2.4. Perturbation theory

a) general considerations

given a Hamiltonian (in Fock space)

$$H(t) = \int d^3r\, \psi^+(\vec{r},t)\left(-\frac{\hbar^2 \Delta_r}{2m}\right)\psi(\vec{r},t)$$

$$+\frac{1}{2}\int d^3r\, d^3r'\, \psi^+(\vec{r},t)\,\psi^+(\vec{r}\,',t)\, V_2(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,',t)\,\psi(\vec{r},t)$$

(i.e. no external forces and no explicit time dependence of interaction)

want to describe scattering event of two particles starting at time t_1 at $\vec{r}_1, \vec{r}_1\,'$ and ending up at time t_2 at $\vec{r}_2, r_2'$

$\leadsto$ want to know probability amplitude

$$p(\vec{r}_2,\vec{r}_2\,',t_2;\vec{r}_1,\vec{r}_1\,',t_1) := \langle \vec{r}_2,\vec{r}_2\,';t_2 | \vec{r}_1,\vec{r}_1\,';t_1\rangle$$

in principle: solution via time translation operator (cf. page 22)

$$|\vec{r}_2,\vec{r}_2\,';t_2\rangle = \mathcal{U}(t_2,t_1)\,|\vec{r}_2,\vec{r}_2\,';t_1\rangle$$

$\mathcal{U}(t,0)$ satisfies $\quad -i\hbar \frac{d}{dt}\mathcal{U}(t,0) = H(t)\,\mathcal{U}(t,0)$

$\Rightarrow \mathcal{U}(t,t') = \mathcal{U}(t,0)\,\mathcal{U}(0,t')$ satisfies

$$-i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t') = H(t)\,\mathcal{U}(t,t')$$

for case at hand H is not time dependent since

$$\frac{d}{dt}H(t) = \frac{i}{\hbar}[H(t),H(t)] = 0$$

(cf. page 23 for arbitrary operator in Heisenberg picture)

$\Rightarrow H(t) = H$

$\Rightarrow \mathcal{U}(t,t') = \exp\left(iH\cdot(t-t')/\hbar\right)$

$$\Rightarrow \rho(\vec{r}_2, \vec{r}_2{}', t_2; \vec{r}_1, \vec{r}_1{}', t_1)$$

$$= \langle \vec{r}_2, \vec{r}_2{}'; t_1 | \mathcal{U}^+(t_2, t_1) | \vec{r}_1, \vec{r}_1{}'; t_1 \rangle$$

$$= \langle 0 | \psi(\vec{r}_2{}', t_1)\, \psi(\vec{r}_2, t_1)\, \exp\left(-i H (t_2 - t_1)/\hbar\right) \psi^+(\vec{r}_1, t_1)\, \psi^+(\vec{r}_1{}', t_1) | 0 \rangle$$

with $H = H(t_1) = \int d^3r\, \psi^+(\vec{r}, t_1) \left(-\frac{\hbar^2 \Delta_r}{2m}\right) \psi(\vec{r}, t_1)$

$$+ \frac{1}{2} \int d^3r\, d^3r'\, \psi^+(\vec{r}, t_1)\, \psi^+(\vec{r}\,', t_1)\, V_2(\vec{r}, \vec{r}\,')\, \psi(\vec{r}\,', t_1)\, \psi(\vec{r}, t_1)$$

now move annihilation operators ψ to the right and creation operators ψ^+ to the left using the equal-time (anti-) commutation relations ... (?)

BUT: $\exp(-iH(t_2 - t_1)/\hbar)$ contains infinitely many terms!

in practice (for small V_2): use perturbation theory, interaction picture

$H = H_0 + H_{int}$

with

$$H_0 = \int d^3r\, \psi^+(\vec{r}) \left(-\frac{\hbar^2 \Delta_r}{2m}\right) \psi(\vec{r})$$

and

$$H_{int} = \frac{1}{2} \int d^3r\, d^3r'\, \psi^+(\vec{r})\, \psi^+(\vec{r}\,')\, V_2(\vec{r}, \vec{r}\,')\, \psi(\vec{r}\,')\, \psi(\vec{r})$$

note: ~~we~~ have dropped the time dependence of the ψ and ψ^+ here since it does not matter at which time they are evaluated - since H is time independent. Below we will choose a convenient time to evaluate the ψ and ψ^+.

b) no interactions

before we continue with the general case, let us first study in more detail the case of no interaction, i.e.

$$V_2 = 0 \Rightarrow H_{int} = 0 \Rightarrow H = H_0$$

we recall from page 19 (for $V_1 = 0$)

$$[H_0(t), \psi(\vec{r},t)] = +\frac{\hbar^2}{2m} \Delta \psi(\vec{r},t)$$

introduce the Fourier transform

$$a(\vec{p},t) := \int d^3r \, e^{-i\vec{p}\cdot\vec{r}/\hbar} \psi(\vec{r},t) \qquad (1)$$

$$\Rightarrow \psi(\vec{r},t) = \int \frac{d^3p}{(2\pi\hbar)^3} e^{+i\vec{p}\cdot\vec{r}/\hbar} a(\vec{p},t) \qquad (2)$$

$$\Rightarrow +\hbar^2 \Delta \psi(\vec{r},t) = -\int \frac{d^3p}{(2\pi\hbar)^3} \vec{p}^2 a(\vec{p},t) e^{-i\vec{p}\cdot\vec{r}/\hbar}$$

$$\Rightarrow [H_0(t), a(\vec{p},t)] = -\underbrace{\frac{\vec{p}^2}{2m}}_{=: E_p} a(\vec{p},t)$$

$\rightsquigarrow$ equation of motion for $a(\vec{p},t)$ for case without interaction (cf. page 23)

$$\frac{\partial}{\partial t} a(\vec{p},t) = \frac{i}{\hbar}[H_0(t), a(\vec{p},t)] = -\frac{i}{\hbar} E_p \, a(\vec{p},t)$$

$$\Rightarrow a(\vec{p},t_2) = e^{-iE_p(t_2-t_1)/\hbar} a(\vec{p},t_1)$$

$$\Rightarrow \psi(\vec{r},t_2) = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-i[E_p(t_2-t_1)-\vec{p}\cdot\vec{r}]/\hbar} a(\vec{p},t_1) \qquad (3)$$

$\Rightarrow$ linear relation between ψ's at time t_1 and t_2 for no-interaction case

note: Also for the case with interaction one can introduce the Fourier decomposition, i.e. equations (1), (2). These equations connect (by construction) ψ and a at equal times. In contrast, equation (3) holds only for the non-interaction case, connecting non-equal times.

With the time translation operator

$$\mathcal{U}_0(t,t') = \exp\left(i H_0 (t-t')/\hbar\right)$$

one can also write (cf. page 23)

$$\rho(\vec{r},t_2) = \mathcal{U}_0(t_2,t_1)\,\rho(\vec{r},t_1)\,\mathcal{U}_0^+(t_2,t_1)$$

and

$$a(\vec{p},t_2) = \mathcal{U}_0(t_2,t_1)\,a(\vec{p},t_1)\,\mathcal{U}_0^+(t_2,t_1)$$
$$\| $$
$$e^{-iE_p(t_2-t_1)/\hbar}\,a(\vec{p},t_1)$$

$\Rightarrow$ in this case, i.e. without interactions, one can evaluate structures like

$$e^{iH_0(t_2-t_1)/\hbar}\,a(\vec{p},t_1)\,e^{-iH_0(t_2-t_1)/\hbar}$$

inspite of the fact that there are infinitely many operators in $e^{iH_0(t_2-t_1)/\hbar}$

$\rightsquigarrow$ much more complicated for interaction case

c) interaction picture

We now come back to the case with interaction, but with V_2 small such that perturbation theory can give reasonable answers.

One can expect that

$$\psi(\vec{r},t_2) = \mathcal{U}(t_2,t_1)\,\psi(\vec{r},t_1)\,\mathcal{U}^+(t_2,t_1)$$

is close to (but <u>not</u> equal to) the object

$$\mathcal{U}_0(t_2,t_1)\,\psi(\vec{r},t_1)\,\mathcal{U}_0^+(t_2,t_1)$$

$$\left(\begin{array}{l}\text{recall } \mathcal{U}(t,t') = \exp(i\underbrace{H}_{H_0+H_{int}}(t-t')/\hbar) \\[2ex] \text{and } \mathcal{U}_0(t,t') = \exp(iH_0(t-t')/\hbar)\end{array}\right)$$

$\rightsquigarrow$ introduce $\psi_D(\vec{r},t) = \mathcal{U}_0(t,t_1)\,\psi(\vec{r},t_1)\,\mathcal{U}_0^+(t,t_1)$

$$\Rightarrow \psi(\vec{r},t) = \mathcal{U}(t,t_1)\,\psi(\vec{r},t_1)\,\mathcal{U}^+(t,t_1)$$

$$= \mathcal{U}(t,t_1)\,\mathcal{U}_0^+(t,t_1)\,\psi_D(\vec{r},t)\,\underbrace{\mathcal{U}_0(t,t_1)\,\mathcal{U}^+(t,t_1)}_{=:\tilde{\mathcal{U}}_D(t,t_1)}$$

note: $\psi_D(\vec{r},t)$ can be calculated exactly from $\psi(\vec{r},t_1)$ via the Fourier decomposition (cf. page 26, 27):

$$a(\vec{p},t_1) = \int d^3r\, e^{-i\vec{p}\cdot\vec{r}/\hbar}\,\psi(\vec{r},t_1)$$

$$\rightsquigarrow \psi_D(\vec{r},t) = \int\frac{d^3p}{(2\pi\hbar)^3}\, e^{-i[E_p(t-t_1)-\vec{p}\cdot\vec{r}]/\hbar}\, a(\vec{p},t_1)$$

if H_0 is evaluated at the same time as a, i.e. at time t_1

$$\Rightarrow \rho(\vec{r}_2,\vec{r}_2{}',t_2;\vec{r}_1,\vec{r}_1{}',t_1) = \langle 0|\psi(\vec{r}_2{}',t_2)\,\psi(\vec{r}_2,t_2)\,\psi^+(\vec{r}_1,t_1)\,\psi^+(\vec{r}_1{}',t_1)|0\rangle$$

$$= \langle 0|\tilde{\mathcal{U}}_D^+(t_2,t_1)\,\psi_D(\vec{r}_2{}',t_2)\,\psi_D(\vec{r}_2,t_2)\,\tilde{\mathcal{U}}_D(t_2,t_1)\,\psi_D^+(\vec{r}_1,t_1)\,\psi_D^+(\vec{r}_1{}',t_1)|0\rangle$$

note: $\psi_D(\vec{r},t_1) = \psi(\vec{r},t_1)$

The operator $\hat{U}_D$ satisfies

$$i\hbar \frac{\partial}{\partial t}\tilde{U}_D(t,t_1) = i\hbar\frac{\partial}{\partial t} e^{iH_0(t-t_1)/\hbar} e^{-iH(t-t_1)/\hbar}$$

$$= -e^{iH_0(t-t_1)/\hbar} H_0 e^{-iH(t-t_1)/\hbar} + e^{iH_0(t-t_1)/\hbar} H e^{-iH(t-t_1)/\hbar}$$

$$= + e^{iH_0(t-t_1)/\hbar} H_{int} e^{-iH(t-t_1)/\hbar}$$

$$= e^{iH_0(t-t_1)} H_{int} e^{-iH_0(t-t_1)/\hbar} \hat{U}_D(t,t_1)$$

$$= \underbrace{U_0(t,t_1) H_{int} U_0^+(t,t_1)}_{=:H_D(t)} \hat{U}_D(t,t_1)$$

note:

H is time independent, but H_0 and H_{int} are not separately time independent

$\rightsquigarrow$ to calculate $\rho_D(\vec{r},t)$ like in the non-interacting case one has to take H_0 at time t_1

$\rightsquigarrow$ take also H_{int} at time t_1

$$\rightsquigarrow H_D(t) = \frac{1}{2}\int d^3r\, d^3r'\, V_2(\vec{r},\vec{r}')\, U_0(t,t_1)\, \rho^+(\vec{r},t_1)\, \rho^+(\vec{r}',t_1)\, \rho(\vec{r}',t_1)\, \rho(\vec{r},t_1) \cdot U_0^+(t,t_1)$$

insert $\mathbb{1} = U_0^+(t,t_1)\, U_0(t,t_1)$ between the operators ρ and ρ^+ and recall $\rho_D(\vec{r},t) = U_0(t,t_1)\, \rho(\vec{r},t_1)\, U_0^+(t,t_1)$

$$\rightsquigarrow H_D(t) = \frac{1}{2}\int d^3r\, d^3r'\, \rho_D^+(\vec{r},t)\, \rho_D^+(\vec{r}',t)\, V_2(\vec{r},\vec{r}')\, \rho_D(\vec{r}',t)\, \rho_D(\vec{r},t)$$

$\Rightarrow$ the expression for $p(\vec{r}_2,\vec{r}_2',t_2;\vec{r}_1,\vec{r}_1',t_1)$ at page 28, last line is entirely expressed in terms of non-interacting operators ρ_D, ρ_D^+; $\tilde{U}_D$ is given by (cf page 28)

$$\tilde{U}_D(t,t_1) = T \exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} dt'\, H_D(t')\right)$$

since $H_D(t)|0\rangle = 0$

$$\Rightarrow \tilde{U}_D(t,t_1)\,|0\rangle = |0\rangle$$

$$\Rightarrow \langle 0|\,\tilde{U}_D^+(t,t_1) = \langle 0|$$

$$\Rightarrow \rho(\vec{r}_2,\vec{r}_2{}',t_2;\vec{r}_1,\vec{r}_1{}',t_1) =$$

$$\langle 0|\,\rho_D(\vec{r}_2{}',t_2)\,\rho_D(\vec{r}_2,t_2)\,T\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2}dt'\,H_D(t')\right)\rho_D^+(\vec{r}_1,t_1)\,\rho_D^+(\vec{r}_1{}',t_1)|0\rangle$$

This expression should be compared to the one at page 25, top: There we had interacting operators ρ, ρ^+ and the full Hamiltonian, here we have non-interacting operators ρ_D, ρ_D^+ and the interaction part H_D only

Write down the first few terms explicitly:

(1) zeroth order in V_2 from $T\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2}dt'\,H_D(t')\right) \to \mathbb{1}$

$$\rightsquigarrow \langle 0|\,\rho_D(\vec{r}_2{}',t_2)\,\rho_D(\vec{r}_2,t_2)\,\rho_D^+(\vec{r}_1,t_1)\,\rho_D^+(\vec{r}_1{}',t_1)|0\rangle$$

strategy for evaluation:

- express $\rho_D(\vec{r},t_2)$ in terms of $a(\vec{p},t_1)$ (cf. page 28)
- express $a(\vec{p},t_1)$ in terms of $\rho_D(\vec{r}\,',t_1)$
- commute $\rho_D(\ldots,t_1)$ with $\rho_D^+(\ldots,t_1)$

$\rightsquigarrow$ not worked out here explicitly

② first order in V_2 from linear term in $T\exp(\ldots)$

$$\rightsquigarrow \langle 0|\psi_D(\vec{r}_2',t_2)\,\psi_D(\vec{r}_2,t_2)\left(-\frac{i}{\hbar}\right)\int_{t_1}^{t_2}dt\,H_D(t)\,\psi_D^+(\vec{r}_1,t_1)\,\psi_D^+(\vec{r}_1',t_1)|0\rangle$$

with $H_D(t)$ given at page 29 bottom in terms of ψ_D, ψ_D^+

strategy for evaluation: same as above (①)

③ second order in V_2:

$$\rightsquigarrow \langle 0|\psi_D(\vec{r}_2',t_2)\,\psi_D(\vec{r}_2,t_2)\,\frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2\int_{t_1}^{t_2}dt\,dt'\,T\left(H_D(t)\,H_D(t')\right)\psi_D^+(\vec{r}_1,t_1)\,\psi_D^+(\vec{r}_1',t_1)|0\rangle$$

⋮

challenge: rewrite time ordered operators such that creation operators are to the left and annihilation operators are to the right

$\rightsquigarrow$ Wick's theorem

note: even the cases ① and ② can be covered by the formalism which we present on the next pages, since

$$P(\vec{r}_2,\vec{r}_2',t_2;\vec{r}_1,\vec{r}_1',t_1) =$$

$$\langle 0|T\left[\psi_D(\vec{r}_2',t_2)\,\psi_D(\vec{r}_2,t_2)\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2}dt\,H_D(t)\right)\psi_D^+(\vec{r}_1,t_1)\,\psi_D^+(\vec{r}_1',t_1)\right]|0\rangle$$

with $T(A(t_1)\,B(t_2)) = \Theta(t_1-t_2)\,A(t_1)\,B(t_2) \pm \Theta(t_2-t_1)\,B(t_2)\,A(t_1)$!

A, B bosonic/fermionic creation or annihilation operators

d) Wick's theorem

let $O_1, O_2, \ldots, O_n$ be arbitrary operators (some might be creation, some annihilation operators) of a non-interacting theory

introduce "normal ordering":
the normal ordered product $:O_1 O_2 \cdots O_n:$ is the expression obtained from $O_1 O_2 \cdots O_n$, if all annihilation operators are moved to the right; for fermions one adds a minus sign for every required interchange

examples:

$$:\psi(\vec{r},t)\,\psi(\vec{r}\,',t'): = \psi(\vec{r},t)\,\psi(\vec{r}\,',t')$$

$$:\psi(\vec{r},t)\,\psi^+(\vec{r}\,',t'): = \pm\,\psi^+(\vec{r}\,',t')\,\psi(\vec{r},t)$$

$$:\psi^+(\vec{r},t)\,\psi(\vec{r}\,',t')\,\psi^+(\vec{r}\,'',t''): = \pm\,\psi^+(\vec{r},t)\,\psi^+(\vec{r}\,'',t'')\,\psi(\vec{r}\,',t')$$

$$H_{int} = \frac{1}{2}\int d^3r\, d^3r'\; V_2(\vec{r},\vec{r}\,')\, :\rho(\vec{r},t)\,\rho(\vec{r}\,',t):$$

Wick's theorem:

$$T(O_1 \cdots O_n) = \,:O_1 \cdots O_n:$$
$$+ \langle 0|T(O_1 O_2)|0\rangle\, :O_3 \cdots O_n: \; + \text{permutations}$$
$$+ \langle 0|T(O_1 O_2)|0\rangle \langle 0|T(O_3 O_4)|0\rangle\, :O_5 \cdots O_n: \; + \text{permutations}$$
$$+ \cdots$$
$$+ \begin{cases} \langle 0|T(O_1 O_2)|0\rangle \cdots \langle 0|T(O_{n-1} O_n)|0\rangle + \text{permutations}, & \text{for } n \text{ even} \\ \langle 0|T(O_1 O_2)|0\rangle \cdots \langle 0|T(O_{n-2} O_{n-1})|0\rangle\, O_n + \text{permutations}, & \text{for } n \text{ odd} \end{cases}$$

proof of Wick's theorem:

$n=1$ trivial: $T(\mathcal{O}_1) = :\mathcal{O}_1: = \mathcal{O}_1$

$n=2$: want to show

$$T(\mathcal{O}_1\mathcal{O}_2) = :\mathcal{O}_1\mathcal{O}_2: + \langle 0|T(\mathcal{O}_1\mathcal{O}_2)|0\rangle$$

either $T(\mathcal{O}_1\mathcal{O}_2)$ and $:\mathcal{O}_1\mathcal{O}_2:$ are the same or they differ by $\left\{\begin{array}{l}\text{a commutator}\\ \text{an anticommutator}\end{array}\right\}$ for $\left\{\begin{array}{l}\text{bosons}\\ \text{fermions}\end{array}\right.$

$$\Rightarrow T(\mathcal{O}_1\mathcal{O}_2) - :\mathcal{O}_1\mathcal{O}_2: = \left\{\begin{array}{c}+1\\-1\\0\end{array}\right\}[\mathcal{O}_1,\mathcal{O}_2]_\mp$$

examples: $t_1 > t_2$, $\mathcal{O}_1 = \psi(\vec{r}_1, t_1)$, $\mathcal{O}_2 = \psi^+(\vec{r}_2, t_2)$

$$\Rightarrow \left\{\begin{array}{l} T(\mathcal{O}_1\mathcal{O}_2) = \mathcal{O}_1\mathcal{O}_2 \\ :\mathcal{O}_1\mathcal{O}_2: = \pm\mathcal{O}_2\mathcal{O}_1 \end{array}\right.$$

$$\Rightarrow T(\mathcal{O}_1\mathcal{O}_2) - :\mathcal{O}_1\mathcal{O}_2: = [\mathcal{O}_1,\mathcal{O}_2]_\mp$$

$t_1 < t_2$, $\mathcal{O}_1 = \psi(\vec{r}_1, t_1)$, $\mathcal{O}_2 = \psi^+(\vec{r}_2, t_2)$

$$\Rightarrow \left\{\begin{array}{l} T(\mathcal{O}_1\mathcal{O}_2) = \pm\mathcal{O}_2\mathcal{O}_1 \\ :\mathcal{O}_1\mathcal{O}_2: = \pm\mathcal{O}_2\mathcal{O}_1 \end{array}\right.$$

$$\Rightarrow T(\mathcal{O}_1\mathcal{O}_2) - :\mathcal{O}_1\mathcal{O}_2: = 0$$

For a non-interacting theory the (anti-)commutator $[\mathcal{O}_1,\mathcal{O}_2]_\mp$ is just a number since operators at different times are linearly related

$$\Rightarrow T(\mathcal{O}_1\mathcal{O}_2) - :\mathcal{O}_1\mathcal{O}_2: = c \,, \quad c \in \mathbb{C}$$

$$\Rightarrow \langle 0|T(\sigma_1\sigma_2)|0\rangle - \underbrace{\langle 0|:\sigma_1\sigma_2:|0\rangle}_{=0} = c\underbrace{\langle 0|0\rangle}_{=1}$$

$$\Rightarrow c = \langle 0|T(\sigma_1\sigma_2)|0\rangle$$

$$\Rightarrow T(\sigma_1\sigma_2) = :\sigma_1\sigma_2: + \langle 0|T(\sigma_1\sigma_2)|0\rangle$$

Wick's theorem for $n=2$

rest of proof by induction:

suppose Wick's theorem is right for n

$\rightsquigarrow$ consider $T(\sigma_1\cdots\sigma_n\,\sigma_{n+1})$ where σ_{n+1} has the earliest (smallest) time argument – otherwise relabel operators

$$\Rightarrow T(\sigma_1\cdots\sigma_n\,\sigma_{n+1}) = T(\sigma_1\cdots\sigma_n)\,\sigma_{n+1}$$

$$= :\sigma_1\cdots\sigma_n:\sigma_{n+1}$$

$+\langle 0|T(\sigma_1\sigma_2)|0\rangle :\sigma_3\cdots\sigma_n:\sigma_{n+1}$ + permutations of $\sigma_1,\dots,\underset{\substack{\uparrow\\ \text{not } \sigma_{n+1}}}{\sigma_n}$

$+\dots$

$$+\begin{cases}\langle 0|T(\sigma_1\sigma_2)|0\rangle\cdots\langle 0|T(\sigma_{n-1}\sigma_n)|0\rangle\,\sigma_{n+1} + \text{perm.}, & n \text{ even}\\ \langle 0|T(\sigma_1\sigma_2)|0\rangle\cdots\langle 0|T(\sigma_{n-2}\sigma_{n-1})|0\rangle\,\sigma_n\sigma_{n+1} + \text{perm.}, & n \text{ odd}\end{cases}$$

case ①: n even, σ_{n+1} is annihilation operator

$$\Rightarrow :\underset{\substack{\uparrow\\ \text{whatever}}}{\cdots}: = :\cdots\sigma_{n+1}:$$

$\rightsquigarrow$ terms start with $:\sigma_1\cdots\sigma_{n+1}:$ and end with

$\langle 0|T(\sigma_1\sigma_2)|0\rangle\cdots\langle 0|T(\sigma_{n-1}\sigma_n)|0\rangle\,\sigma_{n+1}$ + permutations for $\sigma_1,\dots,\underset{\substack{\uparrow\\ \text{not } \sigma_{n+1}}}{\sigma_n}$

$\Rightarrow$ what is missing are terms with $\langle 0|T(\sigma_i\,\sigma_{n+1})|0\rangle$,

but $\langle 0|T(\sigma_i\,\sigma_{n+1})|0\rangle \underset{\substack{\uparrow\\ \sigma_{n+1} \text{ at earliest time}}}{=} \langle 0|\sigma_i\,\sigma_{n+1}|0\rangle \underset{\substack{\uparrow\\ \sigma_{n+1} \text{ annihil. op.}}}{=} 0$

$\Rightarrow$ Wick's theorem proven for case ①

case ②: n odd, σ_{n+1} annihilator

$\rightsquigarrow$ terms start again with $:\sigma_1 \cdots \sigma_{n+1}:$ and end with
$\langle 0|T(\sigma_1\sigma_2)|0\rangle \cdots \langle 0|T(\sigma_{n-2}\sigma_{n-1})|0\rangle :\sigma_n \sigma_{n+1}: +$ permut.

again what is missing are terms $\langle 0|T(\sigma_i \sigma_{n+1})|0\rangle = 0$

$\Rightarrow$ Wick's theorem proven for case ②

case ③: n odd, σ_{n+1} creation operator

last terms involve $\sigma_i \sigma_{n+1}$

$\rightsquigarrow$ difference between $\sigma_i \sigma_{n+1}$ and $:\sigma_i \sigma_{n+1}:$ is again a number

$$\sigma_i \sigma_{n+1} - :\sigma_i \sigma_{n+1}: = \varepsilon$$

$$\Rightarrow \underbrace{\langle 0|\sigma_i \sigma_{n+1}|0\rangle}_{=\langle 0|T(\sigma_i\sigma_{n+1})|0\rangle} - \underbrace{\langle 0|:\sigma_i \sigma_{n+1}:|0\rangle}_{=0} = \varepsilon$$

$$\Rightarrow \sigma_i \sigma_{n+1} = :\sigma_i \sigma_{n+1}: + \langle 0|T(\sigma_i \sigma_{n+1})|0\rangle$$

$\rightsquigarrow$ this produces all permutations between $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}$
for structure $\langle 0|T(\sigma_1\sigma_2)|0\rangle \cdots \langle 0|T(\sigma_{n-2}\sigma_{n-1})|0\rangle\langle 0|T(\sigma_n\sigma_{n+1})|0\rangle$
which is the last term in Wick's theorem for $n+1$

corresponding arguments can be applied to $:\sigma_i \sigma_j \sigma_k: \sigma_{n+1}$ and so on up to $:\sigma_1 \cdots \sigma_n: \sigma_{n+1}$

case ④: n even, σ_{n+1} creation operator

$\rightsquigarrow$ same arguments for $:\sigma_i \sigma_j: \sigma_{n+1}, \ldots$

$\rightsquigarrow$ Wick's theorem for $n+1$

application of Wick's theorem:

for the evaluation of $p(\vec{r}_2, \vec{r}_2', t_2; \vec{r}_1, \vec{r}_1', t_1)$ (cf. page 31, bottom)

we have to calculate objects like

$\langle 0 | T(\sigma_1 \cdots \sigma_n) | 0 \rangle$, n even

$\rightsquigarrow$ all normal ordered terms of Wick's theorem drop out

$$\langle 0 | T(\sigma_1 \cdots \sigma_n) | 0 \rangle = \langle 0 | T(\sigma_1 \sigma_2) | 0 \rangle \cdots \langle 0 | T(\sigma_{n-1} \sigma_n) | 0 \rangle$$
$$+ \text{ permutations}$$

in addition: $\langle 0 | T[\varphi_D(\vec{r}, t)\, \varphi_D(\vec{r}', t')] | 0 \rangle = 0$,

$$\langle 0 | T[\varphi_D^+(\vec{r}, t)\, \varphi_D^+(\vec{r}', t')] | 0 \rangle = 0$$

$\Rightarrow$ the only object one needs is the "propagator" or "two-point function" of the free theory

$$-i \langle 0 | T[\varphi_D(\vec{r}, t)\, \varphi_D^+(\vec{r}', t')] | 0 \rangle =: G(\vec{r}, t; \vec{r}', t')$$

convention

It is straight forward to calculate this object. We do not go through this exercise here. Instead we stress the important general result: To calculate whatever matrix element in perturbation theory, one only needs to use Wick's theorem to reduce any expression to a product of two-point functions of the free theory.

$\rightsquigarrow$ What one needs is a good understanding of the free theory

$\rightsquigarrow$ This motivates the study of free relativistic theories to which we turn next.