Exercise 8 (Critical magnetic field $H_{c1}$ in Type II superconductors):

Consider a Type II superconductor with $\kappa \gg 1$, characterized by $n = N/A$ vortex lines per unit area, each one carrying one flux quantum $\Phi_0 = 2\pi hc/|q^*|$. Assume that the vortex density is sufficiently small so that the average distance between the vortices is much larger than the London length, $n^{-1/2} \gg \lambda$, and the vortices can be treated independently.

a) Assuming that the modulus of the order parameter in the superconducting region is constant, show that the Gibbs free energy per unit volume in the presence of vortices can be approximately written as

$$
\frac{G_S[\psi,H;n]}{V} = \frac{F_S[\psi,B(n);n]}{V} - \frac{B(n)H}{4\pi}
$$

$$
= g_0 + r|\psi|^2 + \frac{u}{2}|\psi|^4 + nE^0_V + \frac{B^2(n)}{2\pi} - \frac{B(n)H}{4\pi},
$$

where

$$
E^0_V = \frac{\Phi_0^2}{16\pi^2\lambda^2} \ln \left(\frac{\lambda}{\xi}\right)
$$

is the energy per unit length of a single vortex line, calculated in Exercise 7, and $B(n) = n\Phi_0$ is the average magnetic induction in the sample.

b) Find the critical value $H = H_{c1}$ for which

$$
\left.\frac{dG_S}{dn}\right|_{n=0} = 0,
$$

corresponding to the value of the magnetic field for which the formation of vortices becomes energetically favorable,

$$
\left.\frac{dG_S}{dn}\right|_{n=0} < 0 \quad \text{for} \; H > H_{c1}.
$$

Show that in terms of the thermodynamic critical field $H_c = \frac{1}{2\pi\sqrt{2}\lambda\xi}$, the critical field $H_{c1}$ reads (for $\kappa \gg 1$)

$$
H_{c1} = \frac{H_c}{\sqrt{2\kappa}} \ln \kappa.
$$
Exercise 9 (Effective repulsion induced by the gravitational force):

As you will learn in class, the microscopic origin of superconductivity in “conventional” BCS-like superconductors is the presence of an effective attraction between electrons, due to their interaction with the lattice degrees of freedom (phonons). In other words, the interaction of the electrons with the environment changes their mutual repulsive interaction into an effective attraction. To easily understand how the environment can change the nature (repulsive or attractive) of a given interaction, consider then the following simple problem.

Two spheres of equal volume $V$ and density $\rho_S < \rho_W$, where $\rho_W$ is the density of water, are held under the surface of an infinitely large water basin at a distance $d$, as shown in Fig. 1. Calculate the net force experienced by the two spheres, assuming that the only relevant interaction in the problem is the gravitational force.

Figure 1: Gravitational interaction between two spheres lying under the surface of a water basin.

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