Tutorial “General Relativity”

Winter term 2016/2017

Lecturer: Prof. Dr. C. Greiner
Tutor: Hendrik van Hees

Sheet No. 1 – Solutions

will be discussed on Nov/04/16

1. Decay of the muon

Muons have been discovered while studying cosmic radiation at Caltech in the thirties of the last century. The muon is an unstable subatomic particle with a mean life time of $\tau \sim 2.2\mu$s (measured in its rest frame). Their decay via the weak interaction is described by

$$N(t) = N_0 \exp\left(-\frac{t}{\tau}\right),$$

where $N(t)$ is the number of muons after the time t, and N_0 is the initial number at $t = 0$. They travel nearly with the speed of light, $v = 0.998c$.

a.) What distance can a muon manage in its proper time[1]?

Solution: With the speed of light $c = 2.99792458 \cdot 10^8\,\text{m/s}$ one finds for the distance travelled in the proper lifetime of the muon $x = \beta c \tau \simeq 658.2\,\text{m}$.

b.) Why does an observer on Earth measure a mean lifetime of around 34.8μs. What distance would a muon travel in this time?

Solution: In the Earth frame the lifetime of the muon is increased by the Lorentz factor (“time dilation”): $\tau_{\text{Earth}} = \gamma = \tau/\sqrt{1-\beta^2} \simeq 34.8\ \mu$s. In this time the travelled distance is $\beta c \tau_{\text{Earth}} \simeq 10.4$ km.

c.) Suppose, that in 9 kilometers above sea level 10^8 muons were produced. How many of them reach the Earth’s surface (non-relativistically)? Why does an observer detect nearly 42% of them nonetheless?

Solution: The time it takes for the muon to travel to sea level is $t_{\text{Earth}} = 9 \cdot 10^3\,\text{m}/(\beta c) \simeq 30.1\,\mu$s, and the number of muons expected to reach the surface of the Earth is $N_{\text{nrel}} = N_0 \exp(-t_{\text{Earth}}/\tau) \simeq 115$. Correctly one has to take into account the dilation of the muon’s lifetime, and thus $N_{\text{rel}} = N_0 \exp(-t_{\text{Earth}}/\tau_{\text{Earth}}) \simeq 0.42 \cdot 10^8$.

2. Addition of velocities

Given a particle in frame Σ, which is moving at $\vec{u} = \frac{3}{4} c \vec{e}_1$ to the right and another observer in frame Σ', which is moving with $\vec{v} = -\frac{3}{4} c \vec{e}_1$ (i.e., to the left with respect to Σ). Why doesn’t the observer in Σ' measure a total speed of $\frac{3}{2}c$ of the particle? What speed does he measure?

Solution: We use Lorentz vectors in the (01) plane of Minkowski space. The proper velocity is

$$U = \gamma_u \begin{pmatrix} 1 \\ \beta_u \end{pmatrix}, \tag{1}$$

[1] Eigenzeit

and the boost to the frame Σ' is given by the Lorentz-boost matrix

$$\hat{\Lambda} = \hat{B}(-v) = \gamma_v \begin{pmatrix} 1 & \beta_v \\ \beta_v & 1 \end{pmatrix}, \tag{2}$$

i.e., the proper velocity in Σ' is

$$U' = \hat{\Lambda} U = \gamma_u \gamma_v \begin{pmatrix} 1 + \beta_u \beta_v \\ \beta_u + \beta_v \end{pmatrix}, \tag{3}$$

and thus the three-velocity of the particle with respect to Σ' is given by

$$u' = \frac{U'^q}{U'^0} = \frac{\beta_u + \beta_v}{1 + \beta_u \beta_v}. \tag{4}$$

For the values $\beta_u = \beta_v = 0.75$ one finds $\beta'_u = 0.96$.

3. **Arrow**
An arrow of length 1 m has been shot. While passing your view, you measure a length of 86.6 cm. At what speed v travels the arrow?

Solution: If the length is measured in the frame, where the arrow moves with a speed $v = \beta c$, it appears shorter by an inverse Lorentz factor, i.e.,

$$L_{\text{lab}} = \frac{L_{\text{proper}}}{\gamma} = \sqrt{1 - \beta^2} L_{\text{proper}} \tag{5}$$

Solved for β we get

$$\beta = \sqrt{1 - \left(\frac{L_{\text{lab}}}{L_{\text{proper}}} \right)^2} \simeq 0.5. \tag{6}$$

4. **Speed of a particle**
If a particle's kinetic energy is n times its rest energy, what is its speed?

Solution: The kinetic energy of a relativistic particle is given by

$$T = mc^2(\gamma - 1), \tag{7}$$

where m is the invariant mass of the particle. Setting $T = nmc^2$ we get

$$\gamma = n + 1 \;\Rightarrow\; \beta = \sqrt{1 - \frac{1}{(n+1)^2}} = \frac{\sqrt{n(n+2)}}{n+1}. \tag{8}$$

5. **Lorentz invariance**
Which of the following quantities is Lorentz-invariant (and which manifestly Lorentz covariant)?

a.) $\vec{x}^2$ b.) $x_\mu x^\mu$ c.) $x^\mu x^\nu$ d.) $\eta_{\mu\nu}$ e.) $\mathrm{d}s^2$ f.) $(\mathrm{d}x^0)^2$ g.) γ

Solution: a.) is the magnitude of a three-vector which is frame dependent; b.) $x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu$ is invariant since it is a Minkowski product of a four-vector with itself; c.) $x^\mu x^\nu$ are the contra-variant components of a 2^{nd}-rank Lorentz tensor; d.) $\eta_{\mu\nu}$ are the invariant components of the Lorentz metric, because for a Lorentz-transformation matrix one has $\eta'_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$; e.) $\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ is Lorentz invariant since it's the Minkowski product of a four-vector with itself, f.) $(\mathrm{d}x^0)^2$ is neither invariant nor covariant; g.) γ is not invariant, because it's the time component of the proper velocity $U^\mu = \gamma(1, \vec{v})$ of a particle with three-velocity $\vec{v}$.

6. **General rotation-free Lorentz boost**

Find the rotation-free Lorentz-boost matrix ${\Lambda^\mu}_\nu$, $x'^\mu = {\Lambda^\mu}_\nu x^\nu$ between two inertial frames Σ and Σ', where Σ' moves with the speed $\vec{v}$ wrt. to Σ ($|\vec{v}| < c$ in arbitrary direction).

Solution: For a boost in 1-direction (Σ' moves with three-velocity $\vec{v} = \beta c\, \vec{e}_1$ relative to Σ) one has

$$x'^0 = \gamma_v(x^0 - \beta x^1), \quad x'^1 = \gamma_v(x^1 - \beta x^0), \quad x'^2 = x^2, \quad x'^3 = x^3. \tag{9}$$

Since for an inertial observer the space is homogeneous and isotropic, we get the boost in a general direction by writing this in a rotation-kovariant form, i.e.,

$$x'^0 = \gamma_v(x^0 - \vec{\beta} \cdot \vec{x}), \quad \vec{x}' = \gamma_v \vec{x}_\parallel + \vec{x}_\perp - \gamma_v \vec{\beta} x^0, \tag{10}$$

where the components of the position vector $\vec{x}$ parallel and perpendicular to the relative velocity of the frames are

$$\vec{x}_\parallel = \frac{\vec{\beta} \cdot \vec{x}}{\beta^2} \vec{\beta}, \quad \vec{x}_\perp = \vec{x} - \vec{x}_\parallel, \tag{11}$$

respectively, and thus we get

$$\vec{x}' = \vec{x} + (\gamma - 1) \frac{\vec{\beta} \cdot \vec{x}}{\beta^2} \vec{\beta} - \gamma_v \vec{\beta} x^0. \tag{12}$$

So the general rotation-free Lorentz boost reads (in (1+3)-notation)

$$\hat{\Lambda} = ({\Lambda^\mu}_\nu) = \hat{B}(\vec{\beta}) = \begin{pmatrix} \gamma & -\gamma \vec{\beta}^{\mathrm{T}} \\ -\gamma \vec{\beta} & \mathbb{1}_{3\times 3} + (\gamma - 1) \frac{\vec{\beta}\vec{\beta}^{\mathrm{T}}}{\beta^2} \end{pmatrix}. \tag{13}$$