Übungsblatt 3

[due: June 06]

Problem 1: Transfer matrix for a q-state Potts model

a) Write down the transfer matrix for the q-state Potts model in $d = 1$ with Hamiltonian

$$\mathcal{H} = -J \sum_i \delta_{s_i s_{I+1}}, \qquad s_i = 1, 2, \ldots q.$$

b) Calculate the eigenvalues of the transfer matrix and give the free energy of the system.

Problem 2: $d = 1$ Ising model with periodic boundary conditions

a) Construct the matrix S which diagonalises the transfer matrix T, such that $T' = S^{-1} T S$ is diagonal. Express the matrix elements in terms of the eigenvalues of T: λ_1, λ_2.

b) Use the expression

$$\langle s_i \rangle = \frac{\mathrm{Tr}\left(S^{-1} \sigma_3 S \, T'^N \right)}{Z}$$

to show that in the thermodynamic limit $N \to \infty$

$$G(i, i+j) \equiv \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = \sin^2(2\phi) \left(\frac{\lambda_2}{\lambda_1} \right)^j$$

ϕ is defined by $\cot(2\phi) = e^{2K} \sinh(h)$. Useful relation: $e^{2K} \sqrt{\sinh^2(h) + e^{-4K}} = 1/\sin(2\phi)$.

c) Calculate the isothermal susceptibility $\chi_T = (\frac{\partial m}{\partial H})_T$ from the formula for $m(H)$ given in the lecture. Verify explicitly that

$$T \, \chi_T = \sum_j G(i, i+j).$$

Caution: in the thermodynamic limit the sum runs from $-\infty$ to ∞.

Problem 3: Radius of a blast wave

Download the associated data file from the lecture page, which gives the radius of the shock wave of an explosion according to photographic snapshots taken at the specified times. Assume that the motion of the shock wave is unaffected by the presence of the ground, and that it is determined only by the energy E released in the blast and the density ρ of the undisturbed air into which the shock is propagating. Derive a scaling law for the radius of the shock wave as a function of time. Test your scaling law by a plot and use it to deduce the energy released in the blast.