Investigation of heavy exotic mesons with lattice QCD

"Quantum Theory Seminar" – FSU Jena

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Introductory remarks

- In this talk only heavy exotic mesons:
  - tetraquarks $\bar{b}\bar{b}qq$,
  - tetraquarks $\bar{b}b\bar{q}q$ (I will most likely not discuss this, due to time limitation),
  - hybrid mesons $\bar{b}b$ + gluons

  (light quarks $q \in \{u, d, s\}$).
- Lattice QCD = numerical QCD.
  - Lattice QCD is not a model, there are no approximations.
  - Results are full and rigorous QCD results.
    $\rightarrow$ Lattice QCD simulations can be seen as computer experiments (based on QCD).
  - The investigation of exotic mesons in lattice QCD is technically very difficult.
    $\rightarrow$ Even though we use lattice QCD, there are assumptions and simplifying approximations (as you will see during the talk) ...

Two types of approaches

- Two types of approaches, when studying **heavy** exotic mesons with lattice QCD:
  - **Born-Oppenheimer approximation** (a 2-step procedure):
    (1) Compute the potential $V(r)$ of the two heavy quarks (approximated as static quarks) in the presence of two light quarks and/or gluons using lattice QCD.
    $\rightarrow$ full QCD results
    (2) Use standard techniques from quantum mechanics and $V(r)$ to study the dynamics of two heavy quarks (Schrödinger equation, scattering theory, etc.).
    $\rightarrow$ an approximation

    $\rightarrow$ **The main focus of this talk.**
  - **Full lattice QCD computations of eigenvalues of the finite-volume QCD Hamiltonian**:
    * Masses of stable hadrons correspond to energy eigenvalues at infinite volume (comparatively easy).
    * Masses and decay widths of resonances can be calculated from the volume dependence of the energy eigenvalues (rather difficult).

    $\rightarrow$ **Only a brief discussion at the end, if there is time.**

Part 1:
Born-Oppenheimer approximation

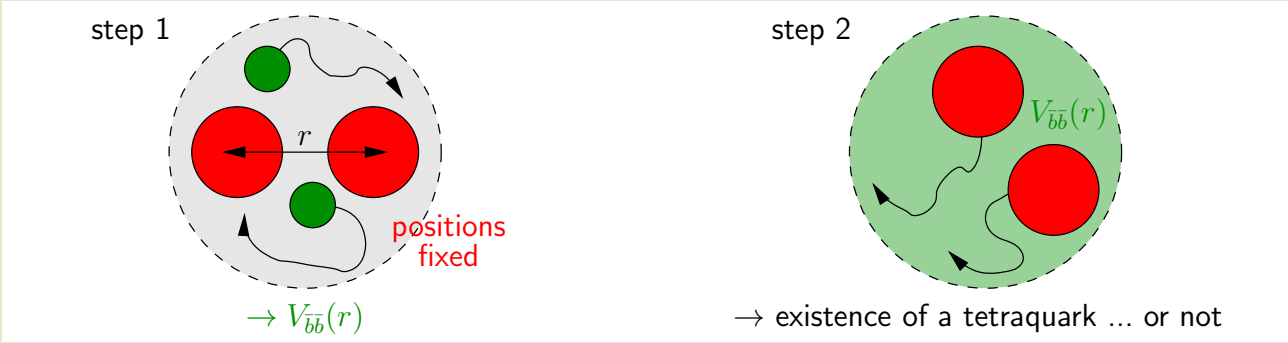
Basic idea: lattice QCD + BO

- Start with $\bar{b}\bar{b}qq$.
- $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^+)$ is the bottom counterpart of the experimentally observed T_{cc}. [R. Aaij *et al.* [LHCb], Nature Phys. **18**, 751-754 (2022) [arXiv:2109.01038]].
- Study such $\bar{b}\bar{b}qq$ tetraquarks in two steps:

  (1) **Compute potentials of the two static quarks $\bar{b}\bar{b}$ in the presence of two lighter quarks qq ($q \in \{u, d, s\}$) using lattice QCD.**

  (2) **Check, whether these potentials are sufficiently attractive to host bound states or resonances** ($\rightarrow$ tetraquarks) **by using techniques from quantum mechanics and scattering theory.**

  (1) + (2) $\rightarrow$ Born-Oppenheimer approximation.

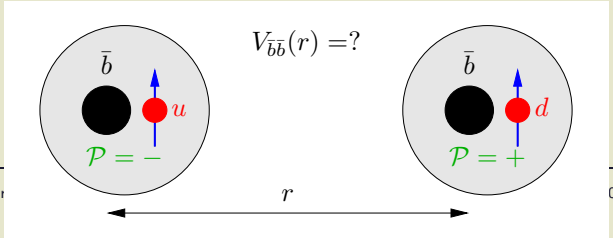


$\bar{b}\bar{b}qq$ / BB potentials (1)

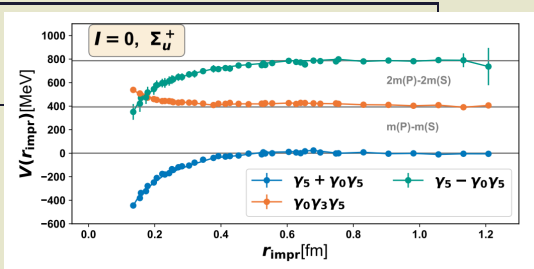
- At large $\bar{b}\bar{b}$ separation r, the four quarks form two static-light mesons $B = \bar{b}q$ and $B = \bar{b}q$.
- Potentials of static quarks are independent of the heavy spins.
- Consider only trial states/operators with vanishing orbital angular momentum, i.e. consider
  - pseudoscalar/vector mesons ($j^P = (1/2)^-$, PDG: B, B^*),
  - scalar/pseudovector mesons ($j^P = (1/2)^+$, PDG: B_0^*, B_1^*),

  which are among the lightest static-light mesons (j: spin of the light degrees of freedom).
- Compute and study the dependence of $\bar{b}\bar{b}$ potentials in the presence of qq on
  - the "light" quark flavors $q \in \{u, d, s\}$ (isospin, flavor),
  - the "light" quark spins (the static quark spin is irrelevant),
  - the types of the mesons B, B^* and/or B_0^*, B_1^* (parity).

  $\rightarrow$ Many different channels: attractive as well as repulsive, different asymptotic values ...



$\bar{b}\bar{b}qq$ / BB potentials (2)



- Rotational symmetry broken by static quarks $\bar{b}\bar{b}$.
- Symmetries and quantum numbers:
  - $|j_z| \equiv \Lambda$: rotations around the separation axis (e.g. z axis).
  - $P \equiv \eta$: parity.
  - $P_x \equiv \epsilon$: reflection along an axis perpendicular to the separation axis (e.g. x axis).
- To determine $\bar{b}\bar{b}$ potentials $V_{\bar{b}\bar{b};I,\Lambda_\eta^\epsilon}(r)$, compute temporal correlation functions

$$\langle\Omega|\mathcal{O}^\dagger_{BB,\Gamma}(t)\mathcal{O}_{BB,\Gamma}(0)|\Omega\rangle \propto_{t\to\infty} e^{-V_{\bar{b}\bar{b};I,\Lambda_\eta^\epsilon}(r)t}$$

of operators

$$\mathcal{O}_{BB,\Gamma} \quad = \quad 2N_{BB}(\mathcal{C}\Gamma)_{AB}(\mathcal{C}\tilde{\Gamma})_{CD}\Big(\bar{Q}^a_C(-\mathbf{r}/2)q^a_A(-\mathbf{r}/2)\Big)\Big(\bar{Q}^b_D(+\mathbf{r}/2)q^b_B(+\mathbf{r}/2)\Big).$$

  - $C = \gamma_0\gamma_2$ (charge conjugation matrix).
  - $qq \in \{ud - du\,,\ uu\,,\ dd\,,\ ud + du\}$ (isospin I, I_z).
  - Γ is an arbitrary combination of γ matrices (spin Λ, parity η, ϵ).
  - $\tilde{\Gamma} \in \{(1-\gamma_0)\gamma_5\,,\ (1-\gamma_0)\gamma_j\}$ (irrelevant).

Lattice setup

- Majority of published results computed on ETMC gauge link ensembles:
  - $N_f = 2$ dynamical quark flavors.
  - Lattice spacing $a \approx 0.079$ fm.
  - $24^3 \times 48$, i.e. spatial lattice extent ≈ 1.9 fm.
  - Three different pion masses $m_\pi \approx 340$ MeV, $m_\pi \approx 480$ MeV, $m_\pi \approx 650$ MeV.

  [R. Baron *et al.* [ETM Collaboration], JHEP **1008**, 097 (2010) [arXiv:0911.5061]

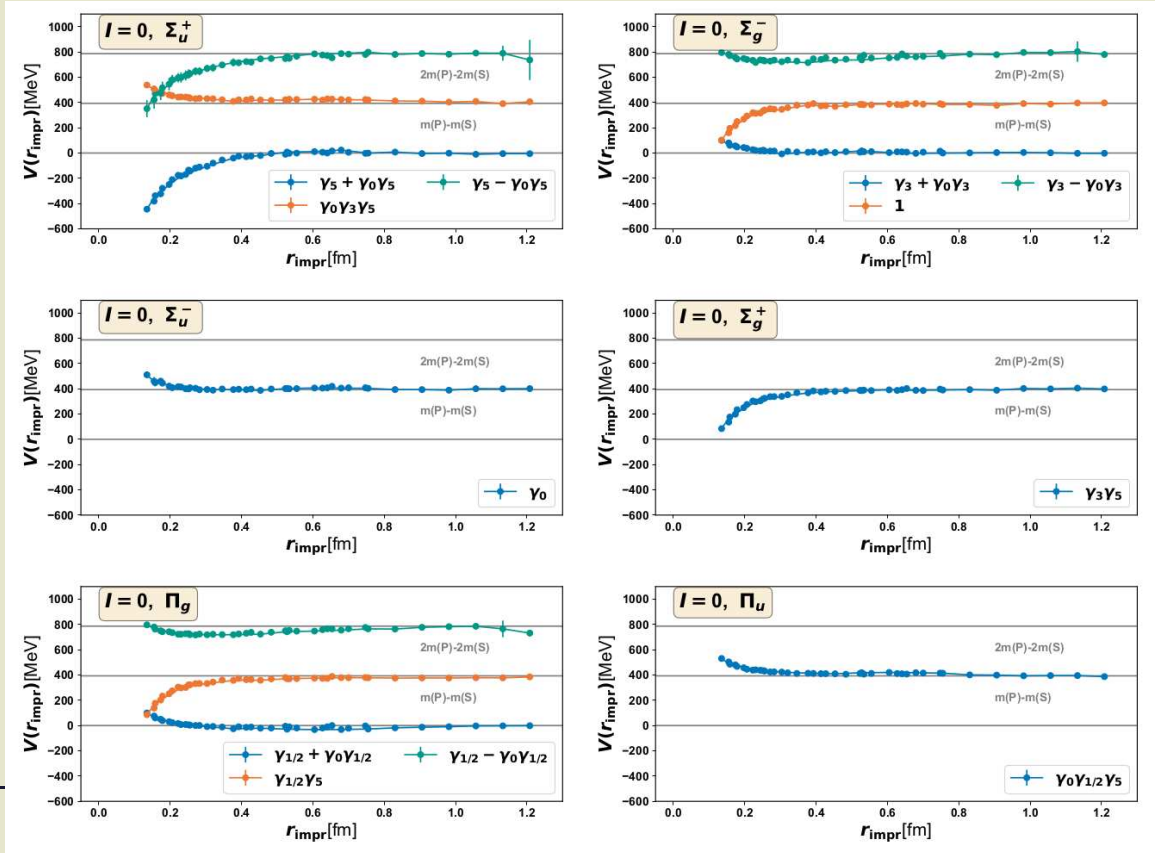
- Recent results (not yet published) computed on CLS gauge link ensembles:
  - $N_f = 2$ dynamical quark flavors.
  - Lattice spacing $a \approx 0.0749$ fm.
  - $32^3 \times 64$, i.e. spatial lattice extent ≈ 2.42 fm.
  - Pion mass $m_\pi \approx 331$ MeV.

  [P. Fritzsch, F. Knechtli, B. Leder, M. Marinkovic, S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B **865**, 397-429 (2012) [arXiv:1205.5380]]

  [G. P. Engel, L. Giusti, S. Lottini and R. Sommer, Phys. Rev. D **91**, no. 5, 054505 (2015) [arXiv:1411.6386]]

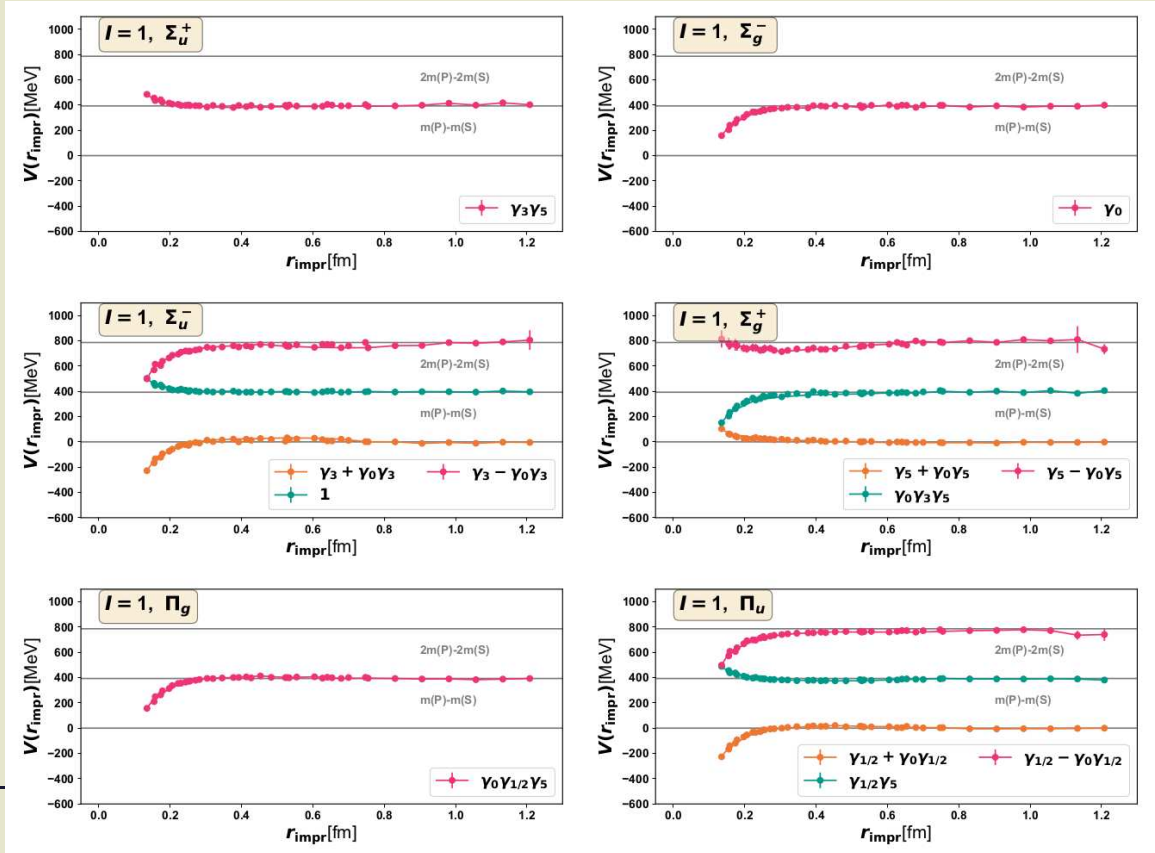
$\bar{b}\bar{b}qq$ / BB potentials (3)

[P. Bicudo, M. Marinkovic, L. Müller, M.W. unpublished ongoing work]



$\bar{b}\bar{b}qq$ / BB potentials (4)

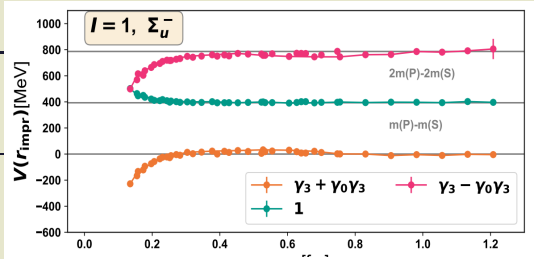
[P. Bicudo, M. Marinkovic, L. Müller, M.W. unpublished ongoing work]



$\bar{b}\bar{b}qq$ / BB potentials (5) to (8)

- **Why are there three different asymtotic values?**
  - They correspond to $B^{(*)}B^{(*)}$ potentials, to $B^{(*)}B^*_{0,1}$ potentials and $B^*_{0,1}B^*_{0,1}$ potentials.
- **Why are certain channels attractive and others repulsive?**
  - $(I=0, j=0)$ and $(I=1, j=1)$ $\rightarrow$ attractive $\bar{b}\bar{b}qq$ / BB potentials.
  - $(I=0, j=1)$ and $(I=1, j=0)$ $\rightarrow$ repulsive $\bar{b}\bar{b}qq$ / BB potentials.
  - Because of the Pauli principle and "1-gluon exchange" at small r.
- **24 different (i.e. non-degenerate) $\bar{b}\bar{b}qq$ / BB potentials.**

$\bar{b}\bar{b}qq$ / BB potentials (5)



Why are there three different asymtotic values?

- Differences ≈ 400 MeV, approximately the mass difference of $B^{(*)}$ $(P = -)$ and $B^*_{0,1}$ $(P = +)$.
- Suggests that the three different asymtotic values correspond to $B^{(*)}B^{(*)}$ potentials, to $B^{(*)}B^*_{0,1}$ potentials and $B^*_{0,1}B^*_{0,1}$ potentials.
- Can be checked and confirmed, by rewriting the $\bar{b}\bar{b}qq$ creation operators in terms of meson-meson creation operators (Fierz transformation).
- Example: $qq = uu$, $\Gamma = (\gamma_3 + \gamma_0\gamma_3)$ (attractive, lowest asymptotic value),

$$\Big(C(\gamma_3 + \gamma_0\gamma_3)\Big)_{AB}\Big(\bar{Q}_C(-\mathbf{r}/2)q_A(-\mathbf{r}/2)\Big)\Big(\bar{Q}_D(+\mathbf{r}/2)q_B(+\mathbf{r}/2)\Big) \quad \propto$$
$$\propto \quad (B^{(*)})_\uparrow(B^{(*)})_\downarrow + (B^{(*)})_\downarrow(B^{(*)})_\uparrow.$$

- Example: $qq = uu$, $\Gamma = 1$ (repulsive, medium asymptotic value),

$$\Big(C1\Big)_{AB}\Big(\bar{Q}_C(-\mathbf{r}/2)q_A(-\mathbf{r}/2)\Big)\Big(\bar{Q}_D(+\mathbf{r}/2)q_B(+\mathbf{r}/2)\Big) \quad \propto$$
$$\propto \quad (B^{(*)})_\uparrow(B^*_{0,1})_\downarrow - (B^{(*)})_\downarrow(B^*_{0,1})_\uparrow + (B^*_{0,1})_\uparrow(B^{(*)})_\downarrow - (B^*_{0,1})_\downarrow(B^{(*)})_\uparrow.$$

$\bar{b}\bar{b}qq$ / BB potentials (6)

Why are certain channels attractive and others repulsive? (1)

- Fermionic wave functions must be antisymmetric (Pauli principle); in quantum field theory/QCD automatically realized.
- qq isospin: $I = 0$ antisymmetric, $I = 1$ symmetric.
- qq angular momentum/spin: $j = 0$ antisymmetric, $j = 1$ symmetric.
- qq color:
  - $(I = 0, j = 0)$ and $(I = 1, j = 1)$: must be antisymmetric, i.e., a triplet $\bar{3}$.
  - $(I = 0, j = 1)$ and $(I = 1, j = 0)$: must be symmetric, i.e., a sextet 6.
- The four quarks $\bar{b}\bar{b}qq$ must form a color singlet:
  - qq in a color triplet $\bar{3}$ $\rightarrow$ static quarks $\bar{b}\bar{b}$ also in a triplet 3.
  - qq in a color sextet 6 $\rightarrow$ static quarks $\bar{b}\bar{b}$ also in a sextet $\bar{6}$.

$\bar{b}\bar{b}qq$ / BB potentials (7)

Why are certain channels attractive and others repulsive? (2)

- Assumption: attractive/repulsive behavior of $\bar{b}\bar{b}$ at small separations r is mainly due to 1-gluon exchange,
  - color triplet 3 is attractive, $V_{\bar{b}\bar{b}}(r) = -2\alpha_s/3r$,
  - color sextet $\bar{6}$ is repulsive, $V_{\bar{b}\bar{b}}(r) = +\alpha_s/3r$

  (easy to calculate in LO perturbation theory).
- Summary:
  - $(I = 0, j = 0)$ and $(I = 1, j = 1)$ $\rightarrow$ attractive $\bar{b}\bar{b}$ potential $V_{\bar{b}\bar{b}}(r)$.
  - $(I = 0, j = 1)$ and $(I = 1, j = 0)$ $\rightarrow$ repulsive $\bar{b}\bar{b}$ potential $V_{\bar{b}\bar{b}}(r)$.
- Expectation consistent with the obtained lattice results.
- **Pauli principle and assuming "1-gluon exchange" at small r explains, why certain channels are attractive and others repulsive.**

$\bar{b}\bar{b}qq$ / BB potentials (8)

- Summary of $\bar{b}\bar{b}qq$ / BB potentials:

$B^{(*)}B^{(*)}$ potentials:	attractive:	$1 \oplus 3 \oplus 6$	(10 states).
	repulsive:	$1 \oplus 3 \oplus 2$	(6 states).
$B^{(*)}B^{*}_{0,1}$ potentials:	attractive:	$1 \oplus 1 \oplus 3 \oplus 3 \oplus 2 \oplus 6$	(16 states).
	repulsive:	$1 \oplus 1 \oplus 3 \oplus 3 \oplus 2 \oplus 6$	(16 states).
$B^{*}_{0,1}B^{*}_{0,1}$ potentials:	attractive:	$1 \oplus 3 \oplus 6$	(10 states).
	repulsive:	$1 \oplus 3 \oplus 2$	(6 states).

  - 2-fold degeneracy due to spin $j_z = \pm 1$.
  - 3-fold degeneracy due to isospin $I = 1$, $I_z = -1, 0, +1$.

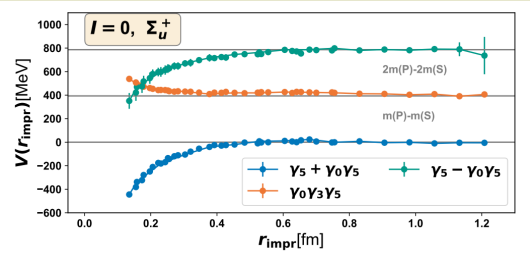
$\rightarrow$ 24 **different $\bar{b}\bar{b}qq$ / BB potentials.**

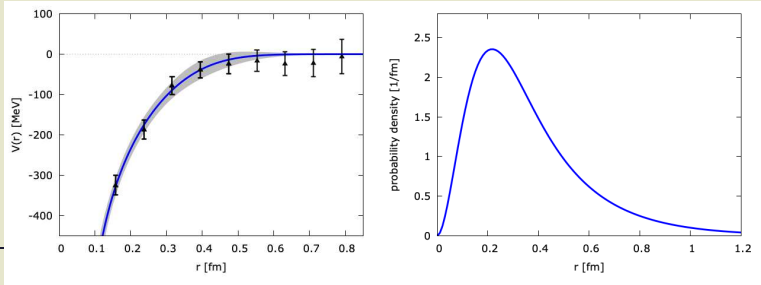
Stable $\bar{b}\bar{b}qq$ tetraquarks (2)

- The most attractive potential of a $B^{(*)}B^{(*)}$ meson pair has $I = 0, \Lambda_\eta^\epsilon = \Sigma_u^+$.
- Parameterize lattice results by

$$V_{\bar{b}\bar{b};0,\Sigma_u^+}(r) \quad = \quad -\frac{\alpha}{r}\exp\left(-\left(\frac{r}{d}\right)^p\right) + V_0$$

(1-gluon exchange at small r; color screening at large r).
[P. Bicudo, K. Cichy, A. Peters, M.W., Phys. Rev. D **93**, 034501 (2016) [arXiv:1510.03441]]





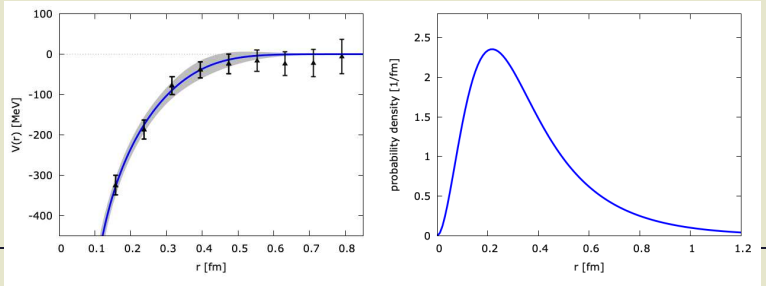
Stable $\bar{b}\bar{b}qq$ tetraquarks (2)

- Solve the Schrödinger equation for the relative coordinate of the heavy quarks $\bar{b}\bar{b}$ using the previously computed $\bar{b}\bar{b}qq$ / BB potentials,

$$\Big(\frac{1}{m_b}\Big(-\frac{d^2}{dr^2}+\frac{L(L+1)}{r^2}\Big)+V_{\bar{b}\bar{b};0,\Sigma_u^+}(r)-2m_B\Big)R(r) \quad = \quad ER(r).$$

- Possibly existing bound states, i.e. $E<0$, indicate QCD-stable $\bar{b}\bar{b}qq$ tetraquarks.
- There is a bound state for orbital angular momentum $L=0$ of $\bar{b}\bar{b}$:
  - Binding energy $E=-90^{+43}_{-36}$ MeV with respect to the BB^* threshold.
  - Quantum numbers: $I(J^P)=0(1^+)$.

[P. Bicudo, M.W., Phys. Rev. D **87**, 114511 (2013) [arXiv:1209.6274]]

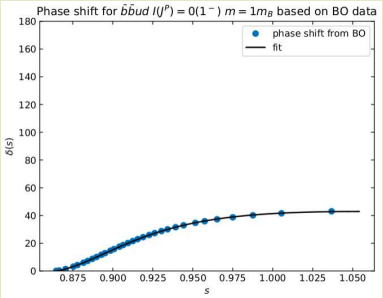


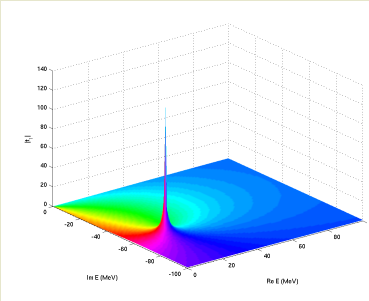
Further $\bar{b}\bar{b}qq$ results (1)

- **Are there further QCD-stable $\bar{b}\bar{b}qq$ tetraquarks with other $I(J^P)$ and light flavor quantum numbers?**
  - → No, not for $qq = ud$ (both $I = 0, 1$), not for $qq = ss$.
    [P. Bicudo, K. Cichy, A. Peters, B. Wagenbach, M.W., Phys. Rev. D **92**, 014507 (2015) [arXiv:1505.00613]]
  - → $\bar{b}\bar{b}us$ was not investigated.
  - – Strong evidence from full QCD computations that a QCD-stable $\bar{b}\bar{b}us$ tetraquark exists (see part 2 of this talk).
- **Effect of heavy quark spins:**
  - – Expected to be $\mathcal{O}(m_{B^*} - m_B) = \mathcal{O}(45\,\text{MeV})$.
  - – Previously ignored (potentials of static quarks are independent of the heavy spins).
  - – In [P. Bicudo, J. Scheunert, M.W., Phys. Rev. D **95**, 034502 (2017) [arXiv:1612.02758]] included in a crude phenomenological way via a BB^* and a B^*B^* coupled channel Schrödinger equation with the experimental mass difference $m_{B^*} - m_B$ as input.
  - → Binding energy reduced from around $90\,\text{MeV}$ to $59\,\text{MeV}$.
  - → Physical reason: the previously discussed attractive potential does not only correspond to a lighter BB^* pair, but has also a heavier B^*B^* contribution.

Further $\bar{b}\bar{b}qq$ results (2)

- **Are there $\bar{b}\bar{b}qq$ tetraquark resonances?**
  - In
    [P. Bicudo, M. Cardoso, A. Peters, M. Pflaumer, M.W., Phys. Rev. D **96**, 054510 (2017) [arXiv:1704.02383]]
    resonances studied via standard scattering theory from quantum mechanics textbooks.





→ Heavy quark spins ignored.

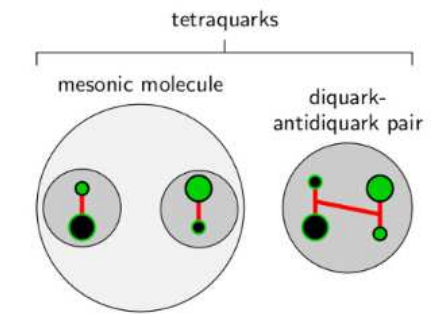
→ Indication for $\bar{b}\bar{b}ud$ tetraquark resonance with $I(J^P) = 0(1^-)$ found, $E = 17^{+4}_{-4}$ MeV above the BB threshold, decay width $\Gamma = 112^{+90}_{-103}$ MeV.

  - In
    [J. Hoffmann, A. Zimermmane-Santos and M.W., PoS **LATTICE2022**, 262 (2023) [arXiv:2211.15765]]
    heavy quark spins included.

→ $\bar{b}\bar{b}ud$ resonance not anymore existent.

→ Physical reason: the relevant attractive potential does not only correspond to a lighter BB pair, but has also a heavier B^*B^* contribution.

Further $\bar{b}\bar{b}qq$ results (3)



- Structure of the QCD-stable $\bar{b}\bar{b}ud$ tetraquark with $I(J^P) = 0(1^+)$: meson-meson (BB) versus diquark-antidiquark (Dd).
  - Use not just one but two operators,

$$\mathcal{O}_{BB,\Gamma} = 2N_{BB}(\mathcal{C}\Gamma)_{AB}(\mathcal{C}\tilde{\Gamma})_{CD}\Big(\bar{Q}^a_C(-\mathbf{r}/2)q^a_A(-\mathbf{r}/2)\Big)\Big(\bar{Q}^b_D(+\mathbf{r}/2)q^b_B(+\mathbf{r}/2)\Big)$$

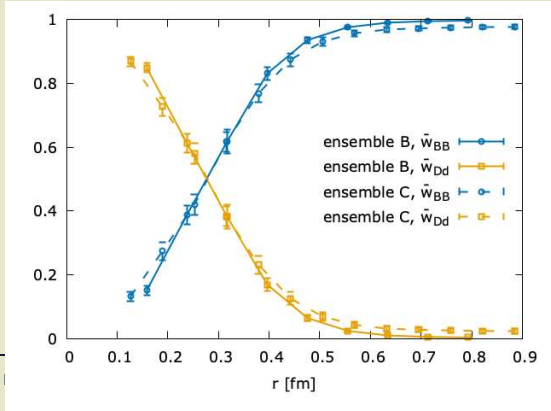
$$\mathcal{O}_{Dd,\Gamma} = -N_{Dd}\epsilon^{abc}\Big(q^b_A(\mathbf{z})(\mathcal{C}\Gamma)_{AB}q^c_B(\mathbf{z})\Big)$$
$$\epsilon^{ade}\Big(\bar{Q}^f_C(-\mathbf{r}/2)U^{fd}(-\mathbf{r}/2;\mathbf{z})(\mathcal{C}\tilde{\Gamma})_{CD}\bar{Q}^g_D(+\mathbf{r}/2)U^{ge}(+\mathbf{r}/2;\mathbf{z})\Big),$$

compare the contribution of each operator to the $\bar{b}\bar{b}$ potential $V_{\bar{b}\bar{b};0,\Sigma_u^+}(r)$.
[P. Bicudo, A. Peters, S. Velten, M.W., Phys. Rev. D **103**, 114506 (2021) [arXiv:2101.00723]]

→ $r \lesssim 0.2$ fm: Clear diquark-antidiquark dominance.

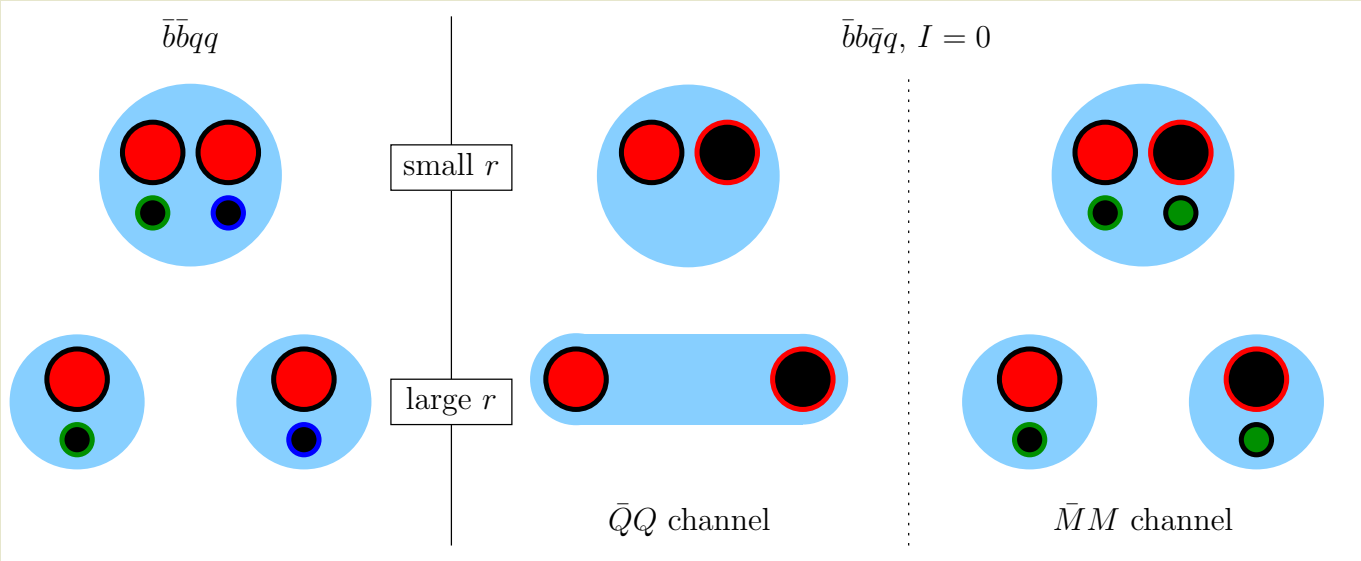
→ $0.5\,\text{fm} \lesssim r$: Essentially a meson-meson system.

→ Integrate over t to estimate the composition of the tetraquark: $\%BB \approx 60\%$, $\%Dd \approx 40\%$.

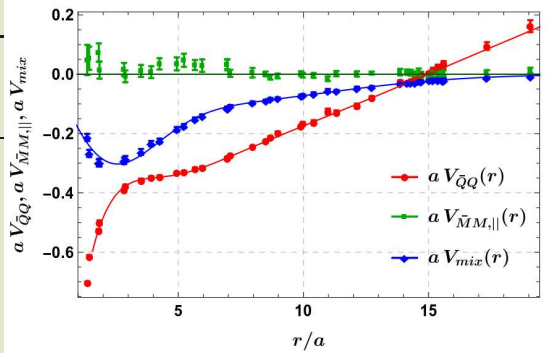


Bottomonium, $I = 0$: difference to $\bar{b}\bar{b}qq$

- Now bottomonium with $I = 0$, i.e. $\bar{b}b$ and/or $\bar{b}b\bar{q}q$ (with $\bar{q}q = (\bar{u}u + \bar{d}d)/\sqrt{2}, \bar{s}s$).
- Technically more complicated than $\bar{b}\bar{b}qq$, because there are two channels:
  - Quarkonium channel, $\bar{Q}Q$ (with $Q \equiv b$).
  - Heavy-light meson-meson channel, $\bar{M}M$ (with $M = \bar{Q}q$), "string breaking".



Bottomonium, $I = 0$: ...



- Lattice computation of potentials for both channels ($\bar{Q}Q$ and $\bar{M}M$) needed, additionally also a mixing potential:
  - Pioneering work:
    [G. S. Bali *et al.* [SESAM Collaboration], Phys. Rev. D **71**, 114513 (2005) [hep-lat/0505012]]
    Rather heavy u/d quark masses ($m_\pi \approx 650\,\text{MeV}$), only 2 flavors, not $2+1$.
  - More recent work:
    [J. Bulava, B. Hörz, F. Knechtli, V. Koch, G. Moir, C. Morningstar and M. Peardon, Phys. Lett. B **793**, 493-498 (2019) [arXiv:1902.04006]]
    Unfortunately, mixing potential not computed.
  - Several assumptions needed to adapt the "Bali results" to $2+1$ flavors and physical quark masses.

$\rightarrow$ Potential for a coupled channel Schrödiger equation (see next slide):

$$V(\mathbf{r}) = \begin{pmatrix} V_{\bar{Q}Q}(r) & V_{\text{mix}}(r)(1 \otimes \mathbf{e}_r) & (1/\sqrt{2})V_{\text{mix}}(r)(1 \otimes \mathbf{e}_r) \\ V_{\text{mix}}(r)(1 \otimes \mathbf{e}_r) & V_{\bar{M}M}(r) & 0 \\ (1/\sqrt{2})V_{\text{mix}}(r)(1 \otimes \mathbf{e}_r) & 0 & V_{\bar{M}M}(r) \end{pmatrix}.$$

Bottomonium, $I = 0$: SE

- Schrödinger equation non-trivial:
  - 3 coupled channels, $\bar{b}b$, BB (3 components), $B_s B_s$ (3 components).
  - Static potentials used as input have other symmetries and quantum numbers than bottomonium states (Λ_η^ϵ versus J^{PC}).

$$\left(-\frac{1}{2}\mu^{-1}\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\mathbf{L}^2}{r^2}\right) + V(\mathbf{r}) + \begin{pmatrix} E_{\text{threshold}} & 0 & 0 \\ 0 & 2m_M & 0 \\ 0 & 0 & 2m_{M_s} \end{pmatrix} - E \right) \psi(\mathbf{r}) \quad = \quad 0.$$

  - Project to definite total angular momentum,
    - 7 coupled PDEs $\rightarrow$ 3 coupled ODEs for $\tilde{J} = 0$,
    - 7 coupled PDEs $\rightarrow$ 5 coupled ODEs for $\tilde{J} \geq 1$

    ($\tilde{J}$: total angular momentum excluding the heavy quark spins).
  - Add scattering boundary conditions.

- Determine scattering amplitudes and T matrices from the Schrödinger equation, find poles of $\mathsf{T}_{\tilde{J}}$ in the complex energy plane to identify bound states and resonances.

- The components of the resulting wave functions provide the compositions of the states, i.e. the quarkonium and meson-meson percentages $\%\bar{Q}Q$ and $\%\bar{M}M$.

		theory		experiment			
$\bar{J}^{PC}$	n	m[GeV]	Γ[MeV]	name	m[GeV]	Γ[MeV]	$I^G(J^{PC})$
0^{++}	1	9.618^{+10}_{-15}	-	$\eta_b(1S)$	9.399(2)	10(5)	$0^+(0^{+-})$
				$\Upsilon_b(1S)$	9.460(0)	≈ 0	$0^-(1^{--})$
	2	10.114^{+7}_{-11}	-	$\eta_b(2S)_{\text{BELLE}}$	9.999(6)	-	$0^+(0^{+-})$
				$\Upsilon(2S)$	10.023(0)	≈ 0	$0^-(1^{--})$
	3	10.442^{+7}_{-9}	-	$\Upsilon(3S)$	10.355(1)	≈ 0	$0^-(1^{--})$
	4	10.629^{+1}_{-1}	$49.3^{+5.4}_{-3.9}$	$\Upsilon(4S)$	10.579(1)	21(3)	$0^-(1^{--})$
	5	10.773^{+1}_{-2}	$15.9^{+2.9}_{-4.4}$	$\Upsilon(10750)_{\text{BELLE II}}$	10.753(7)	36(22)	$0^-(1^{--})$
	6	10.938^{+2}_{-2}	$61.8^{+7.6}_{-8.0}$	$\Upsilon(10860)$	10.890(3)	51(7)	$0^-(1^{--})$
	7	11.041^{+5}_{-7}	$45.5^{+13.5}_{-8.2}$	$\Upsilon(11020)$	10.993(1)	49(15)	$0^-(1^{--})$
1^{--}	1	9.930^{+43}_{-52}	-	$\chi_{b0}(1P)$	9.859(1)	-	$0^+(0^{++})$
				$h_b(1P)$	9.890(1)	-	$?^?(1^{+-})$
				$\chi_{b1}(1P)$	9.893(1)	-	$0^+(1^{++})$
				$\chi_{b2}(1P)$	9.912(1)	-	$0^+(2^{++})$
	2	10.315^{+29}_{-40}	-	$\chi_{b0}(2P)$	10.233(1)	-	$0^+(0^{++})$
				$\chi_{b1}(2P)$	10.255(1)	-	$0^+(1^{++})$
				$h_b(2P)_{\text{BELLE}}$	10.260(2)	-	$?^?(1^{+-})$
				$\chi_{b2}(2P)$	10.267(1)	-	$0^+(2^{++})$
	3	10.594^{+32}_{-28}	-	$\chi_{b1}(3P)$	10.512(2)	-	$0^+(0^{++})$
	4	10.865^{+37}_{-21}	$67.5^{+5.1}_{-4.9}$				
	5	10.932^{+33}_{-54}	$101.8^{+7.3}_{-5.1}$				
	6	11.144^{+52}_{-75}	$25.0^{+1.1}_{-1.3}$				
2^{++}	1	10.181^{+35}_{-46}	-	$\Upsilon(1D)$	10.164(2)	-	$0^-(2^{--})$
	2	10.486^{+32}_{-36}	-				
	3	10.799^{+2}_{-2}	$13.0^{+2.1}_{-2.0}$				
	4	11.038^{+30}_{-44}	$40.8^{+2.0}_{-2.8}$				
3^{--}	1	10.390^{+28}_{-39}	-				
	2	10.639^{+31}_{-25}	$2.4^{+1.5}_{-0.9}$				
	3	10.944^{+20}_{-29}	$46.8^{-4.6}_{+6.2}$				
	4	11.174^{+51}_{-69}	$1.9^{+2.1}_{-1.4}$				

Bottomonium, $I = 0$: results

- Results for masses of bound states and resonances consistent with experimentally observed states within expected errors.
- Errors might be large:
  - Lattice QCD results for the potentials computed with unphysically heavy u/d quarks.
  - Heavy quark spin effects and corrections due to the finite b quark mass not included.
- Several bound states in the sectors $\tilde{J} = 0, 1, 2$ with clear experimental counterparts.
- Two resonance candidates for $\Upsilon(10753)$ recently found by Belle:
  - S wave state, $\tilde{J} = 0$, $n = 5$ ($\%\bar{Q}Q \approx 24$, $\%\bar{M}M \approx 76$).
  - D wave state, $\tilde{J} = 2$, $n = 3$ ($\%\bar{Q}Q \approx 21$, $\%\bar{M}M \approx 79$).
- $\Upsilon(10860)$ confirmed as an S wave state, $\tilde{J} = 0$, $n = 6$ ($\%\bar{Q}Q \approx 35$, $\%\bar{M}M \approx 65$).
  [P. Bicudo, M. Cardoso, N. Cardoso, M.W., Phys. Rev. D **101**, 034503 (2020) [arXiv:1910.04827]]
  [P. Bicudo, N. Cardoso, L. Müller, M.W., Phys. Rev. D **103**, 074507 (2021) [arXiv:2008.05605]]
  [P. Bicudo, N. Cardoso, L. Müller, M.W., Phys. Rev. D **107**, 094515 (2023) [arXiv:2205.11475]]

Bottomonium, $I = 0$: $1/m_Q$ corrections

- Potentials of static quarks are independent of the heavy spins.
  $\rightarrow$ Systematic errors are possibly large, $\mathcal{O}(m_{B^*} - m_B) = \mathcal{O}(45\,\text{MeV})$.
- Such spin effects and further corrections due to the finite b quark mass can be expressed order by order in $1/m_b$.
  [E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981)]
  [N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D **63**, 014023 (2001) [arXiv:hep-ph/0002250]]
- The corresponding correlation functions are Wilson loops with field strength insertions.
- Computations in pure SU(3) lattice gauge theory (no light quarks) up to order $1/m_Q^2$ in
  [Y. Koma and M. Koma, Nucl. Phys. B **769**, 79-107 (2007) [arXiv:hep-lat/0609078]]
- $1/m_Q$ and $1/m_Q^2$ corrections used to predict low lying (stable) bottomonium states with 1st order stationary perturbation theory.
  [Y. Koma and M. Koma, PoS **LATTICE2012**, 140 (2012) [arXiv:1211.6795 [hep-lat]]
  $\rightarrow$ Improvements, but still no satisfactory agreement with experimental results.
- Onging efforts
  - to compute these $1/m_Q$ and $1/m_Q^2$ corrections more precisely using gradient flow,
  - to replace perturbation theory by a non-perturbative coupled channel SE.

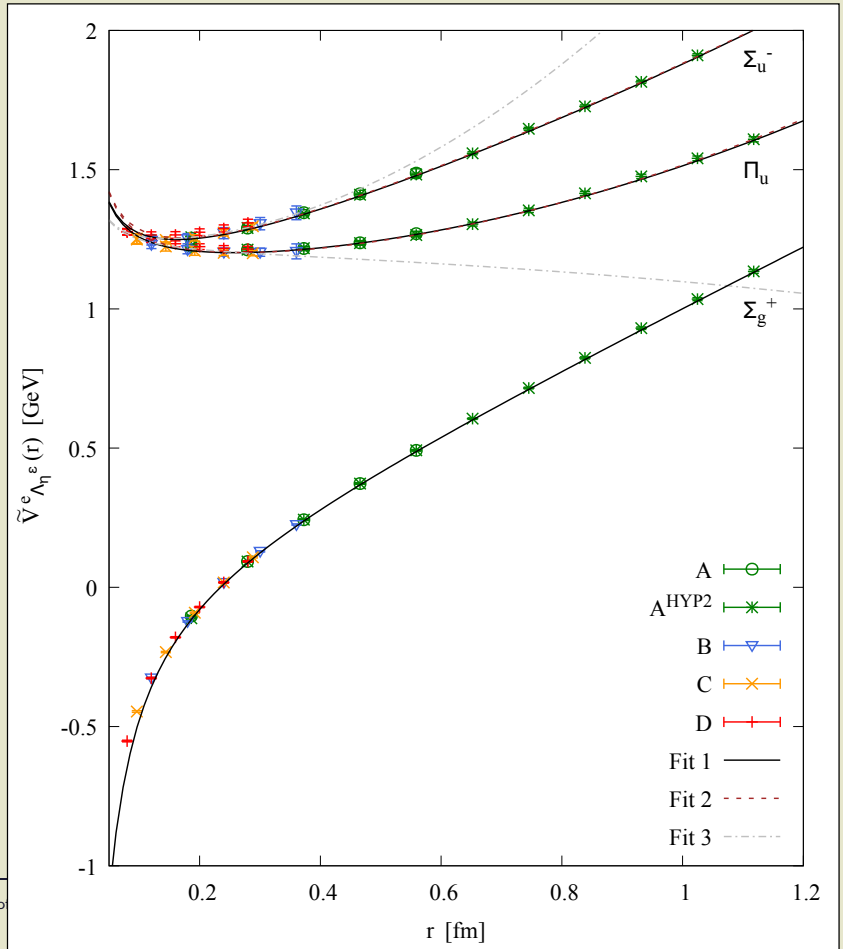
Heavy hybrid mesons: potentials (1)

- Now heavy hybrid mesons, i.e. $\bar{b}b +$ gluons.
- (Hybrid) static potentials can be characterized by the following quantum numbers:
  - Absolute total angular momentum with respect to the $\bar{Q}Q$ separation axis (z axis): $\Lambda = 0, 1, 2, \ldots \equiv \Sigma, \Pi, \Delta, \ldots$
  - Parity combined with charge conjugation: $\eta = +, - = g, u$.
  - Relection along an axis perpendicular to the $\bar{Q}Q$ separation axis (x axis): $\epsilon = +, -$.
- The ordinary static potential has quantum numbers $\Lambda_\eta^\epsilon = \Sigma_g^+$.
- Particularly interesting: the two lowest hybrid static potentials with $\Lambda_\eta^\epsilon = \Pi_u, \Sigma_u^-$.
- References:

[K. J. Juge, J. Kuti, C. J. Morningstar, Nucl. Phys. Proc. Suppl. **63**, 326 (1998) [hep-lat/9709131]
[C. Michael, Nucl. Phys. A **655**, 12 (1999) [hep-ph/9810415]
[G. S. Bali *et al.* [SESAM and TχL Collaborations], Phys. Rev. D **62**, 054503 (2000) [hep-lat/0003012]
[K. J. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. **90**, 161601 (2003) [hep-lat/0207004]
[C. Michael, Int. Rev. Nucl. Phys. **9**, 103 (2004) [hep-lat/0302001]
[G. S. Bali, A. Pineda, Phys. Rev. D **69**, 094001 (2004) [hep-ph/0310130]
[P. Bicudo, N. Cardoso, M. Cardoso, Phys. Rev. D **98**, 114507 (2018) [arXiv:1808.08815 [hep-lat]]]
[S. Capitani, O. Philipsen, C. Reisinger, C. Riehl. M.W., Phys. Rev. D **99**, 034502 (2019) [arXiv:1811.11046 [hep-lat]]]

Heavy hybrid mesons: potentials (2)

- [C. Schlosser, M.W., Phys. Rev. D **105**, 054503 (2022) [arXiv:2111.00741]]



Heavy hybrid mesons: SE

- Solve Schrödinger equations for the relative coordinate of $\bar{b}b$ using hybrid static potentials,

$$\Big(-\frac{1}{2\mu}\frac{d^2}{dr^2}+\frac{L(L+1)-2\Lambda^2+J_{\Lambda_\eta^\epsilon}(J_{\Lambda_\eta^\epsilon}+1)}{2\mu r^2}+V_{\Lambda_\eta^\epsilon}(r)\Big)u_{\Lambda_\eta^\epsilon;L,n}(r) \quad = \quad E_{\Lambda_\eta^\epsilon;L,n}u_{\Lambda_\eta^\epsilon;L,n}(r).$$

  Energy eigenvalues $E_{\Lambda_\eta^\epsilon;L,n}$ correspond to masses of $\bar{b}b$ hybrid mesons.
  [E. Braaten, C. Langmack, D. H. Smith, Phys. Rev. D **90**, 014044 (2014) [arXiv:1402.0438]]
  [M. Berwein, N. Brambilla, J. Tarrus Castella, A. Vairo, Phys. Rev. D **92**, 114019 (2015) [arXiv:1510.04299]]
  [R. Oncala, J. Soto, Phys. Rev. D **96**, 014004 (2017) [arXiv:1702.03900]]

- Important recent and ongoing work to include heavy spin and $1/m_b$ corrections.
  [N. Brambilla, G. Krein, J. Tarrus Castella, A. Vairo, Phys. Rev. D **97**, 016016 (2018) [arXiv:1707.09647]]
  [N. Brambilla, W. K. Lai, J. Segovia, J. Tarrus Castella, A. Vairo, Phys. Rev. D **99**, 014017 (2019) [arXiv:1805.07713]]

Hybrid flux tubes (1)

- We are interested in

$$\Delta F^2_{\mu\nu,\Lambda^\epsilon_\eta}(r;\mathbf{x}) \quad = \quad \langle 0_{\Lambda^\epsilon_\eta}(r)|F^2_{\mu\nu}(\mathbf{x})|0_{\Lambda^\epsilon_\eta}(r)\rangle - \langle\Omega|F^2_{\mu\nu}|\Omega\rangle.$$

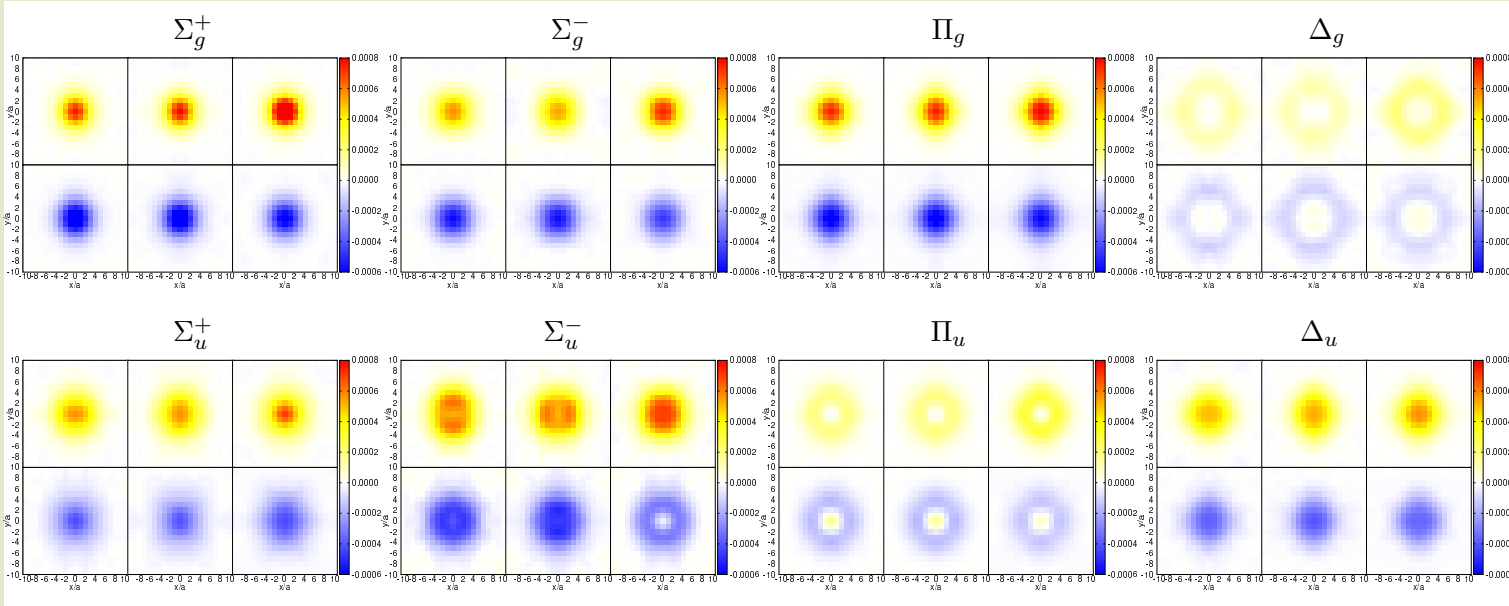
  - $F^2_{\mu\nu}(\mathbf{x})$, $F^2_{\mu\nu}$: squared chromoelectric/chromomagnetic field strength.
  - $|0_{\Lambda^\epsilon_\eta}(r)\rangle$: "hybrid static potential (ground) state" (r denotes the $\bar{Q}Q$ separation).
  - $|\Omega\rangle$: vacuum state.

- The sum over the six independent $\Delta F^2_{\mu\nu,\Lambda^\epsilon_\eta}(r;\mathbf{x})$ is proportional to the chromoelectric and -magnetic energy density of hybrid flux tubes.

Hybrid flux tubes (2)

- $\Delta F^2_{\mu\nu,\Lambda^\epsilon_\eta}(r;\mathbf{x})$, SU(2), mediator plane (x-y plane with Q, $\bar{Q}$ at $(0,0,\pm r/2)$), $r \approx 0.8\,\text{fm}$.
  [L. Müller, O. Philipsen, C. Reisinger, M.W., Phys. Rev. D **100**, 054503 (2019) [arXiv:1907.014820]]]
- For results for $\Lambda^\epsilon_\eta = \Sigma^+_g, \Sigma^+_u, \Pi_u$ see also
  [P. Bicudo, N. Cardoso and M. Cardoso, Phys. Rev. D **98**, 114507 (2018) [arXiv:1808.08815]]

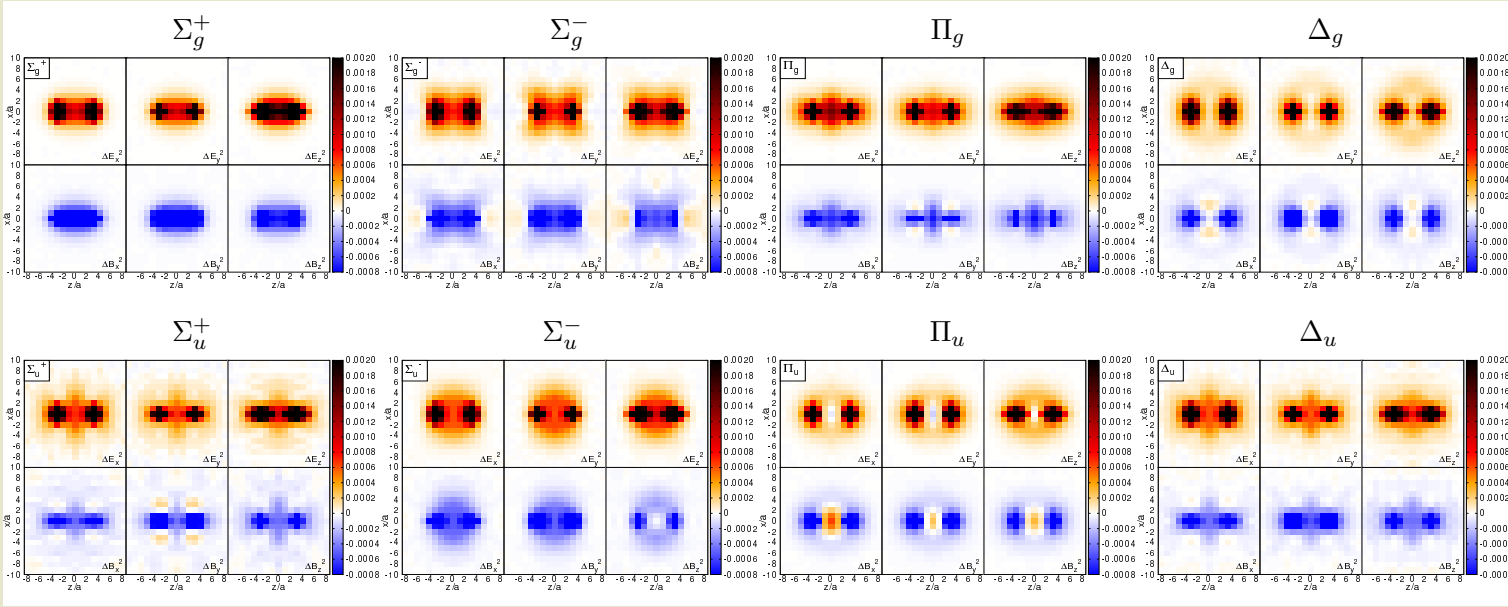
ΔE^2_x	ΔE^2_y	ΔE^2_z
ΔB^2_x	ΔB^2_y	ΔB^2_z



Hybrid flux tubes, $r \approx 0.48\,\mathrm{fm}$ (3)

- $\Delta F^2_{\mu\nu,\Lambda^\epsilon_\eta}(r;\mathbf{x})$, SU(2), separation plane (x-z plane with Q, $\bar{Q}$ at $(0,0,\pm r/2)$), $r \approx 0.8\,\mathrm{fm}$.
  [L. Müller, O. Philipsen, C. Reisinger, M.W., Phys. Rev. D **100**, 054503 (2019) [arXiv:1907.014820]]]
- For results for $\Lambda^\epsilon_\eta = \Sigma^+_g, \Sigma^+_u, \Pi_u$ see also
  [P. Bicudo, N. Cardoso and M. Cardoso, Phys. Rev. D **98**, 114507 (2018) [arXiv:1808.08815]]

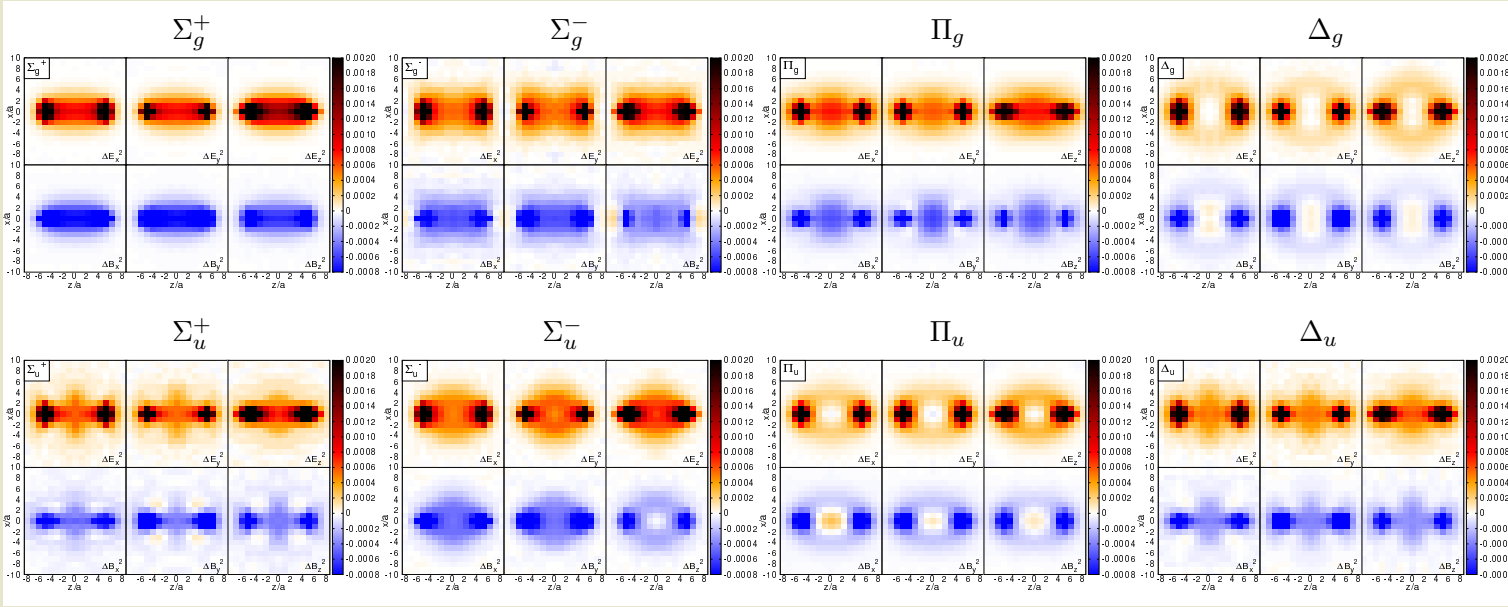
ΔE^2_x	ΔE^2_y	ΔE^2_z
ΔB^2_x	ΔB^2_y	ΔB^2_z



Hybrid flux tubes, $r \approx 0.80\,\mathrm{fm}$ (3)

- $\Delta F^2_{\mu\nu,\Lambda^\epsilon_\eta}(r;\mathbf{x})$, SU(2), separation plane (x-z plane with Q, $\bar{Q}$ at $(0,0,\pm r/2)$), $r \approx 0.8\,\mathrm{fm}$.
  [L. Müller, O. Philipsen, C. Reisinger, M.W., Phys. Rev. D **100**, 054503 (2019) [arXiv:1907.014820]]]
- For results for $\Lambda^\epsilon_\eta = \Sigma^+_g, \Sigma^+_u, \Pi_u$ see also
  [P. Bicudo, N. Cardoso and M. Cardoso, Phys. Rev. D **98**, 114507 (2018) [arXiv:1808.08815]]

ΔE^2_x	ΔE^2_y	ΔE^2_z
ΔB^2_x	ΔB^2_y	ΔB^2_z



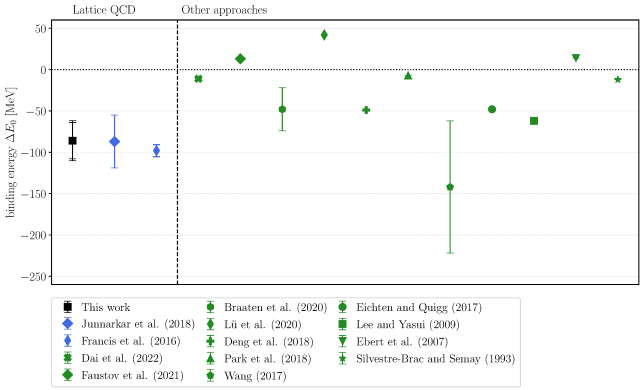
Part 2:
Full lattice QCD computations of eigenvalues of the QCD Hamiltonian

Full lattice QCD computations

- Do not treat the heavy b or c quarks as static.
- Do not separate the computations for heavy and for light quarks, i.e. no potentials.
- Compute eigenvalues of the QCD Hamiltonian at finite spatial volume.
- For QCD-stable states that might already be sufficient.
- For resonances:
  - Relate finite volume energy levels to infinite volume scattering phases (or equivalently scattering amplitudes).
  - Fit an ansatz for the scattering amplitude to the data points (typically only a small number) from the previous step.
  - Find poles in the complex energy plane.

$\bar{b}\bar{b}ud$, $I(J^P) = 0(1^+)$ and $\bar{b}\bar{b}us$, $J^P = 1^+$

- QCD-stable $\bar{b}\bar{b}ud$ tetraquark, $I(J^P) = 0(1^+)$, ≈ 130 MeV below the BB^* threshold.
- QCD-stable $\bar{b}\bar{b}us$ tetraquark, $J^P = 1^+$, ≈ 90 MeV below the BB_s^* threshold.
- Lattice QCD results from independent groups consistent within statistical errors.
  [A. Francis, R. J. Hudspith, R. Lewis, K. Maltman, Phys. Rev. Lett. **118**, 142001 (2017) [arXiv:1607.05214]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)
  [P. Junnarkar, N. Mathur, M. Padmanath, Phys. Rev. D **99**, 034507 (2019) [arXiv:1810.12285]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)
  [L. Leskovec, S. Meinel, M. Pflaumer, M.W., Phys. Rev. D **100**, 014503 (2019) [arXiv:1904.04197]] ($\bar{b}\bar{b}ud$)
  [P. Mohanta, S. Basak, Phys. Rev. D **102**, 094516 (2020) [arXiv:2008.11146]] ($\bar{b}\bar{b}ud$)
  [S. Meinel, M. Pflaumer, M.W., Phys. Rev. D **106**, 034507 (2022) [arXiv:2205.13982]] ($\bar{b}\bar{b}us$)
  [R. J. Hudspith, D. Mohler, Phys. Rev. D **107**, 114510 (2023) [arXiv:2303.17295]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)
  [T. Aoki, S. Aoki, T. Inoue, [arXiv:2306.03565]] ($\bar{b}\bar{b}ud$)
- Strong discrepancies between non-lattice QCD results.



Conclusions

- Significant progress and interesting lattice QCD results in the past ≈ 10 years on heavy exotic mesons ... but still a lot to do and several problems to solve.
- This talk: focus on heavy exotics with two bottom (anti)quarks in the Born-Oppenheimer approximation.
  - Lattice QCD used to compute bb and $\bar{b}b$ potentials in QCD.
  - Majority of presented results obtained with static b quarks.
    $\rightarrow$ Crude, errors of order $\mathcal{O}(m_{B^*} - m_B) = \mathcal{O}(45\,\text{MeV})$ expected.
  - The computation of potentials provides interesting insights, e.g. composition of exotic mesons or hybrid flux tubes.
  - For solid quantitative results heavy spin and finite b quark mass corrections are needed (ongoing work, challenge for the near future).
- Full lattice QCD computations, i.e. not Born-Oppenheimer: mostly studies of $\bar{Q}\bar{Q}qq$.
- At the moment quantitatively reliable and independently confirmed results only for two systems, the QCD-stable tetraquarks $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^+)$ and $\bar{b}\bar{b}us$ with $J^P = 1^+$.