Photon and dilepton emission from a hadronic medium

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1 The photon-production rate

In the following we shall generalize the formula for the photon-production rate off a hot and dense hadronic system to lowest order in the e.m. interaction to the case of a quasi-equilibrium quantum state for a hadronic gas at a certain temperature and with pion-number chemical potentials.

The electromagnetic interaction is treated to lowest order, i.e., is formally incorporated by treating the photon as an external c-number source.

First, we treat the strict equilibrium situation, where the state of the system is described by the statistical operator,

$$\rho = \frac{1}{Z} \exp[-\beta(\mathbf{H} - \mu \mathbf{Q})], \tag{1}$$

where $\beta = 1/T$ is the inverse temperature, $\mathbf{H}$ the full Hamilton operator, $\mathbf{Q}$ the operator of an arbitrary *conserved* charge, and μ the associated chemical potential. The partition sum, Z, is given by

$$Z = \mathrm{Tr} \exp[-\beta(\mathbf{H} - \mu \mathbf{Q})]. \tag{2}$$

According to Noether's theorem the charge Q is conserved if and only if its operator is the generator of a symmetry, i.e.,

$$[\mathbf{H}, \mathbf{Q}] = 0. \tag{3}$$

Now, we assume that the photons decouple from the hadronic system, so that the initial state can be considered always to be free of photons and that the system is dilute enough that only one-photon transitions are important. So we need to take into account only processes $|i\rangle \rightarrow |f'\rangle = |f\gamma\rangle$, where $|i\rangle$ and $|f\rangle$ are purely hadronic initial and final asymptotically free states, respectively, and $|\gamma\rangle$ is an asymptotically free one-photon state. All these asymptotic states are considered to be simultaneous energy-momentum-charge eigenstates and to build a complete orthonormal set of the Fock space.

To leading order in the e.m. coupling, the S-matrix element for such a transition reads

$$S_{f'i} = \left\langle f \left| \int \mathrm{d}^4 x \, \mathbf{J}_\mu(x) A^\mu(x) \right| i \right\rangle, \tag{4}$$

where $\mathbf{J}$ is the hadronic e.m. current operator, and A^μ the outgoing one-photon mode with four-momentum k and polarization ϵ^μ

$$A^\mu(x) = \frac{\epsilon^\mu}{\sqrt{2\omega V}} \exp(\mathrm{i} k x). \tag{5}$$

V is a large (but finite) "quantization Volume" such that the momenta of the particles are discrete. Since the total-four-momentum operators, $\mathbf{P}$, generate the time evolution and spatial translations, we have

$$\mathbf{J}_\mu(x) = \exp(\mathrm{i}\mathbf{P}x)\mathbf{J}_\mu(0)\exp(-\mathrm{i}\mathbf{P}x). \tag{6}$$

Applying this within the matrix element in (4), we get

$$\begin{aligned} S_{f'i} &= \int \mathrm{d}^4 x \frac{\epsilon^\mu}{\sqrt{2\omega V}} \exp(\mathrm{i}kx)\exp[\mathrm{i}(p_f - p_i)x] \left\langle f \left| \mathbf{J}_\mu(0) \right| i \right\rangle \\ &= \frac{\epsilon^\mu}{\sqrt{2\omega V}} \left\langle f \left| \mathbf{J}_\mu(0) \right| i \right\rangle (2\pi)^4 \delta^{(4)}(p_f + k - p_i). \end{aligned} \tag{7}$$

Then the photon-emission rate is

$$R_{f'i} = \frac{|S_{f'i}|^2}{\tau V} = (2\pi)^4 \delta^{(4)}(p_f + k - p_i)|T_{f'i}|^2, \tag{8}$$

where the S-matrix element is written as usual as

$$S_{f'i} = (2\pi)^4 \delta^{(4)}(p_{f'} - p_i) T_{fi}. \tag{9}$$

Now we have to average over the initial equilibrium state and sum over the final state. We are not interested in the photon polarization, and thus the sum over the polarization states can be written as

$$\sum_f \epsilon^\mu (\epsilon^\nu)^* = -g^{\mu\nu} \tag{10}$$

(modulo $k^\mu k^\nu$ terms which do not contribute due to the conservation of the e.m. current). In the large-volume limit the number of momentum eigen states is given by $V \mathrm{d}^3\vec{k}/(2\pi)^3$. Putting Eq.'s (8-10) together and taking the average with respect to the equilibrium state (1), we get for the rate of photons with momentum $\vec{k}$[1]

$$\mathrm{d}R = -\frac{g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{Z} \sum_{i,f} \exp(-\beta K_i)(2\pi)^4 \delta^{(4)}(p_f + k - p_i) \left\langle f \left| \mathbf{J}_\mu(0) \right| i \right\rangle \langle i | \mathbf{J}_\nu(0) | f \rangle. \tag{11}$$

Here, we use the abbreviation

$$K_i = E_i - \mu Q_i, \tag{12}$$

where E_i and Q_i are the eigenvalues of $\mathbf{H}$ and $\mathbf{Q}$ to the eigenstate $|i\rangle$.

2 Relation to the retarded current correlation function

The retarded current-current correlation function is defined as the retarded propagator of the current operators:

$$\Pi_R^{\mu\nu}(k) := \mathrm{i} \int \mathrm{d}^4 x \exp(\mathrm{i}kx) \operatorname{Tr}\left\{\rho \left[\mathbf{J}_\mu(x), \mathbf{J}_\nu(0)\right]\right\} \theta(x^0). \tag{13}$$

[1]Note that the electromagnetic currents contain the elementary-charge factor $e = \sqrt{4\pi\alpha}$. Within the SU(2)-chiral model in the chiral limit $\mathbf{J}_\mu = e\mathbf{J}^3_{V\mu}$, where $\vec{J}_{V\mu}$ denote the components of the iso-vector current. In the SU(3) model the e.m. current contains an additional (strong) hyper-charge contribution.

Writing out the commutator and making use of (6) and inserting complete sets of hadronic initial and final asymptotically free states yields[2]

$$\Pi_R^{\mu\nu}(k) = \mathrm{i} \int \mathrm{d}^4 x \, \theta(x^0) \sum_{i,f} \exp[\mathrm{i}(p_f + k - p_i)x][\exp(-\beta K_f) - \exp(-\beta K_i)] \times \langle i | \mathbf{J}^\mu(0) | f \rangle \langle f | \mathbf{J}^\nu(0) | i \rangle . \tag{14}$$

Here in the contribution from the first term of the commutator the initial states where inserted between the current operators, in the second the final states. For a more detailed derivation, see [CS93].
The spatial integral yields a momentum-conserving $\delta^{(3)}$-factor. The integral over x^0 is to be regularized by adding a small positive imaginary part to the momentum-argument in the exponential:

$$\int \mathrm{d}x^0 \exp[\mathrm{i}(q_0 + \mathrm{i}\eta)x^0]\theta(x^0) = \frac{\mathrm{i}}{q_0 + \mathrm{i}\eta}. \tag{15}$$

Finally, the expression reads

$$\Pi_R^{\mu\nu}(k) = -\sum_{i,f} (2\pi)^3 \delta^{(3)}(\vec{p}_f + \vec{k} - \vec{p}_i) \frac{\exp(-\beta K_f) - \exp(-\beta K_i)}{k_0 + p_i^0 - p_f^0 + \mathrm{i}\eta} \left\langle f \left| \mathbf{J}_\mu(0) \right| i \right\rangle \left\langle i \left| \mathbf{J}_\nu(0) \right| f \right\rangle. \tag{16}$$

The imaginary part is easily taken by making use of Dirac's identity,

$$\operatorname{Im} \frac{1}{q^0 + \mathrm{i}\eta} = -\pi \delta(q^0), \tag{17}$$

yielding

$$2 \operatorname{Im} \Pi_R^{\mu\nu}(k) = \sum_{i,f} (2\pi)^4 \delta^{(4)}(p_f + k - p_i)[\exp(-\beta K_f) - \exp(-\beta K_i)] \left\langle f \left| \mathbf{J}_\mu(0) \right| i \right\rangle \left\langle i \left| \mathbf{J}_\nu(0) \right| f \right\rangle. \tag{18}$$

Now the bracket with the exponentials can be written as follows

$$\exp(-\beta K_f) - \exp(-\beta K_i) = \exp(-\beta K_i)\{\exp[\beta(K_i - K_f)] - 1\}. \tag{19}$$

Making use of the energy-conserving δ-distribution in (18) gives

$$K_i - K_f = \omega - \mu(Q_i - Q_f). \tag{20}$$

Now, since by assumption the charge is conserved, we must have $Q_i = Q_f$, and we have

$$\exp(-\beta K_f) - \exp(-\beta K_i) = \exp(-\beta K_i)[\exp(\beta\omega) - 1]. \tag{21}$$

Since the bracket in this expression is *independent of the hadronic initial and final states*, we obtain the (to order α) *exact* relation

$$\omega \frac{\mathrm{d}R}{\mathrm{d}^3\vec{k}} = -\frac{g_{\mu\nu}}{(2\pi)^3} n_B(\omega) \operatorname{Im} \Pi_R^{\mu\nu}(k), \tag{22}$$

with the Bose-Einstein distribution

$$n_B(\omega) = \frac{1}{\exp(\beta\omega) - 1}. \tag{23}$$

Note that this Bose-Einstein distribution is *independent of the chemical potential*. The only dependence on the chemical potential is included in the *retarded current-current correlation function*.

[2] We insert the $1 = \sum_i |i\rangle \langle i| = \sum_f |f\rangle \langle f|$ between the current-operator products in such a way that we finally obtain the same function $\delta^{(3)}(\vec{p}_f + \vec{k} - \vec{p}_i)$ which is part of the energy-momentum conserving $\delta^{(4)}$ expression in (11).

3 Generalization to pion-number chemical potentials

Now we generalize the application of (18) to an off-equilibrium state, given by the statistical operator

$$\rho = \frac{1}{Z} \exp[-\beta(\mathbf{H} - \mu_\pi \mathbf{N}_\pi)], \tag{24}$$

where $\mathbf{N}_\pi$ is the *pion number* operator, i.e., not a conserved quantity.

Strictly speaking the application of (18) to this situation is not exact since $\mathbf{N}_\pi$ does not commute with $\mathbf{H}$. Note that, however the (perturbative) Feynman rules with the usual Wick theorem still hold, since $\mathbf{N}_\pi$ is a quadratic form of field operators (i.e., a one-particle observable) [Dan84].

Given this approximation, the only difference to (22) is that in (20), generally we have $N_i - N_f \neq 0$. Thus (22) is modified to

$$\omega \frac{\mathrm{d}R}{\mathrm{d}^3\vec{k}} = -\frac{g_{\mu\nu}}{(2\pi)^3} \sum_{i,f} n_B[\omega - \mu(N_i - N_f)] \operatorname{Im} \Pi^{\mu\nu}_{Ri\to f}(k), \tag{25}$$

where the sum is to be taken over all possible processes $|i\rangle \to |f\gamma\rangle$. In Boltzmann approximation the *explicit* fugacity factor thus reads

$$\gamma_\pi^{N_{\pi i} - N_{\pi f}}. \tag{26}$$

4 Application to the Dey-Eletsky-Joffe mixing

In this section we rederive the Dey-Eletsky-Joffe formula for the mixing of vector and axial-vector current correlators at finite temperature due to interactions with thermal pions [DEI90].

For a dilute pion gas, i.e., low temperatures and pion-chemical potentials, one can use current-algebra techniques to systematically develop a virial expansion for the current correlator in the pion density. To that end we make the approximation that the electromagnetic current is dominated by the iso-vector part (neglecting contributions from, e.g., the ω which is iso scalar). Then the e.m. current is the iso-spin-three component of the vector current,

$$\mathbf{J}^\mu = e\mathbf{J}_V^{3\mu}, \tag{27}$$

where $e = \sqrt{4\pi\alpha}$ is the electromagnetic coupling constant. To derive this virial expansion we go back to (13) but do not introduce the $\sum_f |f\rangle\langle f|$ between the currents[3]. To perform the trace, we use initial asymptotic (hadronic) states and distinguish between "soft" and "hard" states. As soft we consider the pions (in the heat bath) and assume that we need to take into account only processes involving at most one soft pion. The expectation values with respect to hard states we can approximate by its vacuum expectation value since the "hard states", $|i_h\rangle$, are suppressed by an additional factor $\exp(-m_{i_h}/T)$ which we consider as very small, where m_{i_h} is the mass of the hard state.

For the single-soft pion states we use the asymptotic initial momentum eigen states $|i_h\rangle = \left|\pi^a(\vec{l})\right\rangle$ ($a \in \{1,2,3\}$: isospin indices), normalized in the usual relativistic convention, i.e.,

$$\left\langle \pi^a(\vec{l}_1) \middle| \pi^b(\vec{l}_2) \right\rangle = (2\pi)^3 2\omega_{\vec{l}_1} \delta^{(3)}(\vec{l}_1 - \vec{l}_2). \tag{28}$$

[3] Note that this cannot be done in (11) for the rate, since the energy-momentum conserving δ function restricts the final states!

Here $\omega_{\vec{l}}=\sqrt{\vec{l}^2+m_\pi^2}$ denotes the pion energy. Then, applying the low-density expansion in leading order, we approximate any trace by[4]

$$\begin{aligned}\mathrm{Tr}(\rho\mathbf{O}) &= \frac{1}{Z}\sum_{i_h}\left\{\exp(-\beta K_{i_h})\langle i_h|\mathbf{O}|i_h\rangle+\sum_{i_s}\exp[-\beta(K_{i_h}+K_{i_s})]\langle i_h i_s|\mathbf{O}|i_h i_s\rangle\right\}\\ &\approx \langle 0|\mathbf{O}|0\rangle+\frac{1}{Z_h}\sum_{i_h}\sum_{a}\int\frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega(\vec{l})}f_B[\omega(\vec{l})-\mu_\pi]\exp(-\beta K_{i_h})\left\langle i_h\pi^a(\vec{l})\middle|\mathbf{O}\middle|i_h\pi^a(\vec{l})\right\rangle.\end{aligned} \tag{29}$$

Now we can apply the soft-pion theorem based on PCAC and current algebra [DGH92]. Let $\mathbf{J}_{V/A}^{a\mu}$ denote the isospin-1 vector/axial-vector currents and $\mathbf{Q}_A^a$ the charges of the axial-vector currents. Then we can rewrite the expectation value in the last expression in (29) by

$$\left\langle i_h\pi^a(\vec{l})\middle|\mathbf{O}\middle|i_h\pi^a(\vec{l})\right\rangle=-\frac{1}{F_\pi^2}\langle i_h|[\mathbf{Q}_A^a,[\mathbf{Q}_A^a,\mathbf{O}]]|i_h\rangle. \tag{30}$$

This we apply to the products of the e.m. current correlators in (13) with the identification (27). Since in the thermal average over $|i_h\rangle$ we finally take into account only the vacuum contribution, in (30) we can substitute $|i_h\rangle$ by the vacuum.

After some current algebra we find

$$\begin{aligned}&\left\langle\pi^a(\vec{l})\middle|\mathbf{J}_V^{b\mu}\mathbf{J}_V^{c\mu}(0)\middle|\pi^a(\vec{l})\right\rangle\\ &=\frac{2}{F_\pi^2}\left\langle 0\middle|\sum_d \mathbf{J}_A^{d\mu}(x)\mathbf{J}_A^{d\nu}(0)\delta^{bc}-\mathbf{J}_A^{b\mu}(x)\mathbf{J}_A^{c\nu}(0)-2\mathbf{J}_V^{b\mu}(x)\mathbf{J}_V^{\nu}(0)\middle|0\right\rangle.\end{aligned} \tag{31}$$

In the chiral limit both the vector and the axial vector current are conserved and the vacuum state is symmetric under chiral transformations, and thus we can simplify (31) by

$$\left\langle\pi^a(\vec{l})\middle|\mathbf{J}_V^{b\nu}(0)\mathbf{J}_V^{c\mu}(0)\middle|\pi^a(\vec{l})\right\rangle=\frac{4}{F_\pi^2}\left[\Pi_A^{\mathrm{vac}}(x)-\Pi_V^{(\mathrm{vac})}(x)\right]\delta^{bc} \tag{32}$$

with

$$\Pi_{V/A}^{(\mathrm{vac})}(x)=\frac{1}{3}\sum_d\left\langle 0\middle|\mathbf{J}_{V/A}^{d\mu}(x)\mathbf{J}_{V/A}^{d\nu}(0)\middle|0\right\rangle \tag{33}$$

Plugging all this into (13), one obtains

$$\Pi_R^{\mu\nu}(k)=e^2\Pi_{VR}^{(\mathrm{vac})}(k)+e^2\epsilon[\Pi_{RA}^{(\mathrm{vac})}(k)-\Pi_{RV}^{(\mathrm{vac})}(k)], \tag{34}$$

where

$$\epsilon=\frac{4}{F_\pi^2}\int\frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}}f_B\left[\omega_{\vec{l}}-\mu)\right]. \tag{35}$$

The *retarded* vacuum-correlation functions are defined by

$$\mathrm{i}\Pi_{RV/A}^{\mathrm{vac}}(q)=\int\mathrm{d}^4x\,\exp(\mathrm{i}kx)\theta(x^0)\left[\Pi_{V/A}^{\mathrm{vac}}(x)-\Pi_{V/A}^{(\mathrm{vac})}(-x)\right], \tag{36}$$

[4]Note that for notational simplicity's sake we include also the vacuum, $|0\rangle$, in the sum over the hard initial states.

where we applied translation invariance of the vacuum state to write

$$\left\langle 0 \left| \mathbf{J}^{b\nu}_{V/A}(0) \mathbf{J}^{c\mu}_{V/A}(x) \right| 0 \right\rangle = \left\langle 0 \left| \mathbf{J}^{b\nu}_{V/A}(-x) \mathbf{J}^{c\mu}_{V/A}(0) \right| 0 \right\rangle, \tag{37}$$

and we also used the fact that the Fourier-transformed correlation functions are even under both exchange of Lorentz and isospin indices.

For the case of *equilibrium situations in the strict sense*, this formalism leads to the rate by just applying (22) since, as shown above, *all* processes (especially including those with no and one initial soft pion) require the same unique factor $f_B(\omega)$ ($\omega = |\vec{k}|$: photon energy), and thus one has not to distinguish different processes in determining this factor.

5 The rate in terms of the correlator

For the case of the *generalized chemical potential* for the pion number in the sense of Sect. 3 one has to be more careful. The difficulty originates in the necessity to apply the same low-density approximation in the rate equation (11) as in the evaluation of the correlators, but this is complicated by the restriction of the final-hadronic-state momenta by the energy-conserving δ-distribution in (11).

For the 0th order in the virial expansion, there is of course no problem, since for this case only over the "hard initial states" is summed, and soft pions are neglected all together. For this partial sum the same manipulations can be applied to the correlation function that have lead to (18), but only over i_h states (*but still arbitrary final* hadronic) states is summed. This leads to

$$\omega \frac{\mathrm{d}R^{(0)}}{\mathrm{d}^3\vec{k}} = -\frac{g_{\mu\nu}}{(2\pi)^3} \sum_{i_h, f} n_B[\omega - \mu(N_{i_h} - N_f)] \operatorname{Im} \Pi^{(\mathrm{vac})\mu\nu}_{RV, i_h \to f}(k), \tag{38}$$

where we have again substituted the thermal average over the hard initial states by the vacuum expectation value in the sense of the virial expansion.

5.1 The first virial correction to the rate

Now we apply the same approximations to the photon-production rate (11) with the goal to obtain the first virial correction. Here, carefully the different types of processes involving *one soft pion* in the initial or final state have to be taken into account.

We distinguish *gain terms*,

(i) Scattering of a soft pion in the initial state off hard initial hadronic states resulting in a hard hadronic final state $|f\rangle$ and a photon with momentum $\vec{k}$,

(ii) Bose enhancement for the decay of a hard initial hadronic state $|i_h\rangle$ to a soft pion, a photon with momentum $\vec{k}$, and eventually in arbitrary further hadrons.

and *loss terms*

(iii) Scattering of a hard initial state (which would decay into a photon in absence of the soft-pion heat bath) off a soft pion into a purely hadronic state which cannot further decay to a photon.

(iv) Bose enhancement for the decay of a hard initial state (which would decay into a photon in absence of the soft-pion heat bath) into a purely hadronic state and a soft pion which cannot further decay to a photon.

Fortunately we need to show the details of the necessary analysis only for case (i). The other cases then follow by simple analogy. Processes of type (i) contribute to the photon rate (11) by

$$\begin{aligned} \mathrm{d}R^{(\mathrm{i})} = & -\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{Z_h} \sum_{i_h, f} \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}} \exp(-\beta K_{i_h})(2\pi)^4 \delta^{(4)}(p_f + k - p_i - l) \\ & \times n_B(\omega_{\vec{l}} - \mu_\pi) \sum_a \left\langle f \left| \mathbf{J}^3_{V\mu}(0) \right| i_h \pi^a(\vec{l}) \right\rangle \left\langle i_h \pi^a(\vec{l}) \left| \mathbf{J}^3_{V\nu}(0) \right| f \right\rangle \end{aligned} \tag{39}$$

Now we can apply the soft-pion theorem to the explicitely written soft-pion states. Using then the approximate conservation of the axial currents (PCAC) under strong interactions, after some current algebra [DGH92], we obtain

$$\sum_a \left\langle f \left| \mathbf{J}^3_{V\mu}(0) \right| i_h \pi^a(\vec{l}) \right\rangle \left\langle i_h \pi^a(\vec{l}) \left| \mathbf{J}^3_{V\nu}(0) \right| f \right\rangle = \frac{2}{3F_\pi^2} \sum_c \left\langle f \left| \mathbf{J}^c_{A\mu}(0) \right| i_h \right\rangle \left\langle i_h \left| \mathbf{J}^c_{A\nu}(0) \right| f \right\rangle. \tag{40}$$

After plugging this into (39), our goal is to perform the sum over the final hadronic states, $|f\rangle$. As already mentioned above this is not directly possible in (39) since the final states are constrained by energy-momentum conservation, indicated by the δ distribution. This problem is solved by using (6) in the opposite direction than in the calculation leading to (11). To that end we write the δ distribution in (39) by a Fourier integral and express the pertinent factor $\exp(\mathrm{i} p_f x)$ by plugging the argument x into the current operator in the first current-matrix element. Then nothing depends on p_f explicitely anymore, and we can perform the sum by using the completeness of the final states, $|f\rangle$, within the hadronic model.

This procedure leads to

$$\begin{aligned} \mathrm{d}R^{(\mathrm{i})} = & -\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{Z_h} \sum_{i_h} \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}} \exp(-\beta K_{i_h}) \\ & \times n_B(\omega_{\vec{l}} - \mu_\pi) \int \mathrm{d}^4 x \exp[\mathrm{i}(k-l)x] \frac{2}{3F_\pi^2} \sum_c \left\langle i_h \left| \mathbf{J}^c_{A\nu}(0) \mathbf{J}^c_{A\mu}(x) \right| i_h \right\rangle. \end{aligned} \tag{41}$$

The hard-hadronic-state partition sum in the leading-order virial approximation can again be approximated by taking into account only the vacuum piece, finally leading to

$$\mathrm{d}R^{(\mathrm{i})} = -\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{2}{F_\pi^2} \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}} n_B(\omega_{\vec{l}} - \mu_\pi) \tilde{\Pi}^{(\mathrm{vac})}_{A\mu\nu}(k-l), \tag{42}$$

where we used the *vacuum-Wightman current-current-correlation functions* defined by

$$\tilde{\Pi}^{(\mathrm{vac})}_{V/A\mu\nu}(q) = \frac{1}{3} \sum_c \int \mathrm{d}^4 x \exp[\mathrm{i}(k-l)x] \left\langle 0 \left| \mathbf{J}^c_{V/A\nu}(0) \mathbf{J}^c_{V/A\mu}(x) \right| 0 \right\rangle. \tag{43}$$

Neglecting the soft-pion momentum in the correlation function, this yields

$$\mathrm{d}R^{(\mathrm{i})} \simeq -\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\epsilon}{2} \tilde{\Pi}^{(\mathrm{vac})}_{A\mu\nu}(k). \tag{44}$$

Processes of type (ii) are treated in a completely analogous manner. The only difference is that here the soft pion is in the final state, and thus the soft-pion-four momentum, l, changes its sign in (42):

$$\mathrm{d}R^{(\mathrm{ii})} = -\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{2}{F_\pi^2} \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}} n_B(\omega_{\vec{l}} - \mu_\pi) \tilde{\Pi}^{(\mathrm{vac})}_{A\mu\nu}(k+l). \tag{45}$$

Negligence of the soft-pion momentum in (45) yields the same result as (44):

$$\mathrm{d}R^{(\mathrm{ii})} \simeq -\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\epsilon}{2} \tilde{\Pi}^{(\mathrm{vac})}_{A\mu\nu}(k), \tag{46}$$

Eqs. (44) and (46) together give a contribution reminiscent to the first correction term in (34)[5].

The processes (iii) are events where a thermal (soft) pion scatters a hard hadronic state which would decay into hadrons and a photon into totally hadronic states which cannot decay further into hadrons and a photon. In the present approximation these are states with overlap with a state with the quantum number of a photon [e.g., a electrically neutral ρ meson which can "oscillate" into a photon (ρ-γ mixing); in this case the loss terms of type (iii) are processes like $\rho + \pi_{\mathrm{soft}} \rightarrow a_1$]. This means we have to calculate the transition of e.m. current, $e\mathrm{J}^3_{V\mu}$, into axial-vector states induced by thermal pions in leading order of the virial expansion. In this leading-order correction we can use the same formula as for the photo-production rate (13) but now with axial currents. When we apply the soft-pion theorem we have to ensure that we pick up only the isospin-three component in the vector-current correlator. Finally, since for the photon rate it is a *loss term* we have of course to subtract it from the rate.

The calculational techniques are the very same as for the derivation of the contribution to processes of type (i). So we can give immediately the result

$$\mathrm{d}R^{(\mathrm{iii})} = +\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{2}{F_\pi^2} \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}} n_B(\omega_{\vec{l}} - \mu_\pi) \tilde{\Pi}^{(\mathrm{vac})}_{V\mu\nu}(k-l). \tag{47}$$

The Bose enhancement of the decay of hard vector-type states into axial-vector-type states and a soft pion (e.g., $\rho \rightarrow \pi + a_1$) is again given by simply switching the sign of the pion-four momentum, l, of the argument of the current-correlation function in (47):

$$\mathrm{d}R^{(\mathrm{iv})} = +\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{2}{F_\pi^2} \int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3 2\omega_{\vec{l}}} n_B(\omega_{\vec{l}} - \mu_\pi) \tilde{\Pi}^{(\mathrm{vac})}_{V\mu\nu}(k+l). \tag{48}$$

Neglecting the thermal motion of the soft pion, we find

$$\mathrm{d}R^{(\mathrm{iii+iv})} \simeq +\frac{e^2 g^{\mu\nu}}{2\omega} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \epsilon \tilde{\Pi}^{(\mathrm{vac})}_{V\mu\nu}(k). \tag{49}$$

For the correct counting of the fugacity factors it is crucial that we can reinterpret the loss terms of types (iii) and (iv) as photon processes.

[5] As shown above, when we derived (22), the difference of the vacuum-Wightman functions in (36) can be expressed by the imaginary part of the retarded correlator modulo the overall Bose factor which in the case of the generalized pion-chemical potentials depends on the process under consideration and which we shall find in the next subsection

For the reinterpretation of (47) we realize that we can express the Wightman functions $\tilde{\Pi}$ in terms of the Wightman functions Π, cf.

$$\tilde{\Pi}^{(\mathrm{vac})}_{V/A\mu\nu}(q) = \Pi^{(\mathrm{vac})}_{V/A\mu\nu}(-q), \tag{50}$$

which identity can immediately be read off by using (37), substitution of x by $-x$ in the Fourier integral, and finally utilizing the symmetry of the correlators under exchange of the Lorentz indices. Substituting (50) into (47) shows that a type-(iii) process is precisely equivalent to the leading-virial order approximation for the Bose enhancement of the *absorption of a photon* of momentum $\vec{k}$ by a hard hadronic state resulting in a soft pion of momentum $\vec{l}$ (e.g., $\gamma + a_1 \rightarrow \pi_{\mathrm{soft}}$).

A type-(iv) process can be directly reinterpreted, namely up to the sign indicating the loss-term character, as the Bose-enhancement correction for the decay of a hard hadronic state to a soft photon and a pion (e.g., $a_1 \rightarrow \gamma + \pi_{\mathrm{soft}}$).

5.2 Connection to the retarded correlator

Now we like to express the different contributions to the differential production rate, $\mathrm{d}R^{(\mathrm{i})}, \ldots, \mathrm{d}R^{(\mathrm{iv})}$, i.e. (42,45,47,48), in terms of the retarded current correlator. To this end, our general calculation (34) is not applicable since we need to extract the process-dependent overall Bose factors which are of the general form

$$N_B = n_B[\omega - \mu_\pi (N_{i_x} - N_{f_x})], \tag{51}$$

and we have to assign the correct processes i_x and f_x in the initial and final state which contribute to the expressions for the rates. As already stated above, for the leading term, without taking into account any processes with soft pions from the heat bath, we have to consider only hard hadronic states in i, and (38) determines the correct pion-number difference.

So our task left is to determine the pertinent processes $i_x \rightarrow f_x$ in (51) for the first virial correction. Of course the same techniques as for the rate apply, and we only need to consider briefly the calculation for process (i). Of course we proceed as described in the beginning of Sect. 2, but now applying the soft-pion theorem first and using the trick of interchanging initial and final states only to the remaining $i_h \rightarrow f$ process. This shows that the difference of initial- and final-pion numbers for this type of processes is

$$(N_{i_x} - N_{f_x})^{(\mathrm{i})} = N_i - 1 - N_f. \tag{52}$$

Example: a_1 annihilation by a thermal pion, i.e., $\pi_{\mathrm{soft}} + a_1 \rightarrow \gamma$. Here we have $N_i = 4$, $N_f = 0$, i.e., the correct difference is 3. In the Boltzmann approximation for the fugacity factors together with the γ_π contained in ϵ this yields an overall factor of γ_π^4 for this process in the *rate*.

In the case of a type-(ii) process the soft pion, treated with the soft-pion theorem is in the *final* state, and we still have to sum over *all* final states. The initial state is a hard state anyway. Thus we get the difference

$$(N_{i_x} - N_{f_x})^{(\mathrm{ii})} = N_i - N_f. \tag{53}$$

Example: $a_1 \rightarrow \pi_{\mathrm{soft}} + \gamma$. This yields $N_i - N_f = 2$, and in the Boltzmann limit the corresponding piece of the rate goes like γ_π^3. This argument shows that any process where the soft pion is in the final state gives a naive difference $N_i - N_f$, and for the Boltzmann approximation the power of γ_π in the rate is $N_i - N_f + 1$ (1 additional power from ϵ in the virial correction to the current correlator).

For type-(iii) processes we have to use the reinterpretation as photo absorption at a hard state. Thus, we can apply the rule just stated,

$$(N_{i_x} - N_{f_x})^{(\mathrm{iii})} = N_i - N_f, \tag{54}$$

and in the Boltzmann approximation the overall power of γ_π in the rate is $N_i - N_f + 1$.

An example is the process $\gamma + a_1 \to \pi_{\mathrm{soft}}$ with $N_i - N_f = 2$ giving an overall power in the Boltzmann approximation γ_π^3. Since this is a loss term we have to take it *out* of the total photon rate (opposite sign!).

Finally, for the type-(iv) processes the correct pion-fugacity power is obtained by the observation that the soft pion in the reinterpretation of the term as Bose enhancement of a hard-state decay to a soft pion, an arbitrary hard hadronic state, and a photon, is in the final state, i.e.,

$$(N_{i_x} - N_{f_x})^{(\mathrm{iv})} = N_i - N_f. \tag{55}$$

Remember, however, that type-(iv) processes are *loss terms*!

An example is again the a_1 decay as in the type-(ii) example. The correct difference is again 2, and the overall fugacity factor in the rate 3.

We summarize the here given examples to the total rate of photons, taking into account only the minimal set of "hard" hadronic states with the lowest pion numbers. Then we have in the Boltzmann limit and neglecting the thermal motion of the soft pion[6]

$$\omega \frac{\mathrm{d}R}{\mathrm{d}^3\vec{k}} = -\frac{\alpha g^{\mu\nu}}{\pi^2} n_B(\omega) \left[\gamma_\pi^2 (1 - \gamma_\pi \epsilon_0) \operatorname{Im} \Pi^{(\mathrm{vac})}_{RV\mu\nu}(k) + \frac{1}{2} \epsilon_0 (\gamma_\pi^3 + \gamma_\pi^4) \operatorname{Im} \Pi^{(\mathrm{vac})}_{RA\mu\nu}(k) \right]. \tag{56}$$

Here, ϵ_0 is defined by (47) with $\mu = 0$. The factor in front of the pure vacuum term stems from the only vacuum process in the context of the here considered hard-particle content, namely the type-(ii) process $\rho \to \gamma$[7].

Remark 1: The here presented formalism immediately can be used to express also dilepton-production rates in leading order in α (which is $O(\alpha^2)$). One has to take the photon formulae simply off shell, i.e., in the time-like region according to the kinematics of dileptons, and multiply with the lepton-tree-level e.m. current correlator for unpolarized leptons. The final result in the same approximation as (56) reads

$$\begin{aligned} \frac{\mathrm{d}R_{ll}}{\mathrm{d}^4 k} = & \frac{\alpha^2}{3\pi^3} \frac{k^2 + 2m_l^2}{(k^2)^2} \sqrt{1 - \frac{4m_l^2}{k^2}} g^{\mu\nu} n_B(k^0) \\ & \times \left[\gamma_\pi^2 (1 - \gamma_\pi \epsilon_0) \operatorname{Im} \Pi^{(\mathrm{vac})}_{RV\mu\nu}(k) + \frac{1}{2} \epsilon_0 (\gamma_\pi^3 + \gamma_\pi^4) \operatorname{Im} \Pi^{(\mathrm{vac})}_{RA\mu\nu}(k) \right]. \end{aligned} \tag{57}$$

Here, k denotes the center-of-mass momentum of the dilepton pair, i.e., the momentum of the virtual photon.

Remark 2: In the chiral limit both vector- and axial-vector currents are conserved, and thus must be transverse. The usual sign convention is that of self-energies[8], i.e., one writes

$$\Pi^{(\mathrm{vac})}_{RV/A\mu\nu}(q) = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \Pi^{\mathrm{vac}}_{RV/A}(q). \tag{58}$$

[6]Note the additional sign on the right-hand side of Eq. (17)!

[7]Using the rule (38), taken in the Boltzmann limit gives an overall factor γ_π^2 for the rate

[8]Note that the current correlators are only one-particle irreducible (1PI) in the Feynman-diagram sense with respect to the corresponding external c-number vector field, but they can be one-particle reducible by cutting a single hadronic line.

Then due to the retardation condition the sign convention is such that

$$\operatorname{sign} \operatorname{Im} \Pi_{RV/A\mu\nu}(q) = -\operatorname{sign}(q^0). \tag{59}$$

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