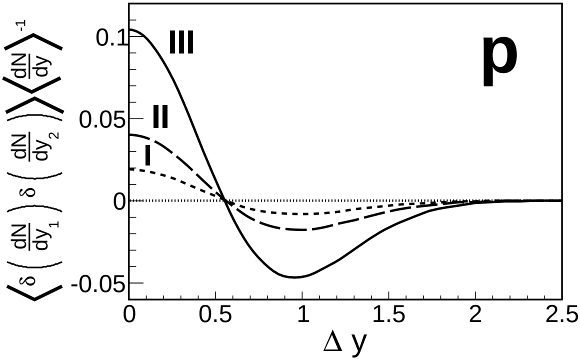
$$\Delta\lambda \sim |T - T_c|^{-0.6}$$

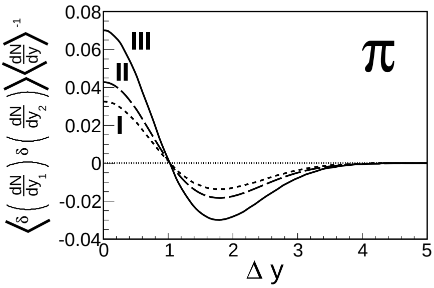
Proton correlation function

$$\left\langle \frac{dN(y_1)}{dy_1}\frac{dN(y_2)}{dy_2} - \left\langle \frac{dN}{dy} \right\rangle^2 \right\rangle \left\langle \frac{dN}{dy} \right\rangle^{-1}$$

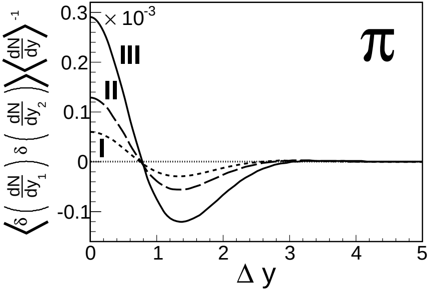


Pion correlation function

$$\left\langle \frac{dN(y_1)}{dy_1}\frac{dN(y_2)}{dy_2} - \left\langle \frac{dN}{dy} \right\rangle^2 \right\rangle \left\langle \frac{dN}{dy} \right\rangle^{-1}$$



$\delta\mu_B \neq 0$



$\delta\mu_B = 0$

Conclusions

- We have developed a model for the thermal conductivity in the vicinity of the QCD critical point.
- We have implemented an equation of state valid close to the critical point as well as in the non-asymptotic region.
- We applied them to a Bjorken expansion of a relativistic heavy-ion collision.
- The singular part of the thermal conductivity induces important two-particle correlations as the critical point is approached.
- Future work: Addition of viscosities will increase these two-particle correlations.
- Future work: First order transition: phase coexistence, nucleation, hadronic equation of state...
- Future work: Correction due to the finite size of the system
- Future work: Comparison to experiment relies on a 3D generalization of the evolution and the ability of heavy-ion colliders to produce trajectories close to the critical point.

Complementary slides

Diffusivity and thermal conductivity

Diffusion equation

$$\frac{\partial n_B}{\partial t} = D_B \nabla^2 n_B$$

Relation to thermal conductivity

$$D_B = \frac{\lambda T}{\chi_B} \left(\frac{n}{w} \right)^2 \ ; \quad \chi_B = (\partial n / \partial \mu)_T$$

Heat equation

$$\frac{\partial T}{\partial t} = D_T \nabla^2 T$$

Thermal diffusivity

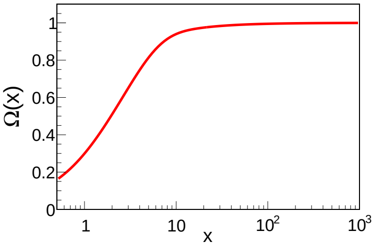
$$D_T = \frac{\lambda}{c_P} \ ; \quad c_P = T(\partial s / \partial T)_P$$

Critical thermal conductivity

Mode-coupling theory

$$\Delta\lambda = c_P \Delta D_T = c_P \frac{R_D T}{6\pi\eta\xi} \, \Omega(q_D \xi)$$

ξ: correlation length $\sim |T - T_c|^{-\nu}$
$R_D = 1.05$ (universal constant)
$\Omega(x) \simeq 0.48 \tanh(0.23x) + \frac{1.04}{\pi} \arctan(0.65x)$
$q_D = \pi T_0 = 534$ MeV



Langevin equation

Equation of motion

$$\tau \frac{\partial \tilde{\mathbf{X}}}{\partial \tau} + \mathbf{D}\tilde{\mathbf{X}} + \tilde{\mathbf{f}} = 0$$

$$\tilde{\mathbf{X}} = \left(\frac{\delta \tilde{s}}{s}, \frac{\delta \tilde{n}}{s}, \tilde{\omega} \right) \quad ; \quad \tilde{\mathbf{f}} = ik \left(-\frac{\mu}{T}, 1, 0 \right)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & ik \\ 0 & 0 & ik\frac{n}{s} \\ ikv_n^2 \frac{Ts}{w} & ikv_s^2 \frac{\mu s}{w} & 1 - v_\sigma^2 \end{pmatrix}$$

Speeds of sound

$$w \ v_\sigma = Ts \ v_n^2 + \mu n \ v_s^2$$

Langevin equation: including dissipative term

Equation of motion

$$\tau \frac{\partial \tilde{\mathbf{X}}}{\partial \tau} + \mathbf{D}\tilde{\mathbf{X}} + \tilde{\mathbf{f}} = 0$$

$$\tilde{\mathbf{X}} = \left(\frac{\delta \tilde{s}}{s}, \frac{\delta \tilde{n}}{s}, \tilde{\omega} \right) \quad ; \quad \tilde{\mathbf{f}} = ik \left(-\frac{\mu}{T}, 1, 0 \right)$$

$$\mathbf{D} = \begin{pmatrix} -\frac{\mu}{T}\left(\frac{\partial T}{\partial n}\right)_\epsilon \frac{\sigma}{\tau} k^2 & -\frac{\mu}{T}\left(\frac{\partial \mu}{\partial n}\right)_\epsilon \frac{\sigma}{\tau} k^2 & ik \\ \left(\frac{\partial T}{\partial n}\right)_\epsilon \frac{\sigma}{\tau} k^2 & \left(\frac{\partial \mu}{\partial n}\right)_\epsilon \frac{\sigma}{\tau} k^2 & ik\frac{n}{s} \\ ikv_n^2 \frac{Ts}{w} & ikv_s^2 \frac{\mu s}{w} & 1 - v_\sigma^2 \end{pmatrix}$$

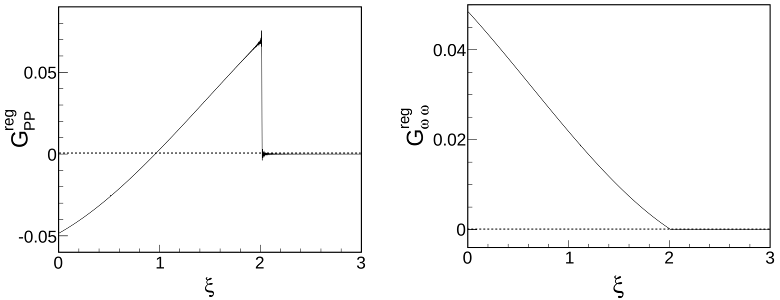
Wiedemann-Franz Law

$$\sigma = \frac{n^2 T}{w^2} \lambda$$

Correlated points

Correlation function

$$\tilde{G}_{XY}(k;\tau_1,\tau_2) = \tilde{G}_X(k;\tau_1,\tau_2)\tilde{G}_Y(-k;\tau_1,\tau_2)$$



A signal can propagate between τ_1 and τ_2 a distance $\Delta\xi = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\tau} v_\sigma(\tau)$

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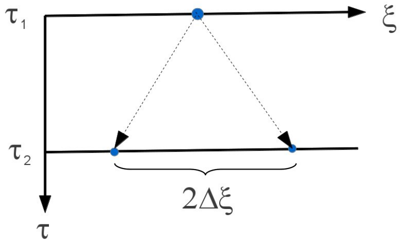

A signal can propagate between τ_1 and τ_2 a distance $\Delta\xi = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\tau} v_\sigma(\tau)$

Final observable

Two-particle correlation

$$\left\langle \frac{dN(y_1)}{dy_1}\frac{dN(y_2)}{dy_2} - \left\langle \frac{dN}{dy} \right\rangle^2 \right\rangle \left\langle \frac{dN}{dy} \right\rangle^{-1} = \frac{d\tau_f\, T_f^2}{2\pi^2} e^{\mu_f/T_f} \frac{C(\Delta y)}{N(m/T_f)}$$

$$C(\Delta y) = \int dk\ e^{ik\Delta y} \sum_{XY \in \delta s, \delta n, \omega} \tilde{F}_X(k)\tilde{F}_Y(-k) \int_{\tau_i}^{\tau_f} \frac{d\tau}{\tau^3}\, \lambda \left(\frac{n_B T}{sw} \right)^2$$

$$\times\ \tilde{G}_X(k;\tau_f,\tau)\tilde{G}_Y(-k;\tau_f,\tau)$$

$$N(m/T_f) = \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x} \Gamma\left(3, \frac{m}{T_f}\cosh x\right)$$

Auxiliary functions

$$F_s(x) \equiv \frac{s\chi_{\mu\mu}}{\Delta \cosh^2 x} \Gamma\left(4, \frac{m}{T_f}\cosh x\right) - \frac{s\chi_{T\mu} + s\chi_{\mu\mu}\mu_f / T_f}{\Delta \cosh^2 x} \Gamma\left(3, \frac{m}{T_f}\cosh x\right)$$

$$F_\omega(x) \equiv \frac{T_f \tanh x}{\cosh^2 x} \Gamma\left(4, \frac{m}{T_f}\cosh x\right)$$

$$F_n(x) \equiv -\frac{s\chi_{T\mu}}{\Delta \cosh^2 x} \Gamma\left(4, \frac{m}{T_f}\cosh x\right) + \frac{s\chi_{TT} + s\chi_{T\mu}\mu_f / T_f}{\Delta \cosh^2 x} \Gamma\left(3, \frac{m}{T_f}\cosh x\right)$$

$$\chi_{ab} = \left(\frac{\partial^2 P}{\partial a \partial b}\right) \quad ; \quad \Delta = \chi_{TT}\chi_{\mu\mu} - \chi^2_{T\mu}$$

Green functions in a static uniform system

Static, uniform system at rest. Using space-time variables:

$$\tilde{G}_s(k;t,t') = -\frac{ik}{v_\sigma^2}\frac{\mu}{T}\left\{v_s^2 + \left(v_\sigma^2 - v_s^2\right)\cos\left[kv_\sigma(t-t')\right]\right\}$$

$$\tilde{G}_n(k;t,t') = \frac{ik}{v_\sigma^2}\left\{v_n^2 + \left(v_\sigma^2 - v_n^2\right)\cos\left[kv_\sigma(t-t')\right]\right\}$$

$$\tilde{G}_v(k;t,t') = \frac{k}{v_\sigma}\frac{s}{n}\left(v_\sigma^2 - v_n^2\right)\sin\left[kv_\sigma(t-t')\right]$$

Physical sound wave and diffusive heat flow

$$\tilde{G}_P(k;t,t') = ik\,\frac{\mu}{T}\left(v_s^2 - v_n^2\right)\cos\left[kv_\sigma(t-t')\right]$$

$$\tilde{G}_\sigma(k;t,t') = ik\,\frac{w}{Ts}$$