The relativistic wave equation

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1 Introduction

We discuss the wave equation

$$\frac{1}{c_{\mathrm{s}}^2}\partial_t^2\Phi - \Delta\Phi = 0 \tag{1}$$

for waves of a phase-velocity $c_{\mathrm{s}} \leq 1$ and Φ some scalar field as a relativistic wave equation. A concrete example is the description of sound waves in a medium, where Φ can be taken as the pressure, which is defined in the (local) rest frame of the fluid and thus as a scalar field. Also, as far as the kinematics of the wave four-vector and four-velocities of the emitter and observer is concerned, our analysis also applies to **plane light waves in a dielectric non-absorptive medium** in the ralm of normal dispersion, i.e., refractive index $n > 1$.

This example also makes it clear that to interpret this wave equation with an arbitrary phase velocity $c_{\mathrm{s}} < c$ we have to take into account that (1) can only be taken as a relativistic equation, when we interpret it to be written down in the one preferred inertial reference frame of the problem, i.e., in the **rest frame** of the fluid. Indeed, as it stands (1) is only manifestly covariant, if $c_{\mathrm{s}} = c$.

We use the following conventions concerning Minkowski space: the pseudo-metric components are taken to be $(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$. Underlined symbols denote four-vectors components, $\underline{k} = (k^\mu)$, and the Minkowski product is written as $\underline{k} \cdot \underline{x} = \eta_{\mu\nu} k^\mu x^\mu$. The space-time four-vector $\underline{x} = (ct, \vec{x})$ and $\partial_\mu = \partial/\partial x^\mu$. Four-velocities $\underline{u}$ are normalized, $\underline{u} = \gamma(1,\vec{\beta})$, $\gamma = 1/\sqrt{1-\vec{\beta}^2}$, and $\vec{\beta} = \vec{v}/c$. Of course $|\vec{\beta}| = \beta < 1$. We also use $\beta_{\mathrm{s}} = c_{\mathrm{s}}/c = 1/n$. One should note that c_{s}, β_{s}, and n are **scalars**, because c_{s} are the phase velocities of the waves in the rest frame of the medium. The objective of the following calculation is to derive a general expression for the "effective" wave velocity in an arbitrary frame of reference, for a general velocity of the medium $\vec{\beta}_{\mathrm{M}}$ and the Doppler effect for arbitrarily moving emitters and observers with velocities $\vec{\beta}_{\mathrm{E}}$ and $\vec{\beta}_{\mathrm{O}}$. I am not aware of any reference, where this in principle straight-forward calculation can be found. The exception is of course the simpler case of $c_{\mathrm{s}} = c$, i.e., (the kinematic part) of light waves in a vacuum, where there is no preferred frame of reference, because there is no medium (aka aether) for light waves in a vacuum. We treat this simpler case first. It can be found in many textbooks on relativity (e.g., [LL96]) as well as already in Sect. 7 of Einstein's famous paper [Ein05].

2 The case $c_s = c$

The case $c_s = c$ is special, because in this case there is no preferred reference frame, and of course the only real-world application are **electromagnetic waves** in the vacuum. Here we shall, however, not bother with the electromagnetic field but only investigate the Doppler effect for plane waves, for which the discussion of a scalar field is entirely sufficient. Indeed in this special case (1) is a manifestly covariant field equation, since it can be written as

$$\Box\Phi = 0, \quad \Box = \underline{\partial} \cdot \underline{\partial} = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\partial_t^2 - \Delta, \quad \Delta = \vec{\nabla} \cdot \vec{\nabla}. \tag{2}$$

For plane waves,

$$\Phi(x) = \Phi_0 \exp(-\mathrm{i}\underline{k} \cdot \underline{x}) \tag{3}$$

we simply get

$$\Box\Phi(x) = -\underline{k}^2\Phi(x) = 0, \tag{4}$$

i.e., the wave four-vector $\underline{k}$ is light-like, and thus the dispersion relation in any frame is

$$k^0 = \omega/c = |\vec{k}|. \tag{5}$$

So in any frame we have $\underline{k} = k(1, \vec{n})$ with $\vec{n}^2 = 1$.
Now it is very easy to treat the **Doppler effect**. Since $\underline{k}$ is a four-vector, in any reference frame moving with a velocity $\vec{v} = c\vec{\beta}$ with respect to our computational frame the frequency of our wave is given by $\omega_{\vec{\beta}} = \underline{u} \cdot \underline{k}/c$, where $\underline{u} = \gamma(1, \vec{\beta})$. If we consider the case that the light emitter moves with a four-velocity $\underline{u}_{\mathrm{E}}$ and an observer with $\underline{u}_{\mathrm{O}}$ the relation between the freqency of the wave in the rest frame of the emitter, ω_{E}, and that measured by the observer ω_{O} is

$$\frac{\omega_{\mathrm{O}}}{\omega_{\mathrm{E}}} = \frac{\underline{u}_{\mathrm{O}} \cdot \underline{k}}{\underline{u}_{\mathrm{E}} \cdot \underline{k}} = \sqrt{\frac{1-\beta_{\mathrm{E}}^2}{1-\beta_{\mathrm{O}}^2}} \frac{1-\beta_{\mathrm{O}}\cos\theta_{\mathrm{O}}}{1-\beta_{\mathrm{E}}\cos\theta_{\mathrm{E}}}. \tag{6}$$

Here θ_{O} (θ_{E}) is the angle between the velocity of the observer (emitter) and the direction of wave propagation $\vec{n}$, as measured in the arbitrary computational reference frame.
The usual Doppler effect follows by taking the computational frame as the observer's frame, i.e., by setting $\beta_{\mathrm{O}} = 0$. Then Θ_{E} is the angle between the velocity of the emitter and the wave vector $\vec{k}$ as measured in the observer's rest frame, and (6) leads to the well-known formula

$$\frac{\omega_{\mathrm{O}}}{\omega_{\mathrm{E}}} = \frac{\sqrt{1-\beta_{\mathrm{E}}^2}}{1-\beta_{\mathrm{E}}\cos\theta_{\mathrm{E}}}. \tag{7}$$

Especially for $\Theta_{\mathrm{E}} = 0$, where the source moves towards the the observer in the same direction as the light, one gets the maximal possible **blue-shift**,

$$\frac{\omega_{\mathrm{O}}}{\omega_{\mathrm{E}}} = \sqrt{\frac{1+\beta_{\mathrm{E}}}{1-\beta_{\mathrm{E}}}}. \tag{8}$$

The other extreme case is $\Theta_E = \pi$, where the source moves away from the observer, i.e., opposite to the direction of the light, leading to the maximal possible **red-shift**,

$$\frac{\omega_O}{\omega_E} = \sqrt{\frac{1-\beta_E}{1+\beta_E}}. \tag{9}$$

Another interesting case is $\Theta_E = \pi/2$, where the light is emitted perpendicular to the velocity of the emitter. In this case one gets a **red-shift**, which is entirely due to **time dilation**,

$$\frac{\omega_O}{\omega_E} = \sqrt{1-\beta_E^2}. \tag{10}$$

Since here the emitter is moving the period of the light wave, $\tau_E = 2\pi/\omega_E$, as measured in the rest frame of the emitter is time-dilated in the observer's frame, we get a **red-shift**. To also derive the **aberration** effect, we can describe the situation in the rest frame of the emitter. In this frame we have

$$\underline{k} = k' \begin{pmatrix} 1 \\ \vec{n}' \end{pmatrix}, \quad \underline{u}_E = \frac{1}{\sqrt{1-\beta_O'^2}} \begin{pmatrix} 1 \\ \vec{\beta}'_O \end{pmatrix} = \frac{1}{\sqrt{1-\beta_E^2}} \begin{pmatrix} 1 \\ -\vec{\beta}_E \end{pmatrix}. \tag{11}$$

Here we have used the fact that $\vec{\beta}'_O = -\vec{\beta}_E$ since the frame Σ (rest frame of the observer) moves with respect to the frame Σ' (rest frame of the emitter) with the opposite velocity than Σ' with respect to Σ, which follows immediately from the corresponding Lorentz boosts.
Seen from Σ' we thus have

$$\frac{\omega_O}{\omega_E} = \frac{\underline{u}'_O \cdot \underline{k}'}{k'^0} = \frac{1+\beta_E \cos\theta'_O}{\sqrt{1-\beta_E^2}}. \tag{12}$$

Comparing this with (7) leads after some algebra to

$$\cos\Theta_E = \frac{\cos\theta'_O + \beta_E}{1+\beta_E \cos\theta'_O}, \tag{13}$$

where $\beta_E = \beta'_O = |\vec{\beta}_E|$.
This shows that the angle between the velocity of the emitter and the wave vector wrt. Σ is different from the angle between the velocity of the observer and the wave vector wrt. Σ'. This effect is called **aberration**. It is noteworthy to observe that the transverse Doppler effect wrt. Σ' is a **blue-shift**, because for $\theta'_O = \pi/2$ we get from (12)

$$\omega_O = \frac{\omega_E}{\sqrt{1-\beta_E^2}}. \tag{14}$$

As above this is also a pure time-dilation effect: since here the observer is moving he sees the proper period $\tau_E = 2\pi/\omega_E$ as $\tau_O = \tau_E/\gamma_O = \tau_E/\gamma_E$, leading to (14).

3 The case $c_s < c$

Now we come back to the general case of a sound wave with phase velocity c_s as measured in the rest frame of the medium. Here (1) is of course not manifestly covariant, and indeed we have to take into account that this equation holds only in the rest frame of the medium. To formulate it in a manifestly covariant way, we simply have to introduce the four-velocity of the medium $\underline{u}_{\text{M}}$. It is also convenient to introduce the tensor

$$h_{\nu}^{\mu} = \delta_{\nu}^{\mu} - u_{\text{M}}^{\mu} u_{\text{M}\nu}, \tag{15}$$

which projects four-vectors to a four-vector that is Minkowski-orthogonal to the fluid four-velocity. It obeys the projection property

$$h_{\nu}^{\mu} h_{\mu}^{\rho} = h_{\nu}^{\rho}. \tag{16}$$

since in the rest frame of the fluid $\underline{u}_{\text{M}} = (1,0,0,0)$ we can write (1) in manifestly covariant form

$$\frac{1}{\beta_{\text{s}}^2} (u_{\text{M}}^{\mu} \partial_{\mu})^2 \Phi + h^{\mu\nu} \partial_{\mu} \partial_{\nu} \Phi = 0. \tag{17}$$

Indeed in the fluid-rest frame we get $u_{\text{M}}^{\mu} \partial_{\mu} = \partial_0 = \partial_t / c$ and $h^{\mu\nu} \partial_{\mu} \partial_{\nu} = \Box - \partial_0^2 = -\Delta$, i.e., (17) reproduces (1). Since it is a manifestly covariant equation, it is the right description in any frame, where the medium flows with a constant velocity $\vec{v}_{\text{M}} = c \vec{\beta}_{\text{M}}$.

With our plane-wave ansatz we (3) we get the equation,

$$\frac{1}{\beta_s^2} (\underline{k} \cdot \underline{u}_{\text{M}})^2 + \underline{k}^2 - (\underline{k} \cdot \underline{u}_{\text{M}})^2 = 0. \tag{18}$$

We note that in the rest-frame of the fluid we get simply $(k^0)^2 / \beta_{\text{s}}^2 - k^2 = 0$ and thus $k^0 = \omega / c = \beta_{\text{s}} k$, i.e., in this frame $\underline{k} = k(\beta_s, \vec{n})$. This shows that $\underline{k}^2 = k^2(\beta_{\text{s}}^2 - 1) < 0$, i.e., $\underline{k}$ is a **space-like vector**, which of course holds true in any frame.

The dispersion relation is found by solving the quadratic equation (18) for k^0, which leads to

$$\frac{k^0}{k} = \beta_{\text{eff}} = \frac{1}{1 - \beta_{\text{M}}^2 \beta_{\text{s}}^2} \left[\beta_{\text{M}} \cos\theta_{\text{M}} (1 - \beta_{\text{s}}^2) + \beta_{\text{s}} \sqrt{1 - \beta_{\text{M}}^2} \sqrt{1 - \beta_{\text{M}}^2 (\cos^2\theta_{\text{M}} + \beta_{\text{s}}^2 \sin^2\theta_{\text{M}})} \right]. \tag{19}$$

Here θ_{M} is the angle between $\vec{\beta}_{\text{M}}$ and $\vec{k}$ and $\beta_{\text{eff}} = c_{\text{eff}} / c$, where c_{eff} is the apparent speed of light wrt. the computational frame of reference. It describes the speed of light, where both the emitter and the observer are at rest wrt. that frame.

We note that in the rest-frame of the medium, i.e., for $\beta_{\text{M}} = 0$ we indeed get the correct solution $k^0 / k = \beta_{\text{s}}$. In the limit $\beta_{\text{s}} = 1$ we also get the correct result $k^0 / k = 1$, and any reference to the velocity of the medium is gone, as expected.

Now, with k^0 given by (19), we can calculate the proper frequencies of the wave for an arbitrarily moving emitter and/or observer as

$$\frac{\omega_{\text{O}}}{\omega_{\text{E}}} = \frac{\underline{u}_{\text{O}} \cdot \underline{k}}{\underline{u}_{\text{E}} \cdot \underline{k}}. \tag{20}$$

In the literature the **Fizeau experiment** is discussed, i.e., the apparent speed of light in a medium with refractive index $n > 1$ (here n denotes the refractive index as measured in the rest frame of the medium). Of course in our notation that means $c_{\text{s}} = c/n < c$ is the speed of light in the medium as measured in the rest frame of the fluid. One should note that for a dispersive medium one has to take $n = n(\omega_{\text{M}})$ with the frequency of the light-wave in the rest frame of the medium, $\omega_{\text{M}} = c \underline{k} \cdot \underline{u}_{\text{M}}$ with k^0 given by (19).

In the literature one usually finds the result for the rest frame of the fluid, which we denote with a $*$, i.e., $\beta_{\mathrm{M}}^{*}=0$ and $\underline{k}^{*}=k^{*}(\beta_{\mathrm{s}},\vec{n}^{*})$, where (20) becomes very simple

$$\frac{\omega_{\mathrm{O}}^{*}}{\omega_{\mathrm{E}}^{*}}=\sqrt{\frac{1-\beta_{\mathrm{E}}^{*2}}{\beta_{\mathrm{s}}-\beta_{\mathrm{O}}^{*2}}\frac{1-\beta_{\mathrm{O}}^{*}\cos\theta_{\mathrm{O}}^{*2}}{\beta_{\mathrm{s}}-\beta_{\mathrm{E}}^{*}\cos\theta_{\mathrm{E}}^{*2}}}. \tag{21}$$

One must however note that of course θ_{O}^{*} and θ_{E}^{*} are not the same as the angles in any other reference frame, where the medium moves with a speed given by $\beta_{\mathrm{O}}\neq 0$ ("aberration effect"). Of course since (21) is a scalar quantity, because it refers to the frequencies of the wave measured in physically determined frames of reference (the rest frame of the observer and the emitter, respectively), it is the same quantity as given in (20) but expressed in terms of different observables referring to the different frames. Of course we can get the relations by the Lorentz boost from the general frame to the rest frame of the fluid with boost velocity $\vec{v}_{\mathrm{M}}$, but here we refrain from working out these lengthy formulae.

In optics it is more common to express the optical properties of the medium in terms of the **effective index of refraction**. Thus we insert $\beta_{\mathrm{s}}=1/n$ in (19), and after some algebra obtain the effective index of refraction of a moving medium, which is of course a frame-dependent quantity,

$$\frac{k}{k^{0}}=n_{\mathrm{eff}}=\frac{(n^{2}-1)\beta_{\mathrm{M}}\cos\Theta_{\mathrm{M}}-\sqrt{(1-\beta_{\mathrm{M}}^{2})[(n^{2}-1)(1-\beta_{\mathrm{M}}^{2}\cos^{2}\theta_{\mathrm{M}})+1-\beta_{\mathrm{M}}^{2}]}}{(n^{2}-1)\beta_{\mathrm{M}}^{2}\cos^{2}\theta_{\mathrm{M}}-(1-\beta_{\mathrm{M}}^{2})}. \tag{22}$$

In 1851 Hippolyte Fizeau (1819-1896) measured the apparent velocity of light c_{eff}. At his time the expectation was that the light, i.e., the aether which is thought to be the fluid whose oscillations are observed as light, should be dragged along with the medium, and the velocity of light should be simply given by the (of course Newtonian) addition of the velocity of light as measured in the rest frame of the medium (including the aether) and the velocity of the medium, i.e., $c_{\mathrm{eff}}=c/n+v_{\mathrm{M}}\cos\theta_{\mathrm{M}}$. Our analysis of course results in (19). To better compare it to Fizeau's expectation, we expand this equation up to first order in $\beta_{\mathrm{M}}=v_{\mathrm{M}}/c$. This leads to

$$c_{\mathrm{eff}}=\frac{c}{n}+v_{\mathrm{M}}\cos\theta_{\mathrm{M}}\left(1-\frac{1}{n^{2}}\right)+\mathcal{O}(\beta_{\mathrm{M}}^{2}), \tag{23}$$

and this was indeed, what Fizeau found in his interference experiment [1]. As any $\mathcal{O}(\beta_{\mathrm{M}})$ effect, also this result agrees with Fresnel's aether theory, which was not well accepted at Fizeau's time. This predicted that the light (or rather the aether) is indeed only "partially dragged" with the medium with "Fresnel's drag coefficient" $f=1-1/n^{2}$. This aether theory also leads to results consistent with the failed attempts to experimentally confirm the "aether wind" in the vacuum, i.e., the motion of the emitter and/or observer wrt. the aether restframe with measurements of effects of order $\mathcal{O}(\beta_{\mathrm{M}})$, for which Fresne's drag coeeficient, $f=1$. This again triggered Michelson and Morley's famous interferometer experiment for measuring the "aether wind" due to the motion of the Earth around the Sun, which was sensitive enough to order $\mathcal{O}(\beta^{2})$. The series expansion of our relativistic result (19) to this order reads

$$c_{\mathrm{eff}}=\frac{c}{n}+v_{\mathrm{M}}\cos\theta_{\mathrm{M}}\left(1-\frac{1}{n^{2}}\right)-\frac{(n^{2}-1)(1+\cos^{2}\theta_{\mathrm{M}})}{2n^{3}}c\beta_{\mathrm{M}}^{2}+\mathcal{O}(\beta_{\mathrm{M}}^{3}). \tag{24}$$

According to our relativistic treatment there is also no "aether wind" at this sensitivity, because also this correction vanishes for $n=1$, and indeed the relativistic result for light propagation in the vacuum must exactly lead to $c_{\mathrm{eff}}=c$.

[1] See `https://en.wikipedia.org/wiki/Fizeau_experiment`

References

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