Ideal relativistic Boltzmann gas

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1 Thermodynamical quantities

The grand-canonical partition sum for a relativistic Boltzmann gas is defined as[1]

$$\begin{aligned} Z(\beta,\alpha) = N(\beta,\alpha) &= \frac{V}{(2\pi)^3} \exp\alpha \int_{\mathbb{R}^3} \mathrm{d}^3 p \exp(-\beta\sqrt{\vec{p}^2+m^2}) \\ &= \frac{4\pi V}{(2\pi)^3} \exp\alpha \int_0^\infty dP P^2 \exp(-\beta\sqrt{P^2+m^2}). \end{aligned} \tag{1}$$

We note that $\beta = 1/T$ with $T > 0$ is the inverse temperature and $\alpha = \mu/T$ with $\mu \in \mathbb{R}$ the chemical potential, and $Z = N$ is the mean particle number of particles in volume, V.
Substitution of $P = m \cosh\eta$ leads to

$$Z(\beta\alpha) = \frac{4\pi V m^3}{(2\pi)^3} \exp\alpha \int_0^\infty \mathrm{d}\eta \cosh\eta \sinh^2\eta \exp(-\beta m \cosh\eta). \tag{2}$$

Since

$$\cosh\eta \sinh^2\eta = \cosh^3\eta - \cosh\eta = \frac{1}{4}[\cosh(3\eta) - \cosh\eta] \tag{3}$$

using (8) and (9 for $\nu = 2$) yields

$$Z(\beta,\alpha) = \frac{\pi V m^3}{(2\pi)^3} \exp\alpha [\mathrm{K}_3(\beta m) - K_1(\beta m)] = \frac{4\pi V m^2}{(2\pi)^3 \beta} \exp\alpha \mathrm{K}_2(\beta m). \tag{4}$$

The internal energy is given by

$$\begin{aligned} U(\beta,\alpha) &= \frac{V}{(2\pi)^3} \exp\alpha \int_{\mathbb{R}^3} \mathrm{d}^3 p \sqrt{\vec{p}^2+m^2} \exp(-\beta\sqrt{p^2+m^2}) \\ &= -\partial_\beta Z(\beta,\alpha) = \frac{4\pi V m^2}{(2\pi)^3} \exp\alpha \left[\frac{1}{\beta^2}\mathrm{K}_2(\beta m) - \frac{m}{\beta} K_2'(\beta m)\right] \\ &= \frac{4\pi V m^2}{(2\pi)^3 \beta} \left[\frac{3}{\beta}\mathrm{K}_2(\beta m) + m\mathrm{K}_1(\beta m)\right]. \end{aligned} \tag{5}$$

In the last step we have used (11).

[1] We use the standard natural units of relativistic thermal field theory, i.e., $\hbar = c = k_{\mathrm{B}} = 0$ and $(\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-)$.

For the average energy per particle, we find

$$\langle E \rangle = \frac{U}{N} = \frac{3}{\beta} + m \frac{\mathrm{K}_1(\beta m)}{\mathrm{K}_2(\beta m)}. \tag{6}$$

The non-relativistic limit follows for $m \gg T$, using the asymptotic expansion (13):

$$\langle E \rangle_{\text{non-rel}} = m + \frac{3}{2\beta}. \tag{7}$$

A The Modified Bessel Functions

We define the modified Bessel functions as the integrals

$$\mathrm{K}_\nu(z) = \int_0^\infty \mathrm{d}y \cosh(\nu y) \exp(-z \cosh y). \tag{8}$$

First we derive a recursion relation:

$$\mathrm{K}_{\nu+1}(z) - \mathrm{K}_{\nu-1}(z) = \frac{2\nu}{z} \mathrm{K}_\nu(z) \tag{9}$$

This is shown by integrating (8) by parts, which gives

$$\begin{aligned} \mathrm{K}_\nu(z) &= \frac{z}{\nu} \int_0^\infty \mathrm{d}y \sinh(\nu y) \sinh y \exp(-z \cosh y) \\ &= \frac{z}{2\nu} \int_0^\infty \mathrm{d}y \{\cosh[(\nu+1)y] - \cosh[(\nu-1)y\} \exp(-z \cosh y) \\ &= \frac{z}{2\nu} \left[\mathrm{K}_{\nu+1}(z) - \mathrm{K}_{\nu-1}(z)\right]. \end{aligned} \tag{10}$$

In a similar way we find for the derivative of the Bessel functions

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} K_\nu(z) &= -\int_0^\infty \mathrm{d}y \cosh y \cosh(\nu y) \exp(-z \cosh y) \\ &= -\frac{1}{2} \int_0^\infty \mathrm{d}y \, \{\cosh[(\nu+1)y] + \cosh[(\nu-1)y]\} \exp(-z \cosh y) \\ &= -\frac{1}{2} \left[K_{\nu+1}(z) + K_{\nu-1}(z)\right] \overset{(10)}{=} -\frac{\nu K_\nu(z) + z K_{\nu-1}}{z}. \end{aligned} \tag{11}$$

Further we need the behavior of the functions for $z \gg 1$. To find the asymptotic behavior for $z \to \infty$ we can use the saddle-point approximation of the defining integral (8). To that end one writes the integrand in the form

$$\begin{aligned} \cosh(\nu y) \exp(-z \cosh y) &= \exp\left[-z\left(1 + \frac{y^2}{2}\right)\right] \cosh(\nu y) \exp\left[-z\left(\cosh y - 1 - \frac{y^2}{2}\right)\right] \\ &= \exp\left[-z\left(1 + \frac{y^2}{2}\right)\right]\left[1 + \frac{\nu}{2} y^2 + \frac{\nu^4 - z}{24} y^4 + \mathcal{O}(y^6)\right] \end{aligned} \tag{12}$$

Plugging this into (8) we find the first two terms of the asymptotic expansion

$$K_\nu(z) \underset{z\to\infty}{\cong} \sqrt{\frac{\pi}{2z}} \exp(-z)\left[1 + \frac{4\nu^2 - 1}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right]. \tag{13}$$