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Exercises to Higher Quantum Mechanics

Sheet 1 (additional voluntary question)

The Physics Nobel Prize 2008 — an analogon in linear algebra
(for those interested only)

Consider the quantity,

$$L = \langle u|\mathcal{U}|d\rangle + \langle d|\mathcal{U}^\dagger|u\rangle + \langle u|\mathcal{M}_u|u\rangle + \langle d|\mathcal{M}_d|d\rangle \,. \tag{1}$$

Herein,

$$|u\rangle = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \qquad \text{and} \qquad |d\rangle = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix} \tag{2}$$

are two N-component vectors with $u_j,\ d_j \in \mathbb{C}$, $j = 1, \ldots, N$. $\mathcal{M}_{u,d}$ are real $N \times N$ diagonal matrices, $\mathcal{U}$ is a unitary $N \times N$ matrix.

Problem: Can one always find, for an arbitrarily given $\mathcal{U}$, $2N$ real numbers, α_j, β_j ($j = 1, \ldots, N$), such that the transformation

$$u_j \to \mathrm{e}^{\mathrm{i}\alpha_j}\, u_j^* \,, \qquad d_j \to \mathrm{e}^{\mathrm{i}\beta_j}\, d_j^* \tag{3}$$

leaves the quantity, L, unchanged?

Hint: Show that the answer depends on N, i.e., show that the answer is "yes" for $N = 1, 2$ but "no" for $N = 3$.[1]

Guidance:

(a) First show that the terms with $\mathcal{M}_{u,d}$ in (1) are invariant under the transformation (3).

(b) Show that L in Eq. (1) is invariant under the transformation (3) if

$$\mathcal{U}\, D_\beta = D_\alpha \left(\mathcal{U}^\dagger\right)^T \,. \tag{4}$$

Herein, D_α is a diagonal matrix,

$$D_\alpha = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha_1} & 0 & 0 & \ldots & \\ 0 & \mathrm{e}^{\mathrm{i}\alpha_2} & 0 & \ldots & \\ \vdots & & \ldots & & \\ & & & & \mathrm{e}^{\mathrm{i}\alpha_N} \end{pmatrix} . \tag{5}$$

The same holds for D_β, respectively.

[1] The answer is also "no" for $N > 3$.

(c) Now, consider the cases, $N = 1$ and $N = 2$, and show that one can always find real numbers, α_j and β_j ($j = 1, \ldots, N$), such that the condition (4) is fulfilled.

(d) To proof that L is in general not invariant for $N = 3$, it is sufficient to give one $\mathcal{U}$, for which one cannot find α_j, β_j such that (4) can be satisfied. Show that this is the case for the matrix,

$$\mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}} - \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\delta} & \frac{1}{2\sqrt{2}} + \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\delta} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}} + \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\delta} & \frac{1}{2\sqrt{2}} - \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\delta} \end{pmatrix} \tag{6}$$

with $\delta \notin \{0, \pi\}$. First show that this matrix, $\mathcal{U}$, is unitary.

Relation to the Physics Nobel Prize

The components of the vectors, $|u\rangle$ and $|d\rangle$, correspond to the different sorts of quarks ("quark flavors") in the standard model of elementary particles. For the physical case, $N = 3$, these are the up, charm, and top and the down, strange, and bottom quarks, respectively, carrying the pertinent electrical charges of $+2/3|e|$ and $-1/3|e|$, where e denotes the charge of one electron.

In a schematic way, the quantity L describes a part of the Hamiltonian of the weak interaction among the quarks. If L is invariant under a transformation, the model obeys a symmetry. Eq. 3 corresponds to a so-called CP transformation. Physically its meaning is that for some reaction of quarks, one replaces each quark with its anti-quark ("charge conjugation") and looks at the same time at the mirror image of this process ("spatial inversion"). If the Hamiltonian was invariant under such CP transformations, the charge conjugated and spatially inverted process would correspond to another possible reaction among quarks in nature.

In 1964, it has been proven experimentally that the CP symmetry is violated. With the then known quark flavors the theory of weak interactions could not be formulated in a way to explain this CP violation. At this time only three quark flavors were known (up, down, and strange) and one had conjectured the existence of a fourth (charm). This corresponds to the case $N = 2$ in our exercise. As we want to prove in this exercise, one cannot find a unitary matrix, $\mathcal{U}$, such that L is changed under the CP transformation (3). Thus, the Hamiltonian of our schematic theory of the weak interactions is always CP invariant. For that reason, two of this year's Nobel Prize winners, Kobayashi and Maskawa, conjectured the existence of two new quark flavors (top and bottom). In our example this is the case, $N = 3$.

For those who are interested in further details in relation to the Physics Nobel Prize 2008, you can find very good material on the Nobel-Prize web site. Later, there also the Nobel Laureates' lectures will be provided as a podcast as well as in written form:

`http://nobelprize.org/nobel_prizes/physics/laureates/2008/`

If you consider this question too difficult, keep in mind that you cannot get a Nobel Prize for nothing!