

$\bar{b}\bar{b}ud$ Tetraquarks with $I(J^P) = 0(1^-)$ and $\bar{b}\bar{c}ud$ Tetraquarks with $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$ from Lattice QCD Antistatic-Antistatic Potentials

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We study heavy spin effects in $\bar{b}\bar{b}ud$ and $\bar{b}\bar{c}ud$ four-quark systems using the Born-Oppenheimer approximation and existing antistatic-antistatic potentials computed with lattice QCD. We report about a recent refined investigation of the $\bar{b}\bar{b}ud$ system with $I(J^P) = 0(1^-)$, where we predicted a tetraquark resonance slightly above the B^*B^* threshold. Furthermore, we extend our Born-Oppenheimer approach to $\bar{b}\bar{c}ud$ four-quark systems. For quantum numbers $I(J^P) = 0(0^+)$ as well as $I(J^P) = 0(1^+)$ we find virtual bound states rather far away from the lowest meson-meson thresholds.

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1. Introduction

In this talk we discuss our investigations of $\bar{b}\bar{b}ud$ and $\bar{b}\bar{c}ud$ four-quark systems using the Born-Oppenheimer approximation, which is a two-step approach. In the first step antistatic-antistatic potentials in the presence of two light quarks are computed with lattice QCD (see Section 2). In the second step, these potentials are used in appropriate coupled-channel Schrödinger equations, where bound states as well as resonances can be predicted using standard techniques from non-relativistic quantum mechanics (see Section 3 and Section 4). Using such coupled-channel Schrödinger equations as well as experimental results for B , B^* , D and D^* mesons allows to take into account effects from the heavy quark spins, even though the antistatic-antistatic potentials are degenerate with respect to these spins.

In the following sections we briefly summarize a completed study of a $\bar{b}\bar{b}ud$ tetraquark resonance with quantum numbers $I(J^P) = 0(1^-)$, where details have recently been published in Ref. [8]. We also present theoretical basics and new results for $\bar{b}\bar{c}ud$ four-quark systems with quantum numbers $I(J^P) = 0(0^+)$ as well as $I(J^P) = 0(1^+)$, which have previously not been investigated within the Born-Oppenheimer approximation.

2. The Antistatic-Antistatic-Light-Light Potentials $V_5(r)$ and $V_j(r)$

Theoretical details of antistatic-antistatic potentials as well as their numerical computation with lattice QCD are extensively discussed in Refs. [1–4]. In this work we use existing potentials from Refs. [2], which were computed using $N_f = 2$ flavor ETMC gauge link ensembles [5, 6] and extrapolated to physically light u and d quark masses. Relevant in the context of this work are the two $I = 0$ potentials $V_5(r)$ and $V_j(r)$ representing the interaction of two pseudoscalar and/or vector static light mesons. Suitable parameterizations of lattice QCD results for these potentials are

$$V_X(r) = -\frac{\alpha_X}{r} e^{-(r/d_X)^2} \quad , \quad X = 5, j \quad (1)$$

with $\alpha_5 = 0.34 \pm 0.03$, $d_5 = 0.45^{+0.12}_{-0.10}$ fm (see Ref. [2]) and $\alpha_j = -0.10 \pm 0.07$ and $d_j = (0.28 \pm 0.017)$ fm (see Ref. [7]). The parameterizations are shown in Figure 1. For details we refer to Section II of our recent publication [8].

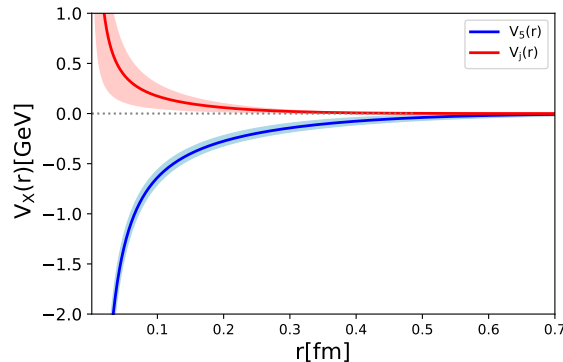


Figure 1: Parametrizations of lattice QCD results from Ref [2] for the $\bar{Q}\bar{Q}qq$ potentials $V_5(r)$ and $V_j(r)$.

3. Coupled-Channel Schrödinger Equations

3.1 Identical Heavy Flavors: $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^-)$

In Ref. [8] we have derived the coupled-channel Schrödinger equation relevant for the $\bar{b}\bar{b}ud$ system with $I(J^P) = 0(1^-)$. It is given by

$$\left(\begin{pmatrix} 2m_B & 0 \\ 0 & 2m_{B^*} \end{pmatrix} - \frac{\nabla^2}{2\mu_{bb}} + H_{\text{int},S=0} \right) \vec{\varphi}_{L=1,S=0}(r) = E \vec{\varphi}_{L=1,S=0}(r) \quad (2)$$

with

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L(L+1)}{r^2} \Big|_{L=1} = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2} \quad (3)$$

(L represents the orbital angular momentum of the heavy antiquarks) and

$$H_{\text{int},S=0} = \frac{1}{4} \begin{pmatrix} V_5(r) + 3V_j(r) & \sqrt{3}(V_5(r) - V_j(r)) \\ \sqrt{3}(V_5(r) - V_j(r)) & 3V_5(r) + V_j(r) \end{pmatrix}, \quad (4)$$

where $\mu_{bb} = m_b/2$ is the reduced b quark mass, $S = 0$ denotes the heavy spin and r is the radial coordinate of the heavy antiquark separation. The 2 components of the wave function represent the following meson-meson combinations:

$$\vec{\varphi}_{L=1,S=0} \equiv \left(BB, \frac{1}{\sqrt{3}} \vec{B}^* \vec{B}^* \right)^T = \left(BB, \frac{1}{\sqrt{3}} (B_x^* B_x^* + B_y^* B_y^* + B_z^* B_z^*) \right)^T. \quad (5)$$

3.2 Different Heavy Flavors: $\bar{b}\bar{c}ud$ with $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$

To derive the coupled-channel Schrödinger equations for the $\bar{b}\bar{c}ud$ systems one can closely follow Refs. [7, 8] as sketched in the following.

$\bar{b}\bar{c}ud$ systems at large $\bar{b}\bar{c}$ separations r are meson pairs, where one of the two mesons is a B or B^* meson and the other meson is a D or D^* meson. Consequently, the free Hamiltonian describing non-interacting meson pairs in the center of mass frame has a 16×16 matrix structure,

$$H_0 = M_B \otimes \mathbb{1}_{4 \times 4} + \mathbb{1}_{4 \times 4} \otimes M_D + \frac{\vec{p}^2}{2\mu_{bc}}, \quad (6)$$

where $M_B = \text{diag}(m_B, m_{B^*}, m_{B^*}, m_{B^*})$ and $M_D = \text{diag}(m_D, m_{D^*}, m_{D^*}, m_{D^*})$ are diagonal matrices containing the meson masses and $\mu_{bc} = m_b m_c / (m_b + m_c)$ is the reduced mass of a b and a c quark. This Hamiltonian acts on a 16-component wave function for the relative coordinate of the heavy quarks \vec{r} , where the components can be interpreted as

$$\vec{\Psi} \equiv \left(BD, BD_x^*, BD_y^*, BD_z^*, B_x^* D, B_x^* D_x^*, B_x^* D_y^*, B_x^* D_z^*, B_y^* D, B_y^* D_x^*, B_y^* D_y^*, B_y^* D_z^*, B_z^* D, B_z^* D_x^*, B_z^* D_y^*, B_z^* D_z^* \right)^T. \quad (7)$$

To include interactions, one has to add the the potentials $V_5(r)$ and $V_j(r)$ discussed in Section 2. One can show that these potentials do not correspond to simple meson pairs, as represented by the components of $\vec{\Psi}$, but to linear combinations containing all four types of mesons, B, B^*, D and D^* .

These linear combinations can be expressed in terms of a 16×16 matrix T using Fierz identities. The interacting part of the Hamiltonian is then

$$H_{\text{int}} = TV_{\text{diag}}T^{-1} \quad , \quad V_{\text{diag}} = \text{diag}(\underbrace{V_5(r), \dots, V_5(r)}_{4 \times}, \underbrace{V_j(r), \dots, V_j(r)}_{12 \times}). \quad (8)$$

Combining Eq. (6) and Eq. (8) leads to the Schrödinger equation

$$H\vec{\Psi}(\vec{r}) = (H_0 + H_{\text{int}})\vec{\Psi}(\vec{r}) = E\vec{\Psi}(\vec{r}). \quad (9)$$

Because H_{int} only depends on the radial coordinate $r = |\vec{r}|$, but not on the direction of \vec{r} , the orbital angular momentum L is conserved. Since the total angular momentum is a conserved quantity, the total spin S is also conserved. Consequently, both L and S can be used as quantum numbers. The former allows to reduce the partial differential equation (9) to an ordinary differential equation in r , while the latter allows to decompose the 16×16 Hamiltonian into smaller blocks.

We are particularly interested in $\bar{b}\bar{c}ud$ systems with quantum numbers $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$, since recent full lattice QCD computations indicate the existence of shallow bound states for these systems [9–11]. After working out the $I(J^P)$ quantum numbers for each block of the decomposed Hamiltonian using QCD symmetries and the Pauli principle (for details see e.g. Section III.III in Ref. [8]), one can read off the relevant coupled channel Schrödinger equations.

Schrödinger Equation for $\bar{b}\bar{c}ud$ with $I(J^P) = 0(0^+)$

The coupled-channel Schrödinger equation for the $\bar{b}\bar{c}ud$ system with $I(J^P) = 0(0^+)$ is

$$\left(\begin{pmatrix} m_B + m_D & 0 \\ 0 & m_{B^*} + m_{D^*} \end{pmatrix} - \frac{1}{2\mu_{bc}} \frac{d^2}{dr^2} + H_{\text{int},S=0} \right) \vec{\varphi}_{L=0,S=0}(r) = E\vec{\varphi}_{L=0,S=0}(r) \quad (10)$$

with $H_{\text{int},S=0}$ as defined in Eq. (4). The 2 components of the wave function represent the following meson-meson combinations:

$$\vec{\varphi}_{L=0,S=0} \equiv \left(BD \quad , \quad \frac{1}{\sqrt{3}}\vec{B}^*\vec{D}^* \right)^T = \left(BD \quad , \quad \frac{1}{\sqrt{3}}(B_x^*D_x^* + B_y^*D_y^* + B_z^*D_z^*) \right)^T. \quad (11)$$

Schrödinger Equation for $\bar{b}\bar{c}ud$ with $I(J^P) = 0(1^+)$

The coupled-channel Schrödinger equation for the $\bar{b}\bar{c}ud$ system with $I(J^P) = 0(1^+)$ is

$$\left(\begin{pmatrix} m_{B^*} + m_D & 0 & 0 \\ 0 & m_B + m_{D^*} & 0 \\ 0 & 0 & m_{B^*} + m_{D^*} \end{pmatrix} - \frac{1}{2\mu_{bc}} \frac{d^2}{dr^2} + H_{\text{int},S=1} \right) \vec{\varphi}_{L=0,S=1,S_z}(r) = E\vec{\varphi}_{L=0,S=1,S_z}(r) \quad (12)$$

with

$$H_{\text{int},S=1} = \frac{1}{4} \begin{pmatrix} V_5(r) + 3V_j(r) & V_j(r) - V_5(r) & \sqrt{2}(V_5(r) - V_j(r)) \\ V_j(r) - V_5(r) & V_5(r) + 3V_j(r) & \sqrt{2}(V_j(r) - V_5(r)) \\ \sqrt{2}(V_5(r) - V_j(r)) & \sqrt{2}(V_j(r) - V_5(r)) & 2(V_5(r) + V_j(r)) \end{pmatrix}. \quad (13)$$

The 3 components of the wave function represent the following meson-meson combinations:

$$\vec{\varphi}_{L=0,S=1,S_z} \equiv \left(B_{S_z}^* D, BD_{S_z}^*, T_{1,S_z}(\vec{B}^*, \vec{D}^*) \right)^T \quad (14)$$

with T_{1,S_z} denoting a spherical tensor coupling the three spin orientations of a B^* and of a D^* meson to a total spin $S = 1$ with z component S_z .

4. Scattering Formalism and the T matrix

Possibly existing bound states and resonances can be studied within the same formalism, by writing the wave function as a sum of an incident plane wave and an emergent spherical wave and by carrying out a partial wave expansion. Then, one can read off the T matrix and determine its poles in the complex energy plane, which signal bound states or virtual bound states (for $\text{Re}(E_{\text{pole}}) < 2m_B$ and $\text{Im}(E_{\text{pole}}) = 0$) or resonances (for $\text{Re}(E_{\text{pole}}) > 2m_B$ and $\text{Im}(E_{\text{pole}}) < 0$). For details we refer to Section IV of our recent publication [8].

4.1 The T matrix for the $\bar{b}\bar{c}ud$ system with $I(J^P) = 0(1^-)$

After the aforementioned partial wave expansion the $L = 1$ wave function (5) becomes

$$\vec{\varphi}_{L=1,S=0}(r) = \begin{pmatrix} A_{BB}j_1(k_{BB}r) + \chi_{BB}(r)/r \\ A_{B^*B^*}j_1(k_{B^*B^*}r) + \chi_{B^*B^*}(r)/r \end{pmatrix}, \quad (15)$$

where A_{BB} and $A_{B^*B^*}$ are the prefactors of the incident BB and B^*B^* waves, respectively, $j_1(k_{BB}r)$ and $j_1(k_{B^*B^*}r)$ denote spherical Bessel functions with scattering momenta $k_{BB} = \sqrt{2\mu(E - 2m_B)}$ and $k_{B^*B^*} = \sqrt{2\mu(E - 2m_{B^*})}$ representing the $L = 1$ contribution to these incident plane waves and $\chi_{BB}(r)/r$ and $\chi_{B^*B^*}(r)/r$ are the radial wave functions of the emergent BB and B^*B^* spherical waves. Inserting $\vec{\varphi}_{L=1,S=0}(r)$ from Eq. (15) into the Schrödinger equation (2) leads to

$$\left(\begin{pmatrix} 2m_B & 0 \\ 0 & 2m_{B^*} \end{pmatrix} - \frac{1}{2\mu_{bb}} \left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right) + H_{\text{int},S=0} - E \right) \begin{pmatrix} \chi_{BB}(r) \\ \chi_{B^*B^*}(r) \end{pmatrix} = -H_{\text{int},S=0} \begin{pmatrix} A_{BB}rj_1(k_{BB}r) \\ A_{B^*B^*}rj_1(k_{B^*B^*}r) \end{pmatrix}. \quad (16)$$

As usual, the boundary conditions for the wave functions close to the origin are $\chi_\alpha(r) \propto r^{L+1}|_{L=1} = r^2$. For large r the wave functions $\chi_\alpha(r)$ exclusively describe emergent spherical waves and, thus, are proportional to spherical Hankel functions,

$$\chi_\alpha(r) \propto ir t_{BB;\alpha} h_1^{(1)}(k_\alpha r) \quad \text{for } r \rightarrow \infty \text{ and } (A_{BB}, A_{B^*B^*}) = (1, 0) \quad (17)$$

$$\chi_\alpha(r) \propto ir t_{B^*B^*;\alpha} h_1^{(1)}(k_\alpha r) \quad \text{for } r \rightarrow \infty \text{ and } (A_{BB}, A_{B^*B^*}) = (0, 1), \quad (18)$$

where $t_{\alpha;\beta}$ denote entries of the T matrix. Thus, Eq. (17) and Eq. (18) allow to determine the 2×2 T matrix,

$$\mathbb{T} = \begin{pmatrix} t_{BB;BB} & t_{BB;B^*B^*} \\ t_{B^*B^*;BB} & t_{B^*B^*;B^*B^*} \end{pmatrix}. \quad (19)$$

4.2 T Matrices for the $\bar{b}\bar{c}ud$ Systems with $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$

One can proceed as sketched in Section 4.1. Because $L = 0$ in both cases one has to replace j_1 by j_0 . Moreover, scattering momenta have to be defined according to the meson types associated with each channel. At the end one arrives at a 2×2 T matrix for $I(J^P) = 0(0^+)$ and at a 3×3 T matrix for $I(J^P) = 0(1^+)$. Because of the page limit, we refrain from providing the corresponding equations.

5. Numerical Results

The following numerical results were generated using quark masses $m_b = 4977$ MeV and $m_c = 1628$ MeV taken from a quark model [12]. For the meson mass splittings we use $m_{B^*} - m_B = 45$ MeV and $m_{D^*} - m_D = 138$ MeV as quoted by the PDG [13]. We solved the coupled-channel radial Schrödinger equations for the wave functions of the emergent wave $\chi_\alpha(r)$ for given complex energy E using a standard fourth order Runge-Kutta integrator (e.g. Eq. (16) in the case of the $\bar{b}\bar{b}ud$ system with $I(J^P) = 0(1^-)$). Then we read off the corresponding T matrix elements from the behavior of the resulting $\chi_\alpha(r)$ at large r , using e.g. Eq. (17) and Eq. (18) for the $\bar{b}\bar{b}ud$ system with $I(J^P) = 0(1^-)$. Finally, we determine the poles of the T matrix by numerically searching for roots of $1/\det(T)$. For details we refer again to our recent publication [8].

5.1 $\bar{b}\bar{b}ud$ Tetraquark Resonance with $I(J^P) = 0(1^-)$

Numerical results for the $\bar{b}\bar{b}ud$ system with $I(J^P) = 0(1^-)$ are extensively discussed in Ref. [8]. Our main findings are the following:

- (1) We found a pole of the T matrix on the $(-, -)$ -Riemann sheet ¹ indicating a tetraquark resonance with mass $2m_B + 94.0^{+1.3}_{-5.4}$ MeV = $2m_{B^*} + 4.0^{+1.3}_{-5.4}$ MeV, i.e. slightly above the B^*B^* threshold, and decay width $\Gamma = 140^{+86}_{-66}$ MeV.
- (2) The coupled channel Schrödinger equation (16), in particular the potential matrix (4), led to a solid physical understanding, why there is a tetraquark resonance close to the B^*B^* threshold, but not in the region of the BB threshold, as naively expected from our previous work [14] using a simplified single-channel approach. The reason is that the attractive potential $V_5(r)$ dominates the B^*B^* channel, but is strongly suppressed in the BB channel, whereas the situation is reversed for the repulsive potential $V_j(r)$.
- (3) This theoretical result is supported by our computation of branching ratios, where we found $\text{BR}_{BB} = 26^{+9}_{-4}\%$ and $\text{BR}_{B^*B^*} = 74^{+4}_{-9}\%$, implying that a decay of the tetraquark resonance is around three times more likely to a B^*B^* pair than to a BB pair.

5.2 $\bar{b}\bar{c}ud$ virtual bound states with $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$

Virtual Bound States

Using the same techniques as for the $\bar{b}\bar{b}ud$ tetraquark resonance with $I(J^P) = 0(1^-)$, we also searched for poles of the T matrix in the complex energy plane for the two $\bar{b}\bar{c}ud$ systems. These pole searches were carried out on all four Riemann sheets for $I(J^P) = 0(0^+)$ and on all eight Riemann sheets for $I(J^P) = 0(1^+)$. For both systems we did neither find bound states nor resonances, but virtual bound states, indicated by poles on the negative real axis on the $(-, +)$ -sheet and on

¹For n scattering channels there are n scattering momenta k_α and 2^n Riemann sheets for the complex energy E . These sheets are labeled by the signs of the imaginary parts of the scattering momenta, e.g. by $(\text{sign}(k_{BB}), \text{sign}(k_{B^*B^*}))$ for the $\bar{b}\bar{b}ud$ system with $I(J^P) = 0(1^-)$. There is a one-to-one correspondence between bound states and poles on the negative real axis of the physical Riemann sheet, which is characterized by having exclusively positive signs, e.g. the $(+, +)$ sheet in the case of 2-channel scattering.

the $(-, +, +)$ -sheet, respectively. These poles are, however, rather far away from the lowest meson-meson thresholds, $\text{Re}(E) - (m_B + m_D) = -106_{-148}^{+65}$ MeV and $\text{Re}(E) - (m_{B^*} + m_D) = -100_{-212}^{+49}$ MeV. Thus, it is questionable, whether they have a sizable effect on physical observables like scattering rates or cross sections. We plan to investigate this in more detail in the near future.

Dependence on the Charm Quark Mass for $I(J^P) = 0(1^+)$

In Ref. [7] we used the same potentials and formalism discussed in Section 2 and Section 3 to predict a $\bar{b}\bar{b}ud$ bound state with quantum numbers $I(J^P) = 0(1^+)$ and binding energy $(m_B + m_{B^*}) - E = 59_{-38}^{+30}$ MeV. This system, which has a BB^* channel and a B^*B^* channel is conceptually similar to the $\bar{b}\bar{c}ud$ system with the same quantum numbers. In particular, one can show that, when setting $m_c = m_b$, $m_D = m_B$ and $m_{D^*} = m_{B^*}$ in the coupled channel Schrödinger equation (12), one equation decouples and the remaining two equations are essentially identical to those solved in Ref. [7]. We have studied this numerically by starting with Eq. (12) and physical quark masses m_b and m_c as provided at the beginning of Section 5 and then continuously increasing m_c from its physical value 1628 MeV to the value of the b quark mass. At the same time we reduce the mass splitting between the D and the D^* meson according to

$$m_{D^*} - m_D = \frac{m_b}{m_c} (m_{B^*} - m_B), \quad (20)$$

which is the leading order in Heavy Quark Effective Theory [15]. The resulting pole energy as function of m_c is shown in Figure 2. One can see the expected transition from a virtual bound state corresponding to a pole on the $(-, +, +)$ -Riemann sheet to a bound state corresponding to a pole on the physical $(+, +, +)$ -Riemann sheet. The transition between the two sheets takes place at $m_c \approx 2930$ MeV, where the pole is located at $E = 0$. For $m_c = m_b$ we recover the binding energy $(m_B + m_{B^*}) - E = 59_{-38}^{+30}$ MeV predicted in Ref. [7], which is an excellent cross check and shows consistency of this work and Ref. [7].

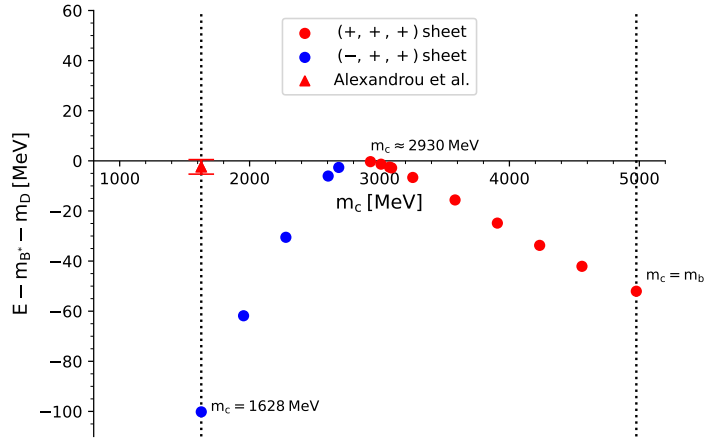


Figure 2: The energy of the T matrix pole as function of the charm quark mass m_c for the $\bar{b}\bar{c}ud$ system with quantum numbers $I(J^P) = 0(1^+)$. The red triangular data point represents the full lattice QCD result from Ref. [10].

Comparison to Full Lattice QCD Results

Recent full lattice QCD studies of $\bar{b}\bar{c}ud$ systems with quantum numbers $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$ have predicted shallow bound states rather close to the BD threshold and the B^*D threshold, respectively [9–11] (the result from Ref. [10] for $I(J^P) = 0(1^+)$ is plotted in Figure 2). It is interesting to note that the study from Ref. [10], which uses a very advanced lattice QCD setup (large symmetric correlation matrices including both local and scattering interpolating operators, Lüscher’s finite volume method to carry out a scattering analysis), cannot rule out the existence of shallow virtual bound states, even though genuine bound states are strongly favored. In any case there is a sizable quantitative difference of these full lattice QCD results and our $\bar{b}\bar{c}ud$ predictions from this work, which are based on lattice QCD potentials and the Born-Oppenheimer approximation. A possible reason for that could be that the attraction of the potential $V_5(r)$ was underestimated in Refs. [1, 2]. To check this, we have recently started a recomputation of these potentials using a significantly improved up-to-date lattice QCD setup [3, 4].

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